Theorem. Between any two real numbers there exists an irrational number.

Proof:

Case 1. Suppose $x$ is positive and $x < y$. There exists a natural number $k$ such that
\[ \frac{\sqrt{2}}{k} < y - x. \]
Let $w = \frac{\sqrt{2}}{k}$. Notice that $w$ is an irrational number. Starting at $x = 0$, we will walk along the positive real line with a step size equal to $w$. We will prove that we must step into the interval $(x, y)$ since its width is greater than $w$.

Let $S = \{ j : jw \geq y \}$. It follows from the Archimedean property that $S$ is not empty. Let $m$ be the least element of $S$. Then, $mw \geq y$ and $(m - 1)w < y$.

We must show that $x < (m - 1)w$. Suppose $(m - 1)w \leq x$. Then $mw - w \leq x$, and $mw \leq x + w < x + y - x = y$. This implies that $mw < y$, a contradiction. Thus, $x < (m - 1)w < y$. Since $w$ is irrational, $(m - 1)w$ is also an irrational number.

Case 2. Suppose $x \leq 0$. Choose a positive integer $n$ such that $x + n > 0$. Consider the positive numbers $x + n$ and $y + n$. From case 1, there exists an irrational number $u$ such that $x + n < u < y + n$. Subtracting $n$ from both sides of this inequality gives $x < u - n < y$. Now, $u - n$ is irrational since $u$ is irrational, and $n$ is rational.

Theorem. Between any two real numbers there exists a rational number.

Proof:

Modify the proof above by replacing $\frac{\sqrt{2}}{k}$ with $\frac{1}{k}$.