Chapter 9  Systems of Equations and Inequalities

9.6  MATRIX ALGEBRA

A genuine discovery should do more than merely conform to the facts; it should feel right, it should be beautiful. Aesthetic qualities are important in science, and necessary, I think, for great science.

Roger Penrose

In Sections 9.2 and 9.5 we introduced some ideas related to matrices, but we did not discuss the algebra of matrices themselves. In this section we present a small portion of matrix algebra for solving systems of linear equations. We limit most of our discussion to $2 \times 2$ or $3 \times 3$ systems, but all of the essential ideas can be applied to larger systems, as well.

Matrix Equality

Since matrices have many entries, we need to know when two matrices are equal. Equality requires not only that the matrices are the same size, but that all corresponding entries be the same.

**Definition: equality of matrices**

Matrices $A$ and $B$ are equal, written $A = B$, if and only if

1. $A$ and $B$ have the same size, and
2. each entry in $A$ is equal to the corresponding entry in $B$: $a_{ij} = b_{ij}$.

Matrix Product

The product of two matrices is probably most easily introduced with an example.

**EXAMPLE 1  Sales by matrix multiplication**  A bicycle dealer has three outlets, one downtown, one in a mall, and one at a nearby resort. A special mountain bike sale features three brands of bikes with these sale prices: Hoppit ($375), Runner ($425), Climber ($315). The numbers of bikes sold at the three outlets during the special promotion are displayed in a matrix:

$$
\begin{array}{ccc}
H & R & C \\
\hline
\text{Downtown} & 8 & 7 & 12 \\
\text{Mall} & 4 & 14 & 9 \\
\text{Resort} & 5 & 8 & 16 \\
\end{array}
$$

Find the sales total in dollars at each outlet.

**Solution**

We could find the desired information without using matrices. The dollar total from the downtown store is $8(375) + 7(425) + 12(315) = 9,755$, and the same operations will give us the gross sales figures for the mall store ($10,285$) and the resort store ($10,315$). Matrix multiplication is defined to do precisely these oper-
9.6 Matrix Algebra

Let $A$ and $B$ be matrices.

$$A = \begin{bmatrix} 8 & 7 & 12 \\ 4 & 14 & 9 \\ 5 & 8 & 16 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 375 \\ 425 \\ 315 \end{bmatrix}$$

The product $AB$ is a $3 \times 1$ matrix $C$:

$$AB = \begin{bmatrix} 8 & 7 & 12 \\ 4 & 14 & 9 \\ 5 & 8 & 16 \end{bmatrix} \begin{bmatrix} 375 \\ 425 \\ 315 \end{bmatrix} = \begin{bmatrix} 8 \cdot 375 + 7 \cdot 425 + 12 \cdot 315 \\ 4 \cdot 375 + 14 \cdot 425 + 9 \cdot 315 \\ 5 \cdot 375 + 8 \cdot 425 + 16 \cdot 315 \end{bmatrix} = \begin{bmatrix} 9,755 \\ 10,285 \\ 10,315 \end{bmatrix} = C$$

From the matrix $C$, read off sales totals: $c_{11} = 9,755$ (downtown), $c_{21} = 10,285$ (mall), and $c_{31} = 10,315$ (resort).

The matrix product in Example 1 is sometimes called a row-by-column product. Each entry in product $AB$ is obtained by multiplying the entries of a row of $A$ by the entries of a column of $B$, and each entry $c_{ij}$ of the product is the sum of the products of the entries in the $i$th row of $A$ with the corresponding entries of the $j$th column of $B$. More specifically, $c_{11}$ is given by $c_{11} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}$. Similarly, $c_{21}$ comes from the second row of $A$ and the first column of $B$: $c_{21} = a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31}$, and $c_{31} = a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31}$. See the illustration in the margin.

