2.2 Graphs of Functions

One can envisage that mathematical theory will go on being elaborated and extended indefinitely. How strange that the results of just the first few centuries of mathematical endeavor are enough to achieve such enormously impressive results in physics.

P.W.C. Davies

Graph of a Function

A function \( f \) assigns a range element to each domain element, so it is often useful to think of the function as pairing numbers (if the domain and range are sets of numbers). The function is completely determined by these ordered pairs. In mathematical notation,

\[
f = \{(x, f(x)) \mid x \in D\}, \text{ where } D \text{ is the domain of } f.
\]

When the function is defined by an equation, \( y = f(x) \), then

\[
f = \{(x, y) \mid y = f(x), x \in D\}.
\]

The ordered pairs that define \( f \) look like coordinates of points in the plane, and we can use this feature to define the graph of a function.

Definition: graph of a function

If \( f \) is a function with domain \( D \), then the graph of \( f \) is the set of points with coordinates \((x, y)\) such that \( x \in D \) and \( y = f(x) \).

An accurate graph of a function makes both the domain and the range of the function apparent. The domain is the set of \( x \) values of points on the graph, and the range is the set of \( y \) values.

Function Properties and Graphs

It is natural to think of drawing a graph by plotting points and connecting them in an appropriate way to get a curve. This is precisely how a computer or graphing calculator shows graphs on a screen. Without the capability to compute hundreds of function values quickly, however, pencil and paper techniques are time consuming and tedious. With or without access to the tools of technology, some additional tools can help us draw graphs and understand the properties of functions.

In this section, we examine symmetry properties of graphs and introduce the notion of even and odd functions. Also, certain core graphs are given. Knowing a single core graph and how it is affected by simple changes, we can draw graphs of a whole family of related functions.

Core Graphs

There are a few graphs that should be familiar to every precalculus student. We will do lots of graphing with calculators and the aid of technology, but you should be able to sketch each of the following core graphs without any help. Knowing the properties of these simple functions and some of their key features will make discussions of all kinds of functional behavior more meaningful. Use your graphing calculator as needed to help absorb the ideas, but make yourself confident that you know the graphs of the functions in Figure 2.
Intercepts, Symmetry, Even and Odd Functions

We are always interested in the points where a graph meets the coordinate axes. If 0 is in the domain of \( f \), then \( f(0) \) is called the y-intercept and the point \((0, f(0))\) is the y-intercept point. A graph need not meet the x-axis, but if it does, any points where it does are called x-intercept points, and if \( f(0) = 0 \), then the number 0 is called an x-intercept.

Given a point \( A(a, b) \), three symmetrically located points can help in graphing. We can reflect \( A \) in the y-axis to the point \( C(-a, b) \), in the x-axis to the point \( D(a, -b) \) or in the origin to the point \( E(-a, -b) \). Figure 3a shows a first-quadrant point \( A \) and its reflections.

The graph of a function may have all sorts of symmetry properties (or none). Suppose that for every arbitrary point \((a, b)\) on the graph of a function, the graph also contains the reflection of \((a, b)\) in the y-axis. That is, whenever \((a, b)\) is on the graph, \((-a, b)\) is also on the graph. Then we say that the graph is symmetric about the y-axis and that the function is an even function. In functional notation, since to say that \((a, b)\) is on the graph means that \( f(a) = b \), a function is even when \( f(-x) = f(x) \). See Figure 3b.

If, whenever \((a, b)\) is on the graph of \( f \), \((-a, -b)\) is also on the graph, then the graph is symmetric about the origin and the function is an odd function. In functional notation, a function is odd if \( f(-x) = -f(x) \) for every \( x \) in \( D \). See Figure 3c.
The catalog of core graphs in Figure 2 has examples that should help you remember the distinction between even and odd functions. The parabola $y = x^2$ and the absolute value function $y = |x|$ are even; the graphs are clearly symmetric about the $y$-axis. The line $y = x$, the cubic $y = x^3$, and the reciprocal $y = \frac{1}{x}$ are all symmetric about the origin; the functions are odd. The square root function $y = \sqrt{x}$ is neither odd nor even. We sum up in the following.

