9.1 Systems of Linear Equations; Gaussian Elimination

TECHNOLOGY TIP

<table>
<thead>
<tr>
<th>Limitations and power</th>
</tr>
</thead>
<tbody>
<tr>
<td>Always remember, when using technology to solve systems of equations or inequalities, because of inherent, unavoidable complexities, technology will sometimes give misleading, incorrect answers. Nonetheless, technology will enable us to get answers that would be impossible to obtain in any other way.</td>
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9.1 SYSTEMS OF LINEAR EQUATIONS; GAUSSIAN ELIMINATION

Mathematics is effective precisely because a relatively compact mathematical scheme can be used to predict over a relatively long period of time the future behavior of some physical system to a certain level of accuracy, and thereby generate more information about the system than is contained in the mathematical scheme to begin with.

P. W. C. Davies

Our focus in this section is linear equations in several variables, such as

\[ 3x - 4y + 2z + w = 5 \quad \text{and} \quad -3s + 2t = 1. \]

The following equations are not linear:

\[ x^2 - y = 4 \quad \text{Not linear in } x \]
\[ x + 3 \left| y \right| - z = 7 \quad \text{Not linear in } y \]
\[ uv + \ln w = 0 \quad \text{Not linear in } u, v, \text{ or } w \]

For a system of linear equations, we indicate both the number of equations and the number of variables. A \( 2 \times 2 \) system consists of two equations in two variables, and a \( 3 \times 3 \) system has three equations in three variables:

\[
\begin{align*}
-3x + 4y &= 11 \\
2x - 3y &= -8
\end{align*}
\]

(2)

\[
\begin{align*}
2a - 5b + 3c &= 8 \\
a + 5b - c &= 4 \\
3a + 2c &= 12
\end{align*}
\]

(3)

A solution to a system of linear equations consists of a value for each variable such that when we substitute these values, every equation becomes a true statement. For system (2) above, the values \( x = -1, y = 2 \) satisfy both equations in the system. A solution to system (3) can be written \((a, b, c) = (6, -1, -3)\), which means that \( a = 6, b = -1, \) and \( c = -3 \). The ordered pair of numbers \((-1, 2)\) is the only solution to system (2), but \((8, -2, -6)\) is one of many solutions to system (3).
Equivalent Systems

We need a systematic procedure to find all solutions to a system of equations. There are several methods, some of which you may have seen in previous courses. We will describe a technique that replaces a system of equations in turn by other, simpler systems with the same solutions until we get a system simple enough that we can read off the solution. For example, consider these $3 \times 3$ systems:

\[
\begin{align*}
2x - 5y + 3z &= -4 \\
x - 2y - 3z &= 3 \\
-3x + 4y + 2z &= -4
\end{align*}
\]  
\tag{4}

\[
\begin{align*}
2x - 5y + 3z &= -4 \\
y - 9z &= 10 \\
-2z &= 1
\end{align*}
\]  
\tag{5}

It is simple to solve system (5) by starting with the last equation to get $z = -1$. Substitute into the second equation and find $y = 1$, and then substitute both $y$ and $z$ values into the first equation to get $x = 2$. In fact, it is easy to see that $(x, y, z) = (2, 1, -1)$ is the only solution for system (5). In Example 1 we will show that the two systems have the same solution, and hence that our solution for system (5) is the solution for system (4). Two systems of linear equations are equivalent if they have identical solutions.

In the process of going from system (4) to system (5), we successively eliminate variables. So $x$ has been eliminated from the second equation in system (5), and both $x$ and $y$ have been eliminated in the third equation. System (5) is called an echelon, or upper triangular, form of system (4).

**Definition: echelon (upper triangular) form**

A system of three linear equations in variables $x$, $y$, $z$ is said to be in echelon form if it can be written as

\[
\begin{align*}
a_1x + a_2y + a_3z &= d_1 \\
b_2y + b_3z &= d_2 \\
c_3z &= d_3
\end{align*}
\]

where the coefficients $a$, $b$, $c$, and $d$ are given numbers, some of which may be zero.