The row-by-column idea defines the product of two matrices in general. The product $AB$ requires that the number of entries in each row of $A$ matches the number of entries in each column of $B$. It is easy to see in a particular example whether or not $A$ and $B$ allow multiplication, but we can also read the information from the dimensions of $A$ and $B$.

**Definition: product of two matrices**

Let $A$ be an $m \times k$ matrix and $B$ be a $k \times n$ matrix. The product $AB$ is an $m \times n$ matrix $C$, where the entry $c_{ij}$ is obtained by multiplying the entries of the $i$th row of $A$ by the corresponding entries of the $j$th column of $B$ and then adding the resulting products:

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{ik}b_{kj},$$

**Strategy:** (a) Using the row-by-column definition, if $AB = C$, then $c_{11} = 1 \cdot 4 + (-2)0 + 0(-2) = 4$, and so on.

**Example 2** $AB \neq BA$ Find the products $AB$ and $BA$ if matrices $A$ and $B$ are given by

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 3 & 2 & -1 \\ 2 & 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 4 & -1 & 2 \\ 0 & 1 & 3 \\ -2 & 1 & -1 \end{bmatrix}.$$
Chapter 9  Systems of Equations and Inequalities

Solution
Follow the strategy.

\[
AB = \begin{bmatrix} 1 & -2 & 0 \\ 3 & 2 & -1 \\ 2 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 4 & -1 & 2 \\ 0 & 1 & 3 \\ -2 & 1 & -1 \end{bmatrix}
\]

\[
= \begin{bmatrix} 1 \cdot 4 + (-2)0 + 0(-2) & 1(-1) + (-2)1 + 0 \cdot 1 & 1 \cdot (-2)3 + 0(-1) \\ 3 \cdot 4 + 2 \cdot 0 + (-1)(-2) & 3(-1) + 2 \cdot 1 + (-1)1 & 3 \cdot 2 + 2 \cdot 3 + (-1)(-1) \\ 2 \cdot 4 + 0 \cdot 0 + (-1)(-2) & 2(-1) + 0 \cdot 1 + (-1)1 & 2 \cdot 2 + 0 \cdot 3 + (-1)(-1) \end{bmatrix}
\]

\[
= \begin{bmatrix} 4 & -3 & -4 \\ 14 & -2 & 13 \\ 10 & -3 & 5 \end{bmatrix}
\]

\[
BA = \begin{bmatrix} 4 & -1 & 2 \\ 0 & 1 & 3 \\ -2 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 & 0 \\ 3 & 2 & -1 \\ 2 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 5 & -10 & -1 \\ 9 & 2 & -4 \\ -1 & 6 & 0 \end{bmatrix}
\]

In the solution to Example 2 note that \(AB \neq BA\), which implies that matrix multiplication is not necessarily commutative.

\section*{Example 3  Associativity}
Matrices \(A\), \(B\), and \(C\) are

\[
A = \begin{bmatrix} -1 & 2 \\ 3 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 4 \\ -1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix}
\]

Find the matrix products (a) \((AB)C\) and (b) \(A(BC)\).

\section*{Strategy:}
(a) First find \(AB\), then multiply the result by \(C\) (with \(C\) on the right) to get \((AB)C\).

\section*{Solution}
Follow the strategy.

(a) \((AB)C = \begin{bmatrix} -3 & 2 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}\)

(b) \(A(BC) = \begin{bmatrix} -1 & 2 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -4 & 6 \end{bmatrix}\)

The solution to Example 3 illustrates a general property of matrix multiplication: matrix multiplication is associative. Whenever the products are defined, \((AB)C = A(BC)\).