**Definition: even and odd functions, symmetry properties**

Suppose $f$ is a function with domain $D$. If, for every $x$ in $D$, we have

- $f(-x) = f(x)$, then $f$ is an **even** function;
- $f(-x) = -f(x)$, then $f$ is an **odd** function.

The graph of an even function is symmetric about the y-axis. The graph of an odd function is symmetric about the origin.

In addition to just looking at a single graph, graphing calculators can be used to see whether or not $f(-x) = f(x)$ or $f(-x) = -f(x)$; compare the graph of $y_1 = f(x)$ with the graph of $y_2 = f(-x)$, where we replace each $x$ in $f(x)$ by $-x$.

**EXAMPLE 1** Even and odd functions  
Sketch the graph and determine, both graphically and algebraically, whether the function is odd or even.

(a) $f(x) = x^2 + 1$  
(b) $g(x) = x^3 - x$

**Solution**

(a) **Algebraic**  
$f(-x) = (-x)^2 + 1 = x^2 + 1$, so $f(-x) = f(x)$, and by definition, $f$ is an even function.

**Graphical**  
The graph, shown in Figure 4(a), is symmetric about the $y$-axis and is clearly the graph of an even function.

(b) **Algebraic**  
For the function $g$, we have $g(-x) = (-x)^3 - (-x) = -x^3 + x = -(x^3 - x) = -g(x)$, and so from the definition, $g$ is an odd function.

**Graphical**  
The calculator confirms that the graph is symmetric about the origin, and hence the graph of an odd function. See Figure 4b.
A function may be neither even nor odd, as the next example shows.

**EXAMPLE 2   Neither even nor odd**  Show that the function is neither even nor odd, and sketch its graph

(a) $F(x) = 2x - 4$  (b) $G(x) = \sqrt{x}$

**Solution**

Apply the definitions for even and odd functions.

$F(-x) = 2(-x) - 4 = -2x - 4$  $-F(x) = -2x + 4$

$G(-x) = \sqrt{-x}$  $-G(x) = -\sqrt{x}$

Since $F(-x) \neq F(x)$ and $F(-x) \neq -F(x)$, $F$ is neither even nor odd, and similarly, $G$ is neither even nor odd.

To graph the two functions, plot points as in Figure 5.
Piecewise, Step Functions, and Calculator Graphs

We have already seen some examples of an important class of functions called piecewise functions because they are defined in pieces (with different rules for different portions of their domains). The first example we encountered is the absolute value function, even though we did not recognize it when we first introduced absolute values. Among the properties of absolute values in Section 1.3 we listed the equality \( |x| = \sqrt{x^2} \), which can be justified by considering values, but it may also be helpful to get calculator reinforcement. Graph both \( y = \text{ABS}(x) \) and \( y = \sqrt{x^2} \) and see that the graphs are identical. The function can also be written in pieces:

\[
y = |x| = \sqrt{x^2} = \begin{cases} 
  x & \text{if } x \geq 0 \\
  -x & \text{if } x < 0 
\end{cases}
\]

The graph consists of part of the line \( y = -x \) and part of the line \( y = x \).

You may want to refer again to the Technology Tip in Section 2.1 to make sure that you know how to graph piecewise-defined functions on your calculator.

**EXAMPLE 3** Calculator graphs of piecewise functions

Draw a calculator graph of the piecewise function from Example 4 of Section 2.1,

\[
f(x) = \begin{cases} 
  x^2 & \text{if } x \leq 1 \\
  2 - x & \text{if } x > 1 
\end{cases}
\]

**Solution**

The graph is shown in Figure 6. You may wish to experiment with different windows to see how changing the view alters the shape of the pieces we see. Whatever the window, however, the graph of \( f \) consists of two pieces that meet at the point \((1, 1)\).