**Elementary Operations and Gaussian Elimination**

The systematic elimination of variables to change a system of linear equations into an equivalent system in echelon form from which we can read the solution is called Gaussian elimination in honor of Carl Friedrich Gauss, one of the most brilliant mathematicians of all time.

The key to Gaussian elimination (which can be done efficiently on computers) is the idea of elementary operation, the replacement of one equation in a system by another in a way that leaves the solution unchanged. Each of the following operations gives an equivalent system, that has the same solution set. $E_k$ denotes the $k$th equation of the system and $-2E_1 + E_2$ is what we get when we multiply both sides of equation $E_1$ by $-2$ and add the result to equation $E_2$. 

9.1 Systems of Linear Equations; Gaussian Elimination

### Elementary operations and equivalent systems

<table>
<thead>
<tr>
<th>Operation</th>
<th>Notation and Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Interchange two equations</td>
<td>$E_2 \leftrightarrow E_3$ means interchange equations $E_2$ and $E_3$.</td>
</tr>
<tr>
<td>2. Multiply by a nonzero constant</td>
<td>$4E_3 \rightarrow E_3$ means replace equation $E_3$ with $4E_3$.</td>
</tr>
<tr>
<td>3. Add a multiple of one equation to another equation</td>
<td>$4E_2 + E_1 \rightarrow E_3$ means replace $E_3$ with $4E_2 + E_1$.</td>
</tr>
</tbody>
</table>

Performing any of the elementary operations on a system of linear equations gives an equivalent system.

Follow the next example closely, performing each operation as indicated, to be certain that you understand both the process by which we reduce the original system to echelon form and the notation by which we keep track of and check each step.

#### EXAMPLE 1 Echelon form

Reduce the following system to echelon form and then find the solution.

\[
\begin{align*}
E_1 & \quad 2x - 5y + 3z = -4 \\
E_2 & \quad x - 2y - 3z = 3 \\
E_3 & \quad -3x + 4y + 2z = -4
\end{align*}
\]

**Solution**

Follow the strategy. We will not repeatedly write the equation numbers, simply assuming in each system that the equations are numbered $E_1$, $E_2$, and $E_3$, from top to bottom. Beginning with the given system, we perform elementary operations as indicated:

\[
\begin{align*}
E_1 & \leftrightarrow E_2 \\
E_1 & \Rightarrow E_1 \\
-2E_1 + E_3 & \Rightarrow E_3 \\
3E_1 + E_3 & \Rightarrow E_3 \\
(-2)E_2 + E_3 & \Rightarrow E_3
\end{align*}
\]

We now have a system in echelon form that is equivalent to the given system.

To solve the echelon-form system, start with the last equation and solve for $z$: $z = \frac{-25}{25} = -1$. Substitute $-1$ for $z$ into $E_2$ and solve for $y$: $-y + 9(-1) = -10$, or $y = 1$. Substitute $-1$ for $z$ and $1$ for $y$ into $E_1$ and solve for $x$: $x - 2(1) - 3(-1) = 3$, or $x = 2$. The solution is given by $x = 2$, $y = 1$, $z = -1$.

The process of solving a system of equations in echelon form has the name **back-substitution**. This suggests the procedure of starting at the bottom and working toward the top, substituting into each successive equation.
Chapter 9 Systems of Equations and Inequalities

**EXAMPLE 2** Eliminate \( x \)  
Use elementary operations to get an equivalent system, eliminating the \( x \)-variable from \( E_2 \) and \( E_3 \).

\[
\begin{align*}
2x - 3y + z &= -1 \\
-3x + 4y - z &= 2 \\
2x - y + 2z &= -3
\end{align*}
\]

**Strategy:** We can easily use \( E_1 \) to eliminate \( x \) in \( E_3 \), but to avoid fractions for \( E_2 \), first multiply \( E_2 \) by 2, then add \( 3E_1 \) to eliminate \( x \).

\[
\begin{align*}
2x - 3y + z &= -1 \\
-6x + 8y - 2z &= 4 \\
2x - 3y + z &= -1
\end{align*}
\]

Complete the solution and verify that \( z = 0, y = -1, \) and \( x = -2 \).