\section*{Example 4  Identity}
Matrices \(A\) and \(B\) are given by

\[
A = \begin{bmatrix} -1 & 2 \\ 3 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

Find the matrix products (a) \(AB\) and (b) \(BA\).
9.6 Matrix Algebra 535

Solution

(a) \( AB = \begin{bmatrix}
-1 & 2 \\
3 & -1
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} = \begin{bmatrix}
-1 & 2 \\
3 & -1
\end{bmatrix} \)

(b) \( BA = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
-1 & 2 \\
3 & -1
\end{bmatrix} = \begin{bmatrix}
-1 & 2 \\
3 & -1
\end{bmatrix} \)

The solution to Example 4 shows that \( AB = BA = A \), so the matrix \( B \) acts much like the number 1 in ordinary arithmetic \((a \cdot 1 = 1 \cdot a = a)\). For any \( 2 \times 2 \) matrix, \( C \), \( CB = BC = C \), and we call \( B \) the identity matrix for the set of \( 2 \times 2 \) matrices. It is customary to denote the identity matrix by the letter \( I \). There is an identity matrix of size \( n \times n \) for every dimension \( n \). The \( 3 \times 3 \) identity is the matrix

\[
I = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

The same letter \( I \) can denote the identity matrix of any size under discussion, but the context should make it clear which size identity is intended.

EXAMPLE 5 Inverses

Find matrix products \( AB \) and \( BA \), where

\[
A = \begin{bmatrix}
1 & 0 & 1 \\
-5 & 1 & -5 \\
-2 & 1 & -1
\end{bmatrix}, \quad B = \begin{bmatrix}
4 & 1 & -1 \\
5 & 1 & 0 \\
-3 & -1 & 1
\end{bmatrix}
\]

Solution

\[
AB = \begin{bmatrix}
1 & 0 & 1 \\
-5 & 1 & -5 \\
-2 & 1 & -1
\end{bmatrix}
\begin{bmatrix}
4 & 1 & -1 \\
5 & 1 & 0 \\
-3 & -1 & 1
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
BA = \begin{bmatrix}
4 & 1 & -1 \\
5 & 1 & 0 \\
-3 & -1 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
1 & 0 \\
1 & 0
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

The product of the two matrices in Example 5 (in either order) is the identity matrix. In the set of real numbers two numbers whose product is 1 are called reciprocals or multiplicative inverses of each other. We use the same terms in matrix algebra. If \( AB = BA = I \), then \( A \) and \( B \) are inverses of each other, \( B = A^{-1} \). In general, \( AA^{-1} = A^{-1}A = I \). Not all matrices have inverses, but every square matrix with a nonzero determinant does have an inverse.

We sum up our discussion so far in a list of some properties of matrix algebra.

Properties of matrix algebra

1. In general, matrix multiplication is not commutative: \( AB \neq BA \).
2. Matrix multiplication is associative: \((AB)C = A(BC)\).
3. The square matrix \( I \) with 1s on the main diagonal and 0s everywhere else is an identity matrix: \( AI = IA = A \).
4. Any square matrix \( A \) with a nonzero determinant has an inverse: \( AA^{-1} = A^{-1}A = I \).
5. The matrix \( kA \) is obtained by multiplying every entry of \( A \) by the number \( k \).
Finding the Inverse of a Square Matrix without Technology

Matrix inverses have several important applications. Among them is another technique for solving systems of linear equations. To use the technique we need a method for finding the inverse of a matrix. The following algorithm is simple and relatively efficient.

**Algorithm to find the inverse of a square matrix**

Suppose $A$ is a square matrix with a nonzero determinant.

1. Adjoin the identity matrix to the right of $A$, getting a matrix with the structure $[A | I]$.
2. Use elementary row operations on $[A | I]$ to get a matrix of the form $[I | B]$.
3. The inverse of $A$ is the matrix $B$.

We illustrate the algorithm with matrix $A$ of Example 5.

