8.5 **MATHEMATICAL INDUCTION**

The idea of mathematical induction is simply that if something is true at the beginning of the series, and if this is “inherited” as we proceed from one number to the next, then it is also true for all natural numbers. This has given us a method to prove something for all natural numbers, whereas to try out all such numbers is impossible with our finite brains. We need only prove two things, both conceivable by means of our finite brains: that the statement in question is true for 1, and that it is the kind that is “inherited.”

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In Section 1.4 we discussed statements (sentences that are either true or false) and open sentences, whose truth value depends on replacing a variable or placeholder with a number. In this section we consider sentences of the type

\[ n! < 8n \]

for every positive integer \( n \).

Such a sentence involves a quantifier: for every positive integer. Looking at the first few statements, we get

\[
\begin{align*}
1! &< 8 \cdot 1 \\
4! &< 8 \cdot 4 \\
6! &< 8 \cdot 6 \\
1 &< 8 \text{ (True)} \\
24 &< 32 \text{ (True)} \\
720 &< 48 \text{ (False)}
\end{align*}
\]

Since \( n! < 8n \) does not yield a true statement for every positive integer \( n \), Sentence (1) is false. To show that a sentence of the type (1) is false, all we need is one value of \( n \) that yields a false statement; any such value of \( n \) is a counterexample.

A statement of the form

\[ P(n) \]

for every positive integer \( n \),

means that the infinite set of statements \( P(1), P(2), P(3), \ldots \) are all true. Establishing the truth of such a statement requires a special method of proof called **mathematical induction.** Consider an example of the type given in Statement (2). An appeal to intuition leads us to the formal statement of the Principle of Mathematical Induction.

Suppose \( P(n) \) is given by

\[ P(n): \text{The sum of the first } n \text{ odd positive integers is } n^2. \]

It is difficult to work mathematically with a statement given verbally. We can restate \( P(n) \) in mathematical terms.

\[ P(n): 1 + 3 + 5 + \cdots + (2n - 1) = n^2 \]

In sigma notation

\[ P(n): \sum_{m=1}^{n} (2m - 1) = n^2. \]
If we claim that \( P(n) \) is true for every positive integer \( n \), then we must somehow show that each of the statements \( P(1) \), \( P(2) \), \( P(3) \), \ldots is true. We list a few of these:

\[
P(1): \sum_{m=1}^{1} (2m - 1) = 1^2 \quad \text{or} \quad 1 = 1^2 \quad \text{True}
\]

\[
P(2): \sum_{m=1}^{2} (2m - 1) = 2^2 \quad \text{or} \quad 1 + 3 = 2^2 \quad \text{True}
\]

\[
P(3): \sum_{m=1}^{3} (2m - 1) = 3^2 \quad \text{or} \quad 1 + 3 + 5 = 3^2 \quad \text{True}
\]

So far we have verified that \( P(1) \), \( P(2) \), and \( P(3) \) are all true. Clearly we cannot continue with the direct verification for the remaining positive integers \( n \), but suppose we can accomplish two things:

(a) Verify that \( P(1) \) is true.

(b) For any arbitrary positive integer \( k \), show that the truth of \( P(k + 1) \) follows from the truth of \( P(k) \).

If (a) and (b) can be done, then we reason as follows: \( P(1) \) is true by (a); since \( P(1) \) is true, then by (b) it follows that \( P(1 + 1) \), or \( P(2) \), must be true; now since \( P(2) \) is true then by (b) it follows that \( P(3) \) is true; and so on. Therefore, it is intuitively reasonable to conclude that \( P(n) \) is true for every positive integer \( n \). This type of reasoning is the basis for the idea of mathematical induction.

Having done (a) for the example, let us see if we can accomplish (b). We can state (b) in terms of a hypothesis and a conclusion and argue that the conclusion follows from the induction hypothesis:

Hypothesis \( P(k) \):

\[
1 + 3 + 5 + \cdots + (2k - 1) = k^2 \quad (3)
\]

Conclusion \( P(k + 1) \):

\[
1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = (k + 1)^2 \quad (4)
\]

Now use Equation (3) and argue that Equation (4) follows from it. To get to Equation (4) from Equation (3), add \( 2k + 1 \) to both sides.

\[
[1 + 3 + 5 + \cdots + (2k - 1)] + (2k + 1) = k^2 + (2k + 1) \quad (5)
\]

The right side of Equation (5) can be written as

\[
k^2 + (2k + 1) = k^2 + 2k + 1 = (k + 1)^2.
\]

Therefore, Equation (5) is equivalent to

\[
1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = (k + 1)^2.
\]

This is precisely Equation (4), so we have accomplished (b). Hence we can conclude that the statement

\[
1 + 3 + 5 + \cdots + (2n - 1) = n^2 \text{ for every positive integer } n
\]

is true.