**EXAMPLE 3** Gaussian elimination  
Solve the system by using Gaussian elimination.

(a) \[
\begin{align*}
x + 2y - 2z &= 3 \\
2x + 3y - 3z &= 1 \\
-4x - 5y + 5z &= 3
\end{align*}
\]

(b) \[
\begin{align*}
x + 2y - 2z &= 3 \\
2x + 3y - 3z &= 1 \\
-4x - 5y + 5z &= 5
\end{align*}
\]

**Solution**

(a) The following elementary operations lead to an echelon form, from which we find \( x, y, \) and \( z \).

\[
\begin{align*}
(-2)E_1 + E_2 &\rightarrow E_2 \\
2x + 3y - 3z &= 1 \\
-4x - 5y + 5z &= 3 \\
x + 2y - 2z &= 3 \\
\end{align*}
\]

We now have an echelon form system in which \( E_1, 0 \cdot z = 0 \), is satisfied by any number \( z \). Therefore, we have infinitely many solutions. Let \( z = t \), where \( t \) is any number. \( E_2 \) implies \( y = z + 5 = t + 5 \). Finally, we get \( x \) from \( E_1 \).

\[
x = 3 - 2y + 2z = 3 - 2(t + 5) + 2t = 3 - 2t - 10 + 2t = -7.
\]
9.1 Systems of Linear Equations; Gaussian Elimination

Infinitely many solutions are given by

\[ x = -7, \quad y = t + 5, \quad z = t, \]

where \( t \) is any number. For instance,

\[ t = 0 \text{ gives } x = -7, \quad y = 5, \quad z = 0 \]
\[ t = -3 \text{ gives } x = -7, \quad y = 2, \quad z = -3. \]

(b) Note that the system of equations given here is the same as that in part (a) except for the right side of \( E_3 \). The same elementary operations performed in the solution to Example 3a yield the following echelon form for the system.

\[
\begin{align*}
\text{Echelon form:} \\
x + 2y - 2z &= 3 \\
y - z &= -5 \\
0 \cdot z &= 2 \\
\end{align*}
\]

Since no number \( z \) satisfies the equation \( 0 \cdot z = 2 \), the system has no solution.

A system of linear equations that has infinitely many solutions is said to be dependent, while a system with no solutions is called inconsistent. The system in Example 3a is dependent and that in 3b is inconsistent. Another advantage of echelon form is that the last equation tells us the nature of the solutions, which must be one of the following possibilities.

**Nature of solutions for a system of linear equations**

1. There is exactly one solution; the solution is unique.
2. There are no solutions; the system is inconsistent.
3. There are infinitely many solutions; the system is dependent.

The next example illustrates the three possibilities for \( 2 \times 2 \) systems. It shows geometrically a unique solution, a dependent system, and an inconsistent system.

**EXAMPLE 4 Solutions and graphs** Graph the pair of equations on the same coordinate system, then solve the system.

\[
\begin{align*}
\text{(a)} & \quad \begin{cases} 
-3x + y &= 5 \\
2x - 3y &= -8 
\end{cases} \\
\text{(b)} & \quad \begin{cases} 
-3x + 6y &= 5 \\
x - 2y &= 4 
\end{cases} \\
\text{(c)} & \quad \begin{cases} 
-3x + 6y &= 15 \\
x - 2y &= -5 
\end{cases}
\end{align*}
\]

**Solution**

The graphs are shown in Figure 3. Use Gaussian elimination to verify the following solutions.

(a) Unique solution; \( x = -1, \quad y = 2 \). The two lines intersect at \((-1, 2)\).

(b) No solution; the system is inconsistent. The two lines are parallel; they have no intersection.

(c) Infinitely many solutions; the system is dependent. Both equations determine the same line; every point of the line satisfies both equations.
Chapter 9  Systems of Equations and Inequalities

A system of any number of linear equations must have either a unique solution, no solution, or be dependent, just as the $2 \times 2$ systems in Example 4. Unfortunately, we cannot see the geometry as easily with larger systems as we can with $2 \times 2$ systems. In the next example we illustrate how linear systems occur in applications.

**EXAMPLE 5  Mixture problem**  Dessert consists of chocolate pudding and whipped cream. We are interested in the energy (calories) and vitamin A content. The necessary information in the table is taken from a handbook on nutrition.