$[A | I] = \begin{bmatrix} 1 & 0 & 1 & | & 1 & 0 & 0 \\ -5 & 1 & -5 & | & 0 & 1 & 0 \\ -2 & 1 & -1 & | & 0 & 0 & 1 \end{bmatrix}$

$5R_1 + R_2 \rightarrow R_2$

$2R_1 + R_3 \rightarrow R_3$

$(-1)R_3 + R_1 \rightarrow R_1$

$0 \quad 1 \quad 0$

$-3 \quad -1 \quad 1$

$(-1)R_2 + R_3 \rightarrow R_3$

$1 \quad 0 \quad 0$

$4 \quad 1 \quad -1$

The last matrix has the form $[I | B]$, so

$A^{-1} = B = \begin{bmatrix} 4 & 1 & -1 \\ 5 & 1 & 0 \\ -3 & -1 & 1 \end{bmatrix}$

as we found in Example 5, which showed that $AB = I$.

Solving Systems of Linear Equations

We stated that a goal of this section was to develop the matrix algebra needed to express an $n \times n$ system of linear equations as a matrix equation and then to use matrix algebra to solve the system. Two examples illustrate this process.

**EXAMPLE 6** Matrix form of linear system

For the matrices

$A = \begin{bmatrix} 1 & 0 & 1 \\ -5 & 1 & -5 \\ -2 & 1 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$

(a) write the matrix product $AX$, and

(b) write the system of linear equations that results if $AX = C$. 
Solution

(a) 
\[
AX = \begin{bmatrix} 1 & 0 & 1 \\ -5 & 1 & -5 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + z \\ -5x + y - 5z \\ -2x + y - z \end{bmatrix}
\]

(b) If \(AX = C\), then 
\[
\begin{bmatrix} x + z \\ -5x + y - 5z \\ -2x + y - z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} \Rightarrow \begin{cases} x + z = 3 \\ -5x + y - 5z = -2 \\ -2x + y - z = 4 \end{cases}
\]

Using Technology to Solve Matrix Equations

Any system of linear equations can be expressed as a matrix equation \(AX = B\) as in Example 6. If the coefficient matrix \(A\) is square and has an inverse, then we can solve the system, at least symbolically. We simply multiply both sides of the equation on the left by \(A^{-1}\) and use the associative property of matrix multiplication.

\[A^{-1}(AX) = A^{-1}B, \quad (A^{-1}A)X = A^{-1}B, \quad IX = A^{-1}B, \quad \text{or} \quad X = A^{-1}B.\]

That is, given the matrix equation \(AX = B\), as long as \(A\) has an inverse, we can solve the system by premultiplying \(B\) by \(A^{-1}\). This is a tremendous boon if we can use technology to find the inverse and perform the matrix multiplication. Therein, of course, lies the rub. Finding the inverse of a large matrix can tax the most sophisticated computer software and finding better ways to manipulate linear systems to improve methods of solution continues as an area of active mathematical research.

If we recognize the limitations, though, the technology we have available to us allows us to solve a great many linear system problems efficiently and easily. Determinants continue to play a role in solving a system. It turns out that the inverse of a matrix involves division by the determinant of the matrix. When the determinant of a matrix is zero, there is no inverse, and if the determinant is near zero, there is a possibility of substantial error in approximations.

A warning example. Consider the following system:

\[
\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}
\]

We mentioned in Section 9.5 that the TI-85 evaluates the determinant of the coefficient matrix \(A\) as \(-2.8E-12\). We should recognize that such a number indicates that the determinant is equal to 0 and that hence that \(A\) has no inverse. If, however, we disregard our warning, enter the matrix \(A\) and the column matrix \(B\) and go ahead and compute \(A^{-1}B\), the calculator immediately returns an answer. We read that \(x = 6.25E12, y = -1.25E12,\) and \(z = 6.25E12\), a highly suspicious result to say the least. If we want to check by multiplying the result by \(A\), we should get \(B\) again, but the calculator indicates that the entries are 1.1, -3, -6, nowhere close to the numbers 2, -1, 1 in \(B\). If we look at \(A^{-1}\), we see huge numbers (suspicious again), and if we calculate \(AA^{-1}\) or \(A^{-1}A\), we get reasonable looking numbers but not the identity matrix.
We emphasize again that this problem is part of any computing technology, not a peculiarity of the TI-85. With a little work we can find such an example for any calculator. What we must do is work within the limitations of our technology. With care, we can take advantage of the power we have available, as outlined below.

