Problem Definition

Problem 33. **Maximum Volume:** A rectangular package to be sent by a postal service can have a maximum combined length and girth of 108 inches. Find the dimensions of the package that contains a maximum volume. Assume the dimensions are $x$ by $x$ by $y$. The girth is the distance around the package perpendicular to the length.

**Solution Step 1:**

The first step is define the variables used to measure the dimensions.

Let’s use the following system. Variable for square end of the package $x$ Variable for the length of the package $w$

The formula for the volume is given by

$$V = xwx = x^2w$$

and the formula for the dimension restrictions is the following.

Girth + Length $= 4x + w = 108$

**Solution Step 2:**

The next step is to solve for width $w$ in terms of the other dimension $x$ as follows.

$$w = 108 - 4w$$

This can be substituted into the volume formula. So, the formula for the volume is

$$V = V(x) = x^2(108 - 4x) = 108x^2 - 4x^3$$

Note that the dimensions are positive and the domain for the volume function is the interval $[0, 27]$.

**Solution Step 3:**
Now we can determine the critical points for the function and then determine the dimensions that maximizes the volume. The derivative of the volume function is

\[
\frac{dV}{dx} = 216x - 12x^2 = 12x(18 - x)
\]

The critical points as \(x = 0\) and \(x = 18\)

**Solution Step 4:**

For this problem, we can compute the volume for the critical points and the end points of the interval that is the domain of the volume function. This gives \(x = 0, w = 108\) \(V(0)=0\)

\(x = 18, w = 36\) \(V(18)=11664\) absolute maximum  As another test \(x = 27, w = 0\) \(V(27)=0\)

we could have used the first or second derivative tests to show that \(x = 18\) is a relative maximum.

**Solution Step 5:**

The maximum volume of 11664 cubic inches is obtained for the dimensions \(x = 18\) and \(w = 36.\)