<table>
<thead>
<tr>
<th>Food</th>
<th>Energy (calories)</th>
<th>Vitamin A (units)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pudding (1 cup)</td>
<td>385</td>
<td>390</td>
</tr>
<tr>
<td>Cream (1 tablespoon)</td>
<td>26</td>
<td>220</td>
</tr>
</tbody>
</table>

How much pudding (in cups) and cream (in tablespoons) will give a dessert with 283 calories and 674 units of vitamin A?

**Strategy:** To find the numbers of cups and tablespoons, assign variables and write equations for the number of calories (283) and units of vitamin A (674).

**Solution**

Follow the strategy. Let $x$ be the number of cups of pudding and $y$ be the number of tablespoons of cream.

Since each cup of pudding contains 385 calories (see the table), $x$ cups must contain $385x$ calories. Similarly, $y$ tablespoons of cream contain 26$y$ calories. Set the sum of these two equal to 283 calories: $385x + 26y = 283$. In a similar manner, to get 674 units of vitamin A, $390x + 220y = 674$. Therefore, solve the following system of equations.

$$E_1: \quad 385x + 26y = 283 \quad \text{Calories}$$
$$E_2: \quad 390x + 220y = 674 \quad \text{Vitamin A}$$

To eliminate $x$ from $E_2$, first multiply $E_1$ by 390 ($390E_1 \rightarrow E_1$) and $E_2$ by $-385$ ($-385E_1 \rightarrow E_2$). Then add the resulting equations ($E_1 + E_2 \rightarrow E_2$). This gives for the last equation

$$-74,560y = -149,120 \quad \text{or} \quad y = 2. \quad \text{Check!}$$

Substitute 2 for $y$ in one of the original equations to get $x = 0.6$. Hence $\frac{3}{5}$ cup of pudding with 2 tablespoons of cream will give the desired proportions of calories and vitamin A.

**Technology Support for $2 \times 2$ Systems (Cramer’s Rule)**

In one sense, all $2 \times 2$ systems of linear equations are the same; all can be solved with exactly the same steps. The results can be summarized in a form that lends itself to convenient implementation on a graphing calculator. A $2 \times 2$ system can be written in the form

$$\begin{align*}
ax + by &= e \\
 cx + dy &= f
\end{align*}$$
We can solve the system by eliminating either $x$ or $y$. To eliminate $x$, replace $E_2$ by $aE_2 - cE_1$, getting $(ad - bc)y = af - ce$. If we choose to eliminate $y$, we replace $E_1$ by $dE_1 - bE_2$, getting $(ad - bc)x = de - bf$. In both cases the coefficient of the variable is identical, $ad - bc$, and the system has a solution if $ad - bc \neq 0$. If $ad - bc = 0$, then we do not use Cramer’s Rule; the system is either dependent or inconsistent.

Furthermore, when $ad - bc$ is nonzero, we can write down the solution:

$$x = \frac{de - bf}{ad - bc}, y = \frac{af - ce}{ad - bc}.$$  \hfill (6)

A simple way to remember the form of this solution comes from determinants, which we will introduce more formally in Section 9.5. At this point, however, since we have solved the system, we only want a convenient way to keep the result in mind.

The denominator and both numerators have the same form in solution (6). Each can be written as a number associated with a 2 by 2 array, called a determinant. The denominator is called the coefficient determinant of the system:

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc;$$

the product $a$ minus the product $b$.

With this notation, the numerator for each variable is also a determinant, where we replace the column of coefficients of each variable in $D$ by the column of constants on the right side:

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{D} = \frac{ed - bf}{D}, \quad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{D} = \frac{af - ec}{D}.$$  

The entire process is known as Cramer’s Rule for 2 by 2 linear systems.

**Cramer’s rule for 2 by 2 linear systems**

Given a system of two linear equations of the form

$$\begin{align*}
ax + by &= e \\
fx + dy &= f
\end{align*}$$

there is a solution if and only if the number $D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ is nonzero, in which case the solution is given by

$$x = \frac{ed - bf}{D}, \quad y = \frac{af - ec}{D},$$

where the numerator in each case is the determinant obtained from $D$ by replacing the coefficients of the variable by the column of constants.