**Solving a matrix system** \( AX = B \)

1. Enter the coefficient matrix \( A \) in your calculator.
2. Compute the determinant of \( A \). If \( \det A = 0 \) or if the calculator shows \( \det A \) as a very small number, stop. Use Gaussian elimination to solve the system.
3. If you are confident that \( \det A \neq 0 \), then enter the column matrix \( B \).
4. On your home screen evaluate \( A^{-1}B \). The result is the solution matrix.
5. Check by premultiplying your result by \( A \). The product should be \( B \).

**TECHNOLOGY TIP**

**Calculating** \( A^{-1}B \)

Note that to solve the system \( AX = B \), we need not necessarily even compute \( A^{-1} \), although some calculators do compute and display \( A^{-1} \) in the process.

*TI-calculators* Having entered the matrix \( A \), on the home screen, we just enter \( A \) [(or \( [A] \)] and the \( x^{-1} \) key, followed by \( B \), and \( \text{ENTER} \).

*Casio fx-7700* only handles three matrices. After entering matrix \( A \), press \( F4 \) to evaluate \( A^{-1} \), which is displayed in the \( C \) register. We want it in the \( A \) register, so \( F1 \) performs the interchange. Then after putting in \( B \), press \( \text{PRE} \). Then \( F5 \) does the multiplication and puts the product in \( C \).

*Casio fx-9700* allows us to store several matrices, so after entering \( A \) and \( B \), we \( \text{EXIT} \) twice and simply press \( F1 \) ALPHA \( A \) \( x^{-1} \) \( F1 \) ALPHA \( B \) to display \( \text{Mat} \ A \times \text{Mat} \ B \) and \( \text{EXE} \).

*HP-38* Having entered matrix \( A \) as \( M1 \) and \( B \) as \( M2 \), return to the home screen. On the command line, type \( M1^{-1} \times M2 \) (use the \( \exists \) key for \( M \)), the \( \text{INV} \) key for \( ^{-1} \), and \( \text{ENTER} \).

*HP-48* Having entered \( A \) on the stack, \( 1/x \) computes the matrix \( A^{-1} \) and enters it in place of \( A \). Then put \( B \) on the stack in Level 1 and press the \( \times \) key (the order of matrix multiplication on the HP-48 is Level 2 \( \times \) Level 1).

We strongly suggest that you check your ability to handle matrices by doing all of the calculations in Example 7.

**EXAMPLE 7** *Matrix solution* Use a graphing calculator to find the inverse of the coefficient matrix, and solve the system.

\[
\begin{align*}
  x + z &= 3 \\
 -5x + y - 5z &= -2 \\
 -2x + y - z &= 4
\end{align*}
\]

**Solution**

For this system, \( A = \begin{bmatrix} 1 & 0 & 1 \\ -5 & 1 & -5 \\ -2 & 1 & -1 \end{bmatrix} \) and \( B = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} \). The calculator shows the
inverse of \( A \) as \( A^{-1} = \begin{bmatrix} 4 & 1 & -1 \\ 5 & 1 & 0 \\ -3 & -1 & 1 \end{bmatrix} \). As a check, see Example 5. The solution is given by \( A^{-1}B = \begin{bmatrix} 6 \\ 13 \\ -3 \end{bmatrix} \), so \( x = 6, y = 13, z = -3 \).

### EXERCISES 9.6

#### Check Your Understanding

**Exercises 1–10**  True or False. Give reasons.

1. If \( A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \), then \( A \cdot A = \begin{bmatrix} 1 & 0 \end{bmatrix} \).