Greatest Integer Function

Another piecewise-defined function that is useful in many different contexts is the greatest integer function, which is programmed into most graphing calculators, defined as the largest integer that is less than or equal to \( x \). Any real number \( x \) is either an integer or it lies between two integers. If \( n \leq x < n + 1 \), the largest integer less than or equal to \( x \) is \( n \), so for such an \( x \), \( \text{Int}(x) = n \). The function is denoted by \( \text{Int}(x) \) on graphing calculators, or sometimes \( \text{Floor}(x) \), or, in older books, by \( [x] \).

**Definition: The greatest integer function**

The greatest integer function of \( x \), denoted by \( \text{Int}(x) \), is the largest integer that is less than or equal to \( x \).

As examples,

\[
\text{Int}(1) = 1, \quad \text{and for any integer } n, \quad \text{Int}(n) = n, \\
\text{Int}(0.3) = 0, \quad \text{and for any number } x \text{ between } 0 \text{ and } 1, \quad \text{Int}(x) = 0, \\
\text{Int}(\pi) = 3 \text{ because } 3 \text{ is the largest integer less than } \pi, \\
\text{Int}(-\pi) = -4 \text{ because } -\pi \text{ is between } -4 \text{ and } -3.
\]

Postal rates are examples of functions defined piecewise, where the definition can make use of \( \text{Int}(x) \). Mailing cost is a function of the weight of a letter. It remains constant for a while and then suddenly jumps to a new value.
**EXAMPLE 4** The postage function  
In 1995, postal rates for first class letters delivered within the United States were set as follows: the cost is 32 cents for anything less than 1 ounce; for each additional ounce (up to 11 ounces) the cost increases in increments of 23 cents. Express the cost \( C \) (in cents) of first-class postage as a function of the weight \( W \) (ounces) and draw a graph.

**Solution**

In mathematical language, we can express the cost \( C \) as a function of the weight \( W \) either piecewise, or by using the greatest integer function.

\[
C = 32 + 23 \lfloor W \rfloor, \quad 0 < W < 11.
\]

The graph of the cost function is shown in Figure 7.

**TECHNOLOGY TIP**

In connected mode, a calculator graph of a piecewise-defined function can make it appear as if there is a vertical piece of the graph connecting pieces that should not be connected. Don’t be fooled.

There can be a “jump” from one piece to another *between pixels*. When \( y \)-values differ in adjacent pixel columns, the calculator connects pixels in vertical columns joining separated points.

It is often helpful to change to a non-connected format to see jumps in graphs of piecewise functions. In calculator graphs, remember that unless the calculator is in dot mode, the calculator connects separated pixels in adjacent columns.

Not all equations define functions, and some familiar graphs are not graphs of functions. A function assigns to each domain element exactly one element of the range. This implies that for any function \( f \) and any given domain number \( c \), there is exactly one point, \((c, f(c))\), on the graph of \( f \). A vertical line can meet the graph of \( f \) in at most one point, which is the basis for the following handy test.

**Vertical line test**

For a given graph, if at each number \( c \) of the domain, the vertical line \( x = c \) intersects the graph in exactly one point, then the graph represents the graph of a function. If some vertical line meets a graph in more than a single point, then the graph is not the graph of a function.

**EXAMPLE 5** Vertical line test  
Use a calculator to sketch a graph of all points that satisfy the equation \( y^2 = x \). Use the vertical line test to verify that the graph is not that of a function.

**Solution**

In function mode, we cannot enter equations except in the form of \( y = \ldots \), so when we solve the given equation for \( y \), we get \( y = \pm \sqrt{x} \). On the same screen we graph \( y_1 = \sqrt{x} \) and \( y_2 = \sqrt{x} \). The two graphs together form a parabola opening to the right, as shown in Figure 8. Since any vertical line through the positive \( x \)-axis meets the graph twice, the vertical line test tells us that the graph is not the graph of a function.
EXERCISES 2.2

Check Your Understanding

Exercises 1–6 True or False. Give reasons.