We illustrate in the next example by using Cramer’s Rule for two systems we have already solved.
EXERCISES 9.1

Check Your Understanding

Exercises 1–7 True or False. Give reasons.

1. The equation \(3x - \sqrt{2}y = 5\) is linear in \(x\) and \(y\).
2. The equation \(3\sqrt{x} + 4y = 7\) is linear in \(x\) and \(y\).
3. The graphs of \(2x - 3y = 3\) and \(x + y = 3\) intersect in the first quadrant.
4. Both \((0, 0, 0)\) and \((-3, 2, 1)\) are solutions to the system
   \[
   \begin{align*}
   x + y + z &= 0 \\
   y - 2z &= 0 \\
   x - 2y - z &= 0
   \end{align*}
   \]
5. The solution to the system
   \[
   \begin{align*}
   2x + y &= 5 \\
   x + 3y &= -4
   \end{align*}
   \]
   consists of a pair of positive integers.
6. The system
   \[
   \begin{align*}
   2x + y &= 0 \\
   x - 3y &= 5
   \end{align*}
   \]
   is dependent.
7. In the solution to the following system, \(x\) and \(y\) are negative and \(z\) is positive.
   \[
   \begin{align*}
   x + y - z &= 4 \\
   y + 2z &= 0 \\
   3x + y &= 5
   \end{align*}
   \]

Exercises 8–10 Fill in the blank so that the resulting statement is true. Lines \(L_1\), \(L_2\), and \(L_3\) are given by \(L_1\):
\(x - 3y = 0\), \(L_2\): \(x + 3y = 6\), \(L_3\): \(x - 9y = 6\).

8. Lines \(L_1\) and \(L_2\) intersect at _______.
9. Lines \(L_1\) and \(L_3\) intersect at _______.
10. Lines \(L_2\) and \(L_3\) intersect at _______.

Develop Mastery

Exercises 1–6 Pairs of Lines Solve the system of equations and graph the pair of lines on the same system of coordinates. (See Example 4.)

1. \(x + y = 4\)
   \[
   \begin{align*}
   3x - 2y &= -3 \\
   -x + 2y &= 4
   \end{align*}
   \]
2. \(3x + y = -5\)
   \[
   \begin{align*}
   3x - 2y &= 4 \\
   -5x + 2y &= 8
   \end{align*}
   \]
3. \(3x + 4y = -1\)
   \[
   \begin{align*}
   -3x + 5y &= -2 \\
   -5x + 2y &= 8
   \end{align*}
   \]
4. \(4x - 2y = 3\)
   \[
   \begin{align*}
   -2x + y &= 5 \\
   x + 2y &= 1.5
   \end{align*}
   \]

Exercises 7–36 Linear Systems Solve the system of equations.

7. \(2x - y + z = 6\)
   \[
   \begin{align*}
   3y + 2z &= 3 \\
   -z &= 3
   \end{align*}
   \]
8. \(x + 3y - z = 4\)
   \[
   \begin{align*}
   2y - 3z &= 8 \\
   3z &= -6
   \end{align*}
   \]
9. \(x + y + z = 1\)
   \[
   \begin{align*}
   2x - y - z &= 5 \\
   -x + 2y - 3z &= -4
   \end{align*}
   \]
10. \(2x - 3y + z = 6\)
    \[
    \begin{align*}
    x + 2y + 2z &= -5 \\
    -3x - y - z &= 6
    \end{align*}
    \]
Exercises 37–42 Cramer’s Rule Use Cramer’s Rule to solve the system. Then find a window in which you can see the intersection of the graphs.

37. $15x + 37y = 19$
   $17x + 14y = 245$

38. $192x - 135y = 2709$
   $64x + 83y = 519$

39. $72x + 43y = 141$
   $129x - 22y = -1233$

40. $429x - 362y = -5285$
   $611x + 243y = -1306.8$

41. $17x + 43y = -118$
   $12x - 28y = -200$

42. $42x - 36y = -113.4$
   $61x - 24y = 72.9$

9.1 Systems of Linear Equations; Gaussian Elimination

Exercises 43–46 Substitution Solve for $x$ and $y$. (Hint: First let $\frac{1}{x} = u$ and $\frac{1}{y} = v$.)