2. The inverse of \( \begin{bmatrix} 1 & 4 \\ 1 & 5 \end{bmatrix} \) is \( \begin{bmatrix} 5 & -4 \\ -1 & 1 \end{bmatrix} \).

3. If \( A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \), then \( A^{-1} = A \).

4. If \( A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \) and \( B = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \), then \( BA = AB \).

5. If \( A = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \) and \( B = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \), then \( BA = AB \).

**Exercises 6–10**  Let \( A = [-1, 3] \)

6. The only entry in \( AB \) is positive.

7. \( AC \) is a square matrix.

8. All entries in \( BA \) are negative.

9. \( (BA)C = B(AC) \)

10. \( A(BC) \) is undefined.

#### Develop Mastery

**Exercises 1–4**  **Matrix Notation**  (a) Give the dimension of matrix \( A \) and (b) find \( a_{13} \) and \( a_{31} \) when possible. If this is not possible, explain why.

1. \( A = \begin{bmatrix} 2 & -3 \\ -1 & -4 \end{bmatrix} \)

2. \( A = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix} \)

3. \( A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 2 \\ 1 & 2 & 4 \end{bmatrix} \)

4. \( A = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix} \)

**Exercises 5–12**  **Matrix Products**  Evaluate the matrix product when possible; if the product is not defined, explain why. Use the matrices

\[
A = \begin{bmatrix} 2 & -3 \\ -1 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \\
C = \begin{bmatrix} 3 & -1 & 4 \\ -2 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \\
E = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & -1 & 2 \\ 3 & -2 & 1 \\ 4 & 0 & 2 \end{bmatrix}
\]

5. \( AB \)

6. \( BA \)

7. \( CD \)

8. \( AE \)

9. \( EA \)

10. \( CF \)

11. \( FC \)

12. \( A(EA) \)

**Exercises 13–20**  **Matrix Inverse**  Use the algorithm of this section to find the inverse of the matrix if it has an inverse; if it has no inverse, explain how you know. Check by technology.

13. \( A = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix} \)

14. \( B = \begin{bmatrix} 3 & 5 \\ -1 & -2 \end{bmatrix} \)

15. \( C = \begin{bmatrix} 1 & 0 \\ 6 & 2 \end{bmatrix} \)

16. \( A = \begin{bmatrix} -2 & 1 \\ -4 & 3 \end{bmatrix} \)

17. \( B = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \)

18. \( A = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \)

19. \( A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 3 \\ 1 & 2 & 3 \end{bmatrix} \)

20. \( B = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ -2 & 6 & -3 \end{bmatrix} \)

**Exercises 21–28**  **Inverses by Calculator**  Use your calculator to find \( A^{-1} \) if \( A \) has an inverse; if not, explain why. (Hint: It may help to evaluate \( \det(A)A^{-1} \).)

21. \( A = \begin{bmatrix} -4 & 2 & -3 \\ 10 & -5 & 8 \\ -1 & 1 & -1 \end{bmatrix} \)

22. \( A = \begin{bmatrix} 4 & 1 & -1 \\ 5 & 1 & 0 \\ -3 & -1 & 1 \end{bmatrix} \)
Chapter 9  Systems of Equations and Inequalities

Exercises 29–40  Matrix Systems  (a) Evaluate |A|. (b) If A has an inverse, find $A^{-1}$ and solve the system of equations by solving the matrix equation $AX = C$.

29. \[3x + 4y = 2\]
   \[-7x - 9y = 3\]

30. \[x + 3y = 4\]
   \[3x + 5y = -2\]

31. \[-3x + 2y = 4\]
   \[5x - 3y = -1\]

32. \[-2x + y = 3\]
   \[-5x + 3y = 1\]

33. \[x - y = 0\]
   \[-y + z = 4\]
   \[-2x + 6y - 3z = 1\]

34. \[x + 2y + 2z = -1\]
   \[x + 3y + 2z = -2\]
   \[2x + 6y + 5z = 3\]

