Limit of a Function (More Newton and Leibniz)

Example 1.

The function $f$ is defined by $f(x) = \frac{x^2 - 1}{x - 1}$.

What happens to $\frac{x^2 - 1}{x - 1}$ as $x$ gets closer and closer to 1?
We say “The limit of $f(x)$ as $x$ approaches 1 is equal to 2.”

We write $\lim_{x \to 1} f(x) = 2$ or $\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$.

Example 2.

Find $\lim_{x \to 0} \frac{\sqrt{x^2 + 4} - 2}{x}$
Example 3.

Find \( \lim_{x \to 0} \left( x^3 + 9x^2 + 43 \right) \).

Example 4.

Find \( \lim_{x \to 2} \frac{x^4 + x^2 + 1}{x^2 - 3} \).

Example 5.

Find \( \lim_{t \to 3} \sqrt{4t^2 + 5} \).
How can the limit of a function fail to exist as $x$ approaches a value $a$?

Consider the following three functions:

a) $f$ where $f(x) = \frac{|x|}{x}$

b) $g$ where $g(x) = \frac{1}{x^2}$

c) $h$ where $h(x) = \sin \frac{1}{x}$

THEOREM 1—Limit Laws  
If $L$, $M$, $c$, and $k$ are real numbers and

$$\lim_{x \to c} f(x) = L$$  and  $$\lim_{x \to c} g(x) = M,$$  then

1. **Sum Rule:**  $$\lim_{x \to c} (f(x) + g(x)) = L + M$$

2. **Difference Rule:**  $$\lim_{x \to c} (f(x) - g(x)) = L - M$$

3. **Constant Multiple Rule:**  $$\lim_{x \to c} (k \cdot f(x)) = k \cdot L$$

4. **Product Rule:**  $$\lim_{x \to c} (f(x) \cdot g(x)) = L \cdot M$$

5. **Quotient Rule:**  $$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$$

6. **Power Rule:**  $$\lim_{x \to c} [f(x)]^n = L^n, \quad n \text{ a positive integer}$$

7. **Root Rule:**  $$\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}, \quad n \text{ a positive integer}$$

(If $n$ is even, we assume that $\lim f(x) = L > 0$.)
What does $|a-b|$ represent?

Definition:

$$\lim_{x \to c} f(x) = L$$ means

For every distance $\varepsilon > 0$ there exists a distance $\delta > 0$ such that if $0 < |x - c| < \delta$ then $0 < |f(x) - L| < \varepsilon$.

Consider $f$ to be a correspondence or mapping:
Consider $f$ graphically:
Example:

Suppose $f$ is defined by $f(x) = \sqrt{x}$. For $\varepsilon = \frac{1}{10}$, find a corresponding $\delta$ so that if $0 < |x - 4| < \delta$ then $0 < |f(x) - 2| < \frac{1}{10}$. 
Example:

Suppose $f$ is defined by $f(x) = \sqrt{x}$. For $\varepsilon > 0$, find a corresponding $\delta$ so that if $0 < |x - 4| < \delta$ then $0 < |f(x) - 2| < \varepsilon$. (This will show that $\lim_{x \to 4} \sqrt{x} = 2$.)