43. $\frac{1}{x} + \frac{1}{y} = 4$
   $\frac{3}{x} + \frac{1}{y} = -5$

44. $\frac{3}{x} + \frac{1}{y} = -5$
   $\frac{1}{x} + \frac{2}{y} = -4$

45. $\frac{3}{x} + \frac{2}{y} = 4$
   $\frac{1}{x} + \frac{3}{y} = 0$

46. $\frac{1}{x} + \frac{3}{y} = 0$
   $\frac{4}{x} + \frac{1}{y} = 6$

47. Find the point of intersection of the two lines given by $2x - 3y = 4$ and $3x + y = -3$.

48. Find the point of intersection of the two lines given by $y = 2x - 5$ and $2y = 3x - 8$.

Exercises 49–50 Nonlinear Systems Follow the procedure in the introductory section to solve the system; then draw graphs of both equations on the same screen.

49. $x + y = 25$
   $x^2 + y^2 = 289$

50. $x + y = 21$
   $x^2 + y^2 = 289$

Exercises 51–54 Perimeter and Area One vertex of a triangle is the point of intersection of lines $L_1$ and $L_2$, and the other two vertices are the $x$-intercept points of $L_1$ and $L_2$. Find (a) the perimeter of the triangle and (b) the area of the triangular region.

51. $L_1: x + y = 6$
   $L_2: x - 3y = -2$

52. $L_1: x + 2y = 4$
   $L_2: 3x - y = -9$

53. $L_1: y = -0.5x + 2.5$
   $L_2: y = x - 2$

54. $L_1: y = 0.5x + 0.5$
   $L_2: y = -3x$

Exercises 55–60 Systems Solve the system of equations.

55. $\frac{xy}{x + y} = 3$, $\frac{xz}{x + z} = 4$, $\frac{yz}{z + y} = 6$
   (Hint: If $\frac{xy}{x + y} = 3$, then $\frac{x + y}{xy} = \frac{1}{y}$ and $x = \frac{1}{3}$.)

56. $\log(xyz) = 2$, $\log\left(\frac{xy}{z}\right) = 0$, $\log\left(\frac{x}{y}\right) = 0$
   (Hint: $\log(xyz) = \log x + \log y + \log z$.)

57. $\ln(xy) = 0.5$, $\ln(x^2y) = 1$, $\ln\left(\frac{yz}{x}\right) = -1.5$
   (Hint: See Exercise 56.)

58. $2^{x+y} = 4$, $4 \cdot 2^{x+2} = 8$
   $32 \cdot 2^{x+y} = 4$
   (Hint: Use properties of exponents.)

59. $4^y = 8 \cdot 2^{x+y}$
   $9^{-2y} = 9 \cdot 3^{-y}$

60. $\log(2x - y) + \log 5 = 1$
   $\log x - \log y = 0$
61. **Triangle** Suppose lines \( L_1, L_2, L_3 \) are given by the equations:

\[
L_1: -x + 2y = 1 \\
L_2: x + 2y = 3 \\
L_3: 3x + 2y = 13.
\]

(a) Draw a graph to show lines \( L_1, L_2, \) and \( L_3. \)

(b) Find the points of intersection for each pair of the three lines.

(c) For the triangle formed by the three lines in (a), find the largest angle to the nearest degree.

62. **Rectangle** The area of a rectangle remains unchanged if its width is increased by 2 and its length is decreased by 2, or if its width is decreased by 2 and its length is increased by 3. What is the perimeter of the rectangle?

63. **Rectangle** The perimeter of a rectangle is 24 cm. If its length is 2 cm greater than its width, what is the area of the rectangular region?

64. **Gardening** A gardener wants to buy two kinds of flowers to plant a border. Ajugas are $1.10 each, and Lilliput Zinnias are $0.85 each. The gardener wants to spend exactly $200 to purchase exactly 200 plants. Can some combination of ajugas and zinnias meet this need? If so, how many of each should be bought?

65. **Investing** A total of $2500 is invested at simple interest in two accounts. The first pays 8 percent interest and the second pays 10 percent interest per year. The total interest earned from the two accounts after one year is $234. How much is invested in each account?