1. The graph of a function cannot have more than one y-intercept point.
2. The graph of \( y = \frac{x}{|x|} \) is identical to the graph of \( y = \frac{1}{x} \).
3. For the greatest integer function \( f(x) = \lfloor x \rfloor \), (a) \( f(-2.5) = -f(2.5) \) (b) \( f(-3) = -f(3) \).
4. The distance between the x- and y-intercept points of the graph of \( y = 1 - x \) is \( \sqrt{2} \).
5. For any function \( f \), the function \( g(x) = f(x^2) \) is an even function.
6. For any even function \( f \), if \( (-2, -4) \) is on the graph of \( f \), then \( (2, 4) \) must also be on the graph of \( f \).

Exercises 7–10 Fill in the blank so that the following statement is true. “If you draw a graph of function \( f \) using \([-10, 10] \times \{-10, 10\} \), then the number of x-intercept points shown in the display is ______.”

7. \( f(x) = 0.3x^2 - 4x - 5 \)
8. \( f(x) = 0.5x^2 + 4x + 4 \)
9. \( f(x) = 2|x - 3|x - 3| \)
10. \( f(x) = 3|x - 6| - 2|x - 1| \)

Develop Mastery

Exercises 1–4 Isolated Points A function is given along with its domain. Draw a graph of the function. The graph consists of isolated points. State the range of the function.

1. \( f(x) = 2x - 1; D = \{-1, 2, 3\} \)
2. \( f(x) = 4 - x^2; D = \{-1, 0, 1, 2\} \)
3. \( f(x) = x^3 - x; D = \{-2, -1, 0, 1, 2\} \)
4. \( f(x) = \sqrt{x}; D = \{1, 2, 4\} \)

Exercises 5–8 Make a table of several \((x, y)\) ordered pairs that satisfy the equation. Plot the points in your table and draw a graph.

5. \( y = 2x - 4 \)
6. \( y = 4 - 2x \)
7. \( y = x^2 - x \)
8. \( y = 2x^2 + 4x \)

Exercises 9–12 Find the value of \( x \) or \( y \) so that point \( P \) is on the graph of \( f \).

9. \( f(x) = x^2 - 4x - 3; P(2, y) \)
10. \( f(x) = \sqrt{1 - 4x}; P(-3, y) \)
11. \( f(x) = 3x - 2; P(x, 4) \)
12. \( f(x) = x^2 - 2x - 8; P(x, -5) \)

Exercises 13–16 Odd, Even Determine whether function \( f \) is odd, even, or neither. Do the same for \( g \). First draw a graph then use algebra to support your conclusion.

13. \( f(x) = x^3 - 3x^2, g(x) = \sqrt{x - 1} \)
14. \( f(x) = x - x^3, g(x) = x^2 + 2|x| - 3 \)
15. \( f(x) = x^3 - 1, g(x) = x^3 - 2x \)
16. \( f(x) = (x + 1)(x - 1), g(x) = 3 - |x| \)

Exercises 17–28 Graphs Draw a graph of \( f \). Give the coordinates of the x- and y-intercept points.

17. \( f(x) = (x + 2)^2 \)
18. \( f(x) = x^3 - 2 \)
19. \( g(x) = \sqrt{x + 2} \)
20. \( g(x) = \sqrt{-x} \)
21. \( f(x) = |x - 1| - 1 \)
22. \( g(x) = \frac{1}{2} \sqrt{16 - x^2} \)
23. \( g(x) = (x + 1)^2 - 2 \)
24. \( g(x) = -2\sqrt{4 - x^2} \)
25. \( f(x) = -|x + 2|^2 \)
26. \( g(x) = \sqrt{x + 2} + 1 \)
27. \( g(x) = -|x| + 2 \)
28. \( f(x) = \sqrt{8 + 2x - x^2} \)

Exercises 29–30 Calculator Graph Suppose you are interested in using a graph to help you get information about the zeros of \( f \). Which of the given windows would you use?

29. \( f(x) = 2x^2 + 47x - 75 \)
   (i) \([-10, 10] \times [-10, 10]\)
   (ii) \([-20, 10] \times [-100, 100]\)
   (iii) \([-40, 10] \times [-400, 200]\)

30. \( f(x) = x - 5|x| + 100 \)
   (i) \([-10, 10] \times [-10, 10]\)
   (ii) \([-16, 20] \times [-5, 10]\)
   (iii) \([-20, 30] \times [-5, 100]\)

Exercises 31–34 Window Dimensions The graph of \( f \) has two x-intercept points and either a highest or lowest point. Give the dimensions of a window for which the graph of \( f \) will show this information. (Answers may vary.)