The mathematical basis for our conclusion follows from the Principle of Mathematical Induction.
Principle of mathematical induction

Suppose $P(n)$ is an open sentence that gives statements $P(1), P(2), P(3), \ldots$. If we can accomplish the following two things:

(a) verify that $P(1)$ is true,
(b) for any arbitrary positive integer $k$, show that the truth of $P(k + 1)$ follows from the truth of $P(k),$

then $P(n)$ is true for every positive integer $n$.

**EXAMPLE 1 Truth values**  For the open sentence, write out $P(1), P(2),$ and $P(5).$ Determine the truth value of each.

(a) $P(n): n^3 + 11n = 6(n^2 + 1)$.
(b) $P(n): 5^a - 1$ is divisible by 4.

**Solution**

(a) $P(1): 1^3 + 11 \cdot 1 = 6(1^2 + 1)$ or $1 + 11 = 6(1 + 1)$ True
$P(2): 2^3 + 11 \cdot 2 = 6(2^2 + 1)$ or $8 + 22 = 6(5)$ True
$P(5): 5^3 + 11 \cdot 5 = 6(5^2 + 1)$ or $125 + 55 = 6(26)$ False

(b) $P(1): 5^1 - 1$ is divisible by 4, $5^1 - 1 = 4$ True
$P(2): 5^2 - 1$ is divisible by 4, $5^2 - 1 = 24$ True
$P(5): 5^5 - 1$ is divisible by 4, $5^5 - 1 = 3124$ True

**EXAMPLE 2 Proof by mathematical induction**  Let $P(n)$ be the open sentence

$P(n): 5^a - 1$ is divisible by 4.

Prove that $P(n)$ is true for every positive integer $n$.

**Solution**

Proof will follow if we can accomplish (a) and (b) of the Principle of Mathematical Induction. For (a) we must show that $P(1)$ is true. This has already been done in Example 1b.

For (b), state the induction hypothesis and conclusion.

Hypothesis $P(k): 5^k - 1$ is divisible by 4.  (6)

Conclusion: $P(k + 1): 5^{k+1} - 1$ is divisible by 4.  (7)

Since by hypothesis, $5^k - 1$ is divisible by 4, there is an integer $m$ such that

$5^k - 1 = 4m$ or $5^k = 4m + 1$.

Therefore,

$5^{k+1} - 1 = 5 \cdot 5^k - 1 = 5(4m + 1) - 1$
$= 20m + 4 = 4(5m + 1)$.

Hence if $5^k - 1$ is divisible by 4, then $5^{k+1} - 1$ is also divisible by 4. This establishes (b), and proves that $5^n - 1$ is divisible by 4 for every positive integer $n$.  

**Strategy:** After verifying that $P(1)$ is true, for mathematical induction, show that $P(k)$ implies $P(k + 1)$; relate $5^{k+1} - 1$ to $5^k - 1$. 
EXERCISES 8.5

Check Your Understanding

Exercises 1–10  True or False. Give reasons.
1. There is no positive integer $n$ such that $(n + 1)! = n! + 1!$.
2. There is no positive integer $n$ such that $n^2 + n = 6$.
3. For every positive integer $n$, $(n + 1)^2 \geq 2^n$.
4. For every positive integer $n$, $\sin n\pi = 0$.
5. For every positive integer $n$, $(2n - 1)(2n + 1)$ is an odd number.
6. For every positive integer $n$, $n^2 + n$ is an even number.
7. For every positive integer $n$, $n^2 + 1 \geq 2n$.
8. For every positive integer $n$, $(n + 1)^3 - n^3 - 1$ is divisible by 6.
9. For every positive integer $n$, $n^2 - n + 17$ is a prime number.
10. For every integer $n$ greater than 1, $\log_2 n \geq \log_n 2$.