35. \[x + 2y + 4z = -1\]
   \[x + 3y + 3z = 0\]
   \[x + 2y + 3z = -4\]

36. \[x + 2y + 3z = -4\]
   \[2x + 4y + z = -4\]
   \[-4x - 8y - 9z = 22\]

37. \[-3x - y + z = 2\]
   \[2x + y + 2z = -1\]
   \[x + y - z + 2w = -6\]
   \[-x + 3z = 0\]

38. \[2x + y + 2z = -1\]
   \[x + y - z + 2w = -6\]
   \[2x + y - z + 3w = -6\]

39. \[x - y + 4z - w = 4\]
   \[2x + y - 3z + 5w = -1\]
   \[4x + 3z - 2w = 13\]
   \[-2x + 4y - 3z = 5\]

40. \[x + y + z + w = 4\]
   \[2x - y + 3z + w + v = -2\]
   \[5x + y + w + 2v = 11\]
   \[-x + 2y - z - 2w = -7\]
   \[-x + 3y + 2z - 6w + 4v = 3\]

Exercises 41–43  Find a Circle  For the circle that passes through the three points, (a) write an equation in the form $x^2 + y^2 + bx + cy = d$ and (b) find the radius and the coordinates of the center.

41. \((-1, -2), (5, 6), (6, 5)\)
42. \((-1, -1), (0, 2), (2, 2)\)
43. \((0, 2), (7, 1), (8, -2)\)

Exercises 44–46  Find a Parabola  For the parabola that passes through the three points, (a) write an equation in the form $y = ax^2 + bx + c$ and (b) find the coordinates of the x-intercept points and the vertex.

44. \((0, 1), (1, -2), (2, -3)\)
45. \((-1, -1), (0, 2), (2, 2)\)
46. \((0, 7), (1, 1), (2, -1)\)

47. The height from ground level of an object is given by an equation of the form $h(t) = at^2 + bt + c$, where $t$ is the time in seconds and $h$ is measured in feet. (a) Find $a$, $b$, and $c$, if $h(1) = 240$, $h(2) = 246$, and $h(3) = 248$. (b) At what time will the object be at ground level?

Exercises 48–51  Inverse of a Product  For matrices $A$ and $B$, find (a) $AB$, (b) $(AB)^{-1}$, (c) $A^{-1}B^{-1}$ and $B^{-1}A^{-1}$.

48. \[A = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}\]
49. \[A = \begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}\]
50. \[A = \begin{bmatrix} -4 & 2 \\ -1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}\]
51. \[A = \begin{bmatrix} 3 & 2 \\ 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ 5 & 2 \end{bmatrix}\]

Exercises 52–63  Powers of a Matrix  For matrix $A$, find (a) $A^2 = A \cdot A$, (b) $A^3$, (c) $A^n$, and (d) $A^{-t}$.

52. \[A = \begin{bmatrix} -2 & -1 \\ 3 & 2 \end{bmatrix}\]
53. \[A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}\]
54. \[A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\]
55. \[A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\]
56. \[A = \begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix} \quad A = \begin{bmatrix} -2 & -6 \\ -6 & -12 \end{bmatrix}\]
57. \[A = \begin{bmatrix} 2 & -1 \\ -3 & -5 \end{bmatrix}\]
58. \[A = \begin{bmatrix} 2 & -1 \\ -3 & -5 \end{bmatrix}\]
59. \[A = \begin{bmatrix} 2 & -1 \\ -3 & -5 \end{bmatrix}\]
Exercises 64–65  Multiple of 1  For the given matrix, show that the given expression is equal to some multiple of the identity matrix.