66. **Mixture Problem** A mixture of 36 pounds of peanuts and cashews costs a total of $33. If peanuts cost $0.80 per pound and cashews cost $1.10 per pound, how many pounds of each does the mixture contain?

67. **Two Numbers** The sum of two numbers is 63 and the first is twice the second. What is the product of the two numbers?

68. **Fencing** A rectangular lot has a length-to-width ratio of 4 to 3. If 168 meters of fence will enclose it, what are the dimensions of the lot?

69. **Mixture Problem** Suppose \( x \) grams of food \( A \) and \( y \) grams of food \( B \) are mixed and the total weight is 2000 grams. Food \( A \) contains 0.25 units of vitamin D per gram, and food \( B \) contains 0.50 units of vitamin D per gram. Suppose the final mixture contains 900 units of vitamin D. How many grams of each type of food does the mixture contain?

70. **Filling a Tank** Two pipelines \( A \) and \( B \) are used to fill a tank with water. The tank can be filled by running \( A \) for three hours and \( B \) for six hours, or it can be filled by having both of the supply lines open for four hours. How long would it take for \( A \) to fill the tank alone? How long would it take for \( B \) to fill the tank alone? (Hint: If \( x \) is the number of hours it takes \( A \) to fill the tank alone, then in one hour, \( A \) will fill \( \frac{1}{x} \) of the total capacity of the tank.)

71. **Airspeed** When flying with the wind, it takes a plane 1 hour and 15 minutes to travel 600 kilometers; when flying against the wind it takes 1 hour 40 minutes to travel 600 kilometers. What is the airspeed of the plane and the speed of the wind?

72. **Mixture Problem** One cup of half-and-half cream contains 28 g of fat and 7 g of protein, while one cup of low-fat milk contains 5 g of fat and 8 g of protein. How many cups of half-and-half and how many cups of low-fat milk should be combined to get a mixture that contains 71 g of fat and 38 g of protein?

73. **Finding Costs** The cost of a sandwich, a drink, and a piece of pie is $2.50. The sandwich costs a dollar more than the pie, and the pie costs twice as much as the drink. What is the cost of each?

74. **Investing** A total of $3600 is invested in three different accounts. The first account earns interest at a rate of 8 percent, the second at 10 percent, and the third at 12 percent. The amount invested in the first account is twice as much as that in the second account. If the total amount of simple interest earned in one year is $388, how much is invested in each account?

75. **Mixture Problem** Suppose \( x \) grams of food \( A \), \( y \) grams of food \( B \), and \( z \) grams of food \( C \) are mixed together for a total weight of 2400 grams. The vitamin D and calorie content of each food is given in the table.

<table>
<thead>
<tr>
<th>Food</th>
<th>Units of Vitamin D per Gram</th>
<th>Calories per Gram</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>0.75</td>
<td>1.4</td>
</tr>
<tr>
<td>( B )</td>
<td>0.50</td>
<td>1.6</td>
</tr>
<tr>
<td>( C )</td>
<td>1.00</td>
<td>1.5</td>
</tr>
</tbody>
</table>

The 2400-gram mixture contains a total of 1725 units of vitamin D and 3690 calories. How many grams of each type of food does it contain?

76. **Finding a Quadratic**

(a) Find an equation for the quadratic function whose graph passes through the three points \((-1, 8), (0, 5), \) and \((1, -4). \) (Hint: Let the parabola have equation \( y = Ax^2 + Bx + C \); substitute coordinates of the given points, and solve for \( A, B, \) and \( C. \))

(b) What is the distance between the \( x \)-intercept points of the parabola?

77. **Filling a Tank** A large tank full of water has three outlet pipes, \( A, B, \) and \( C. \) If only \( A \) and \( B \) are opened, the tank empties in three hours. If only \( A \) and \( C \) are open, the tank drains in four hours. If only pipes \( B \) and \( C \) are open, the tank drains in six hours. How long does it take to empty the tank if all three pipes are open? (Hint: If outlet \( A \) can empty the tank in \( x \) hours, how much drains through \( A \) in one hour?)