31. \( f(x) = x^3 - 25x + 150 \)
32. \( f(x) = 40 - 18x - x^3 \)
33. \( f(x) = x - 4|x| + 45 \)
34. \( f(x) = 2|x + 8| - x - 40 \)

Exercises 35–38 Limited Domain (a) Draw a graph of \( f \) with the indicated domain. (b) Find the range of \( f \).

35. \( D = \{x \mid x \geq 0\}; f(x) = 2x - 3 \)
36. \( D = \{x \mid x < 0\}; f(x) = 1 - x \)
37. \( D = \{x \mid x > 1\}; f(x) = x^2 \)
38. \( D = \{x \mid x \leq 2\}; f(x) = x^2 \)
Exercises 39–41  Piecewise Graph  Draw a graph of the given function and determine the range.

39. \( f(x) = \begin{cases} 
  x^2, & \text{if } x \geq 0 \\
  2 - x, & \text{if } x < 0 
\end{cases} \)

40. \( f(x) = \begin{cases} 
  x & \text{if } x \leq 0 \\
  1 + x & \text{if } x > 0 
\end{cases} \)

41. \( f(x) = \begin{cases} 
  x^2 + 2x & \text{if } x < 1 \\
  4 - x & \text{if } x \geq 1 
\end{cases} \)

Exercises 42–48  Refer to the function \( f \) whose graph is shown in the diagram with domain \([-2, 6]\).

42. From the graph, give \( f(-2), f(0), \) and \( f(4) \).

43. Order the following numbers from smallest to largest:
\( f(-1), f(4), f(3), f(\frac{3}{2}) \).

44. (a) What is the maximum value (the largest value) of \( f(x) \)?
   (b) What is the minimum value (the smallest value) of \( f(x) \)?
   (c) What is the range of \( f \)?

45. Give the coordinates of the highest and the lowest point on the graph.

46. Give the coordinates of the \( y \)-intercept point and the \( x \)-intercept point.

47. (a) For what values of \( x \) is \( f(x) \) negative?
   (b) For what values of \( x \) is \( f(x) \) positive?

48. True or false. Explain.
   (a) \( f(-2) \) is less than \( f(3) \).
   (b) \( f(4.3) \) is a negative number.
   (c) \( f(-1) - f(\sqrt{3}) \) is a negative number.
   (d) There are three \( x \)-intercept points for the graph of \( f \).

49. Use the vertical line test to determine which of these graphs are graphs of functions that have \( x \) as the independent variable.

Exercises 50–51  Interesting Functions

(a) Give the domain for \( f \) and for \( g \). Evaluate \( f \) at \( x = 0, 4, 10, 20, 40, 56 \).

(b) Draw graphs of \( f \) and \( g \) on separate screens.

(c) Look at the graphs and describe any interesting features.

(d) Are functions \( f \) and \( g \) identical? Explain.

(e) What is the solution set for \( f(x) = 3 \)? \( f(x) = 4 \)?

50. \( f(x) = \frac{1}{2}(\sqrt{x} + \sqrt{x} + 64 - 16\sqrt{x}) \)
\( g(x) = \frac{1}{2}(\sqrt{x} + |\sqrt{x} - 8|) \)

51. \( f(x) = \frac{1}{5}(\sqrt{x} + 200 + \sqrt{x} + 600 - 40\sqrt{x} + 2000) \)
\( g(x) = \frac{1}{5}(\sqrt{x} + 200 + |\sqrt{x} + 200 - 20|) \)

Exercises 52–53  Functions Involving Abs

(a) Draw a graph of \( f \). Use a decimal window.

(b) Use the graph to find the solution set for \( f(x) \leq 0 \).