64. \[
   A = \begin{bmatrix}
   2 & 2 & 0 \\
   2 & 1 & 1 \\
   -7 & 2 & -3
   \end{bmatrix} \quad A = 13A - A^3
\]

65. \[
   B = \begin{bmatrix}
   1 & 0 & -2 \\
   2 & 2 & 4 \\
   0 & 0 & 2
   \end{bmatrix} \quad 3B - B^2
\]

Exercises 66–69  Your Choice  Find two \(2 \times 2\) matrices \(A\) and \(B\) such that the first row of \(A\) is \(1, - 1\) and such that \(A\) and \(B\) satisfy the given condition.

66. \(AB = I\)  
67. \(AB = BA\)  
68. \(AB \neq BA\)  
69. \(A\) and \(B\) have no zero entries but \(AB\) is the zero matrix.

70. Explore: Generating the Fibonacci Sequence  Let \(F\) be the matrix \(
   \begin{bmatrix}
   0 & 1 \\
   1 & 1
   \end{bmatrix}
\) and let \(A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \).

(a) Compute the first few powers of \(F\) by entering \(F\) and then iterating \(F \cdot \text{ANS}\).

(b) Guess a formula for \(F^n\) in terms of the Fibonacci sequence.

(c) Enter \(A\) and iterate \(F \cdot \text{ANS}\). How would you describe the entries you observe?

(d) Describe how to generate the Lucas sequence, which is defined recursively by \(L_0 = 2, L_1 = 1, \) and \(L_{n+1} = L_n + L_{n-1}, n \geq 1\).

CHAPTER 9 REVIEW

Test Your Understanding

True or False. Give reasons.

1. The equation \(3x - 4y + z = 7\) is linear in \(x, y, \) and \(z\).
2. The equation \(\sqrt{3}x - \sqrt{3}y = \sqrt{6}\) is not a linear equation in \(x\) and \(y\).
3. The system
   \[
   \begin{align*}
   2x - 3y &= 5 \\
   -4x + 6y &= 7
   \end{align*}
   \]
   has infinitely many solutions. It is dependent.
4. The solution for the system
   \[
   \begin{align*}
   x - 2y - 3z &= 4 \\
   y - 2z &= 6 \\
   3z &= -9
   \end{align*}
   \]
   is given by \(x = -5, y = 0, z = -3\).

Exercises 5–8  Refer to the system of inequalities:

\[
2x - y \geq 0 \\
2x + y \geq 4
\]

5. Point \((0, 1)\) is in the solution set.
6. Point \((2, 4)\) is not in the solution set.
7. Point \((1, 2)\) is a corner point.
8. The solution set contains no points in Quadrants III or IV.

Exercises 9–12  Lines \(L_1\) and \(L_2\) are given by

\[
\begin{align*}
L_1: & \quad x + 2y = 0 \\
L_2: & \quad 3x - 4y = -5
\end{align*}
\]

9. Point \((-2, 1)\) is on both \(L_1\) and \(L_2\).
10. Point \((-1, \frac{1}{2})\) is on both \(L_1\) and \(L_2\).
11. Point \((1, 2)\) is on \(L_2\), but not on \(L_1\).
12. Point \((0, 1)\) is above \(L_1\) and below \(L_2\).

Exercises 13–18  Let \(G\) be the set of all points \((x, y)\) that satisfy the system

\[
\begin{align*}
x - 2y &\geq -6 \\
x + y &\geq -3 \\
7x - 2y &\leq 6
\end{align*}
\]

13. Point \((0, 3)\) is in \(G\).
14. Point \((0, 0)\) is in \(G\).
15. Point \((0, -3)\) is a corner point of \(G\).
16. Point \((-4, 1)\) is not a corner point of \(G\).
17. Point \((2, 4)\) is not in \(G\).
18. There is no point on the line \(2x + y = 0\) that is also in \(G\).

Exercises 19–23  Line \(L\) and parabola \(P\) are given by

\[
\begin{align*}
L: & \quad x - 2y = -1 \\
P: & \quad y = x^2 - 1
\end{align*}
\]

19. There is exactly one point that is on both \(L\) and \(P\).
20. There are exactly two points that are on both \(L\) and \(P\).