Limit of a Function (More Newton and Leibniz)

Example 1.

The function $f$ is defined by $f(x) = \frac{x^2 - 1}{x - 1}$.

What happens to $\frac{x^2 - 1}{x - 1}$ as $x$ gets closer and closer to 1?

\[
\frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{(x-1)} = x+1
\]

when $x \neq 1$. So $\frac{x^2 - 1}{x - 1}$ gets ever closer to 2 as $x$ gets closer and closer to 1.
We say "The limit of \( f(x) \) as \( x \) approaches 1 is equal to 2."

We write \( \lim_{x \to 1} f(x) = 2 \) or \( \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2 \).

Example 2.

Find \( \lim_{x \to 0} \frac{\sqrt{x^2 + 4} - 2}{x} \)

If we substitute \( x = 0 \), we get \( \frac{\sqrt{4} - 2}{0} \) or \( \frac{0}{0} \) which is not defined.

*Algebra to the rescue:*

\[
\frac{\sqrt{x^2 + 4} - 2}{x} \cdot \frac{\sqrt{x^2 + 4} + 2}{\sqrt{x^2 + 4} + 2} = \frac{x^2 + 4 - 4}{\sqrt{x^2 + 4} + 2} = \frac{x^2}{\sqrt{x^2 + 4} + 2} \rightarrow \frac{0}{4} = 0
\]

We write \( \lim_{x \to 0} \frac{\sqrt{x^2 + 4} - 2}{x} = 0 \).
Example 3.

Find \( \lim_{x \to 10} (x^3 + 9x^2 + 43) \).

\[
= 10^3 + 9 \cdot 10^2 + 43 = 1943
\]

Example 4.

Find \( \lim_{x \to 2} \frac{x^4 + x^2 + 1}{x^2 - 3} \).

\[
= \frac{2^4 + 2^2 + 1}{2^2 - 3} = \frac{16 + 4 + 1}{1} = 21
\]

Example 5.

Find \( \lim_{t \to 3} \sqrt{4t^2 + 5} \).

\[
= \sqrt{4 \cdot (3)^2 + 5} = \sqrt{41}
\]
THEOREM 1—Limit Laws  If \( L, M, c, \) and \( k \) are real numbers and

\[
\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \quad \text{then}
\]

1. **Sum Rule:** \( \lim_{x \to c} (f(x) + g(x)) = L + M \)
2. **Difference Rule:** \( \lim_{x \to c} (f(x) - g(x)) = L - M \)
3. **Constant Multiple Rule:** \( \lim_{x \to c} (k \cdot f(x)) = k \cdot L \)
4. **Product Rule:** \( \lim_{x \to c} (f(x) \cdot g(x)) = L \cdot M \)
5. **Quotient Rule:** \( \lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0 \)
6. **Power Rule:** \( \lim_{x \to c} [f(x)]^n = L^n, \quad n \) a positive integer
7. **Root Rule:** \( \lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}, \quad n \) a positive integer

(If \( n \) is even, we assume that \( \lim_{x \to c} f(x) = L > 0 \).)

How can the limit of a function fail to exist as \( x \) approaches a value \( a \)?

Consider the following three functions:

a) \( f \) where \( f(x) = \frac{|x|}{x} \)

b) \( g \) where \( g(x) = \frac{1}{x^2} \)

c) \( h \) where \( h(x) = \sin \frac{1}{x} \)
$y = \frac{1\times|}{x}$

\[
\lim_{x \to 0} \frac{|x|}{x} \text{ does not exist}
\]

"there is a jump at \(x = 0\)"

\[
\text{Note: } \lim_{x \to 0^+} \frac{|x|}{x} = 1
\]

\[
\lim_{x \to 0^-} \frac{|x|}{x} = -1
\]
\[ y = g(x) = \frac{1}{x^2} \]

\[ \lim_{x \to 0} \frac{1}{x^2} \text{ does not exist} \]

"unbounded behavior at } x = 0"

Note: we will write \( \lim_{x \to 0} \frac{1}{x^2} = \infty \)
\lim_{x \to 0} f(x) \text{ does not exist}

"wildly oscillatory behavior"
What does $|a - b|$ represent?

Definition:

$$\lim_{x \to c} f(x) = L$$ means

For every distance $\varepsilon > 0$ there exists a distance $\delta > 0$ such that if $0 < |x - c| < \delta$ then $0 < |f(x) - L| < \varepsilon$.

Consider $f$ to be a correspondence or mapping:

If we choose any $x$ such that $x \neq c$ and the distance from $x$ to $c$ is less than $\delta$, we can conclude that the distance from $f(x)$ to $L$ is less than $\varepsilon$. 
Consider $f$ graphically:

The graph of $f$ over $(c-s, c+s)$ is between the lines $y = L + \varepsilon$ and $y = L - \varepsilon$. 
Example:

Suppose $f$ is defined by $f(x) = \sqrt{x}$. For $\varepsilon = \frac{1}{10}$, find a corresponding $\delta$ so that if $0 < |x - 4| < \delta$ then $0 < |f(x) - 2| < \frac{1}{10}$.

\[
\sqrt{x_1} = 1.9, \quad x_1 = (1.9)^2 = 3.61
\]
\[
\sqrt{x_2} = 2.1, \quad x_2 = (2.1)^2 = 4.41
\]

Choose $\delta = \text{minimum of } 4.41 - 4$ and $4 - 3.61$, the minimum of

$.41$ and $.39$

$\delta = .39$ will work,

so will any $\delta < .39$
Example:

Suppose $f$ is defined by $f(x) = \sqrt{x}$. For $\varepsilon > 0$, find a corresponding $\delta$ so that if $0 < |x - 4| < \delta$ then $0 < |f(x) - 2| < \varepsilon$. (This will show that $\lim_{x \to 4} \sqrt{x} = 2$)

\[
\begin{align*}
\sqrt{x_1} &= 2 - \varepsilon, \quad & x_1 &= (2 - \varepsilon)^2, \quad & S_1 &= 4 - (2 - \varepsilon)^2 \\
\sqrt{x_2} &= 2 + \varepsilon, \quad & x_2 &= (2 + \varepsilon)^2, \quad & S_2 &= (2 + \varepsilon)^2 - 4 \\
\delta &= \min \{ S_1, S_2 \}.
\end{align*}
\]

Now if $4 - S < x < 4 + S$, then

\[
\begin{align*}
4 - S_1 < x < 4 + S_2, &\quad 4 - [4 - (2 - \varepsilon)^2] < x < 4 + (2 + \varepsilon)^2 - 4 \\
(2 - \varepsilon)^2 < x < (2 + \varepsilon)^2, &\quad 2 - \varepsilon < \sqrt{x} < 2 + \varepsilon.
\end{align*}
\]