52. \( f(x) = x - |x + 3| + |x - 4| - 1 \)

53. \( f(x) = |x + 4| - |x - 3| - x - 1 \)

Exercises 54–55  (a) Evaluate \( f \) at \( x = -5, -2, 0, 3.5 \).

(b) Give a formula for \( f \) in piecewise form.

(c) Use part (b) to draw a graph of \( f \).

54. \( f(x) = \min(2x - 3, 6 - x) \)

55. \( f(x) = \max(x - 2, -2x + 7) \)
60. \( f(x) = \lfloor x \rfloor - 1; f(5) = 2 \)
61. \( f(x) = \lfloor x - 1 \rfloor; f(x) = 2 \)
62. \( f(x) = \lfloor -1 \rfloor; f(x) = 2 \)
63. \( f(x) = 0.5x - \lfloor 0.5x \rfloor; f(3) = 0 \)

Exercises 64–67 Solution Set Find the solution set for the open sentence.
64. \( 2\lfloor x \rfloor^2 - 5\lfloor x \rfloor - 12 = 0 \) (Hint: Factor.)
65. \( \frac{3x}{2} - 1 = 0 \)
66. \( \lfloor \sqrt{p} \rfloor = 4 \) where \( p \) is a prime.
67. \( \lfloor x \rfloor - 3 \leq 0 \)

68. Postage Costs This section discussed postage charges. When a parcel or letter exceeds 11 ounces, a different rule applies for determining mailing cost as a function of weight. The rule depends upon mailing zones as well as weight and is given in tabular form. For example, when mailing from Zone 1 to Zone 8, the table below lists charges where \( w \) is the weight (not exceeding the number of pounds) and \( c \) is the cost in dollars.

<table>
<thead>
<tr>
<th>( w ) (Pounds)</th>
<th>( c ) (Dollars)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.00</td>
</tr>
<tr>
<td>2</td>
<td>3.00</td>
</tr>
<tr>
<td>3</td>
<td>4.00</td>
</tr>
<tr>
<td>4</td>
<td>5.00</td>
</tr>
<tr>
<td>5</td>
<td>6.00</td>
</tr>
<tr>
<td>6</td>
<td>8.00</td>
</tr>
<tr>
<td>7</td>
<td>9.80</td>
</tr>
<tr>
<td>8</td>
<td>11.60</td>
</tr>
</tbody>
</table>

69. Bug on a Ladder A bug starts at point \( (1, 0) \) and travels along the line segment \( AB \) toward point \( (0, 2) \) as shown in the diagram. If \( P(x, y) \) denotes the location of the bug when it has traveled a distance \( d \) from \( (1, 0) \), express the coordinates \( x \) and \( y \) as functions of the distance \( d \).

Exercises 70–71 Parking Costs
70. A parking garage charges $2.00 for parking up to one hour and $0.50 for each additional hour (or fraction thereof), with a maximum of $8.00 if you park 12 hours or longer. Suppose \( x \) denotes the number of hours you park and \( y \) (dollars) the corresponding cost. Then \( y \) is a function of \( x \) given in piecewise form:

\[
y = \begin{cases} 
2 + 0.50 \cdot \lfloor x \rfloor & \text{if } x < 12 \\
8 & \text{if } x \geq 12
\end{cases}
\]

(a) Draw a graph of this function. Use dot mode.
(b) If you have only $5.00, how long can you park?

71. A parking garage charges $3.00 for parking up to one hour and $0.30 for each additional fifteen minutes (or fraction thereof), with a maximum of $10. Suppose \( x \) denotes the number of hours you park and \( y \) (dollars) the corresponding cost. Then \( y \) is a function of \( x \) given in piecewise form:

\[
y = \begin{cases} 
3 & \text{if } x < 1 \\
2.10 + 0.30 \cdot \lfloor 4x \rfloor & \text{if } 1 \leq x < 6.75 \\
10 & \text{if } x \geq 6.75
\end{cases}
\]

(a) Use several values of \( x \) to check this formula.
(b) What is the cost for 4 hours and 20 minutes?
(c) For what values of \( x \) is the cost $4.50? Check using the graph. Use dot mode and a decimal window.