Develop Mastery

Exercises 1–8  Truth Values  Denote the given open sentence as $P(n)$. Write out $P(1)$, $P(2)$, and $P(5)$, and determine the truth value of each.
1. $n^2 - n + 11$ is a prime number.
2. $4n^2 - 4n + 1$ is a perfect square.
3. $n^2 < 2n + 1$
4. $3^n > n^2$
5. $n! \leq n^2$
6. $4^n - 1$ is a multiple of 3.
7. $1^3 + 2^3 + 3^3 + \ldots + n^3$ is a perfect square.
8. The sum of the first $n$ even positive integers is equal to $n(n + 1)$.
9. $P(n): n! \leq n^3$
10. $P(n): n^2 - n + 5$ is a prime number.
11. $P(n): n^3 < 3^n$
12. $P(n): n! < 3^n$

Exercises 9–12  When is $P(n)$ False?  Find the smallest positive integer $n$ for which $P(n)$ is false.
13. $1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n + 1)^2}{4}$
14. $2 + 5 + 8 + \ldots + (3n - 1) = \frac{n(3n + 1)}{2}$
15. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + n(n + 1) = \frac{n(n + 1)(n + 2)}{3}$
16. $3^n > n^2$
17. $4^n - 1$ is a multiple of 3.
18. The sum of the first $n$ even positive integers equals $n(n + 1)$.
Exercises 19–32  Give a Proof Use mathematical induction to prove that the given formula is valid for every positive integer $n$.

19. $2 + 5 + 8 + \cdots + (3n - 1) = \frac{n(3n + 1)}{2}$

20. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n + 1) = \frac{n(n + 1)(n + 2)}{3}$

21. The sum of the first $n$ positive integers is equal to $rac{n(n + 1)}{2}$.

22. The sum of the first $n$ even positive integers is equal to $n(n + 1)$. 

23. $2 \cdot 1 + 2 \cdot 4 + 2 \cdot 7 + \cdots + 2(3n - 2) = 3n^2 - n$.

24. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n + 1)(2n + 1)}{6}$.

25. $1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2}$.

26. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n + 1)} = \frac{n}{n + 1}$.

27. $2 + 2^2 + 2^3 + \cdots + 2^n = 2(2^n - 1)$.

28. $1 + 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n! = (n + 1)! - 1$

29. $\sum_{n=1}^{n} (3n^2 + m) = n(n + 1)^2$

30. $\sum_{n=1}^{n} (2n - 3) = n(n - 2)$

31. $\sum_{n=1}^{n} (2^{n-1} - 1) = 2^n - n - 1$

32. $\sum_{n=1}^{n} \ln m = \ln(n!)$

Exercises 33–40  Give a Proof  Prove that $P(n)$ yields a true statement when $n$ is replaced by any positive integer $n$.

33. $P(n): 4^n - 1$ is divisible by 3.

34. $P(n): \cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$

35. $P(n): n^3 + 2n$ is divisible by 3.

36. $P(n): 2^{n+1} + 1$ is divisible by 3.

37. $P(n): 2^n \leq (n + 1)!$

38. $P(n): 3^n \leq (n + 2)!$

39. $P(n): 2^n \geq n + 1$

40. $P(n): 2^{n+1} > 2n + 1$

Exercises 41–49  Is it True?  Let $P(n)$ denote the open sentence. Either find the smallest positive integer $n$ for which $P(n)$ is false, or prove $P(n)$ is true for every positive integer $n$.

41. $n(n^2 - 1)$ is divisible by 6.

42. $n^2 + n$ is an even number.

43. $n^2 - n + 41$ is an odd number.

44. $5n^2 + 1$ is divisible by 3.

45. $5n^2 + 1$ is not a perfect square.

46. $2^x < (n + 1)^2$.

47. $n^2 - n + 41$ is a prime number.

48. $n^4 + 35n^2 + 24 = 10n^2 + 25$.

49. $\left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right) \cdots \left(1 + \frac{1}{n}\right) = n + 1$.

Exercises 50–51  Explore  (a) Evaluate the sum when $n$ is 1, 2, 3, and 4.  (b) Guess a formula for the sum.  (c) Prove that your formula is valid for every positive integer $n$.

50. $\sum_{n=1}^{n} \left(\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n + 1)}\right)$

51. $1 + 2 + 3 + \cdots + n + \cdots + 3 + 2 + 1$

52. Let $f(n) = 5^n - 4$.

(a) Evaluate $f$ when $n$ is 1, 2, 3, 4, and 5.

(b) For what positive integers $n$ is $f(n)$ divisible by 3?  Prove that your guess is correct.

(c) For what positive integers $n$ is $f(n)$ divisible by 21?  Prove that your guess is correct.

53. Explore  Sequence $(a_n)$ is defined recursively by $a_1 = 6$, $a_{n+1} = 5a_n/(a_n - 5)$ for $n \geq 1$ and $(b_n)$ is defined by $b_n = a_n/a_{n+1}$.  (a) Find the first four terms of $(b_n)$.  (b) Give a simpler formula for $b_n$.  (c) Is mathematical induction necessary to prove that your formula for $b_n$ is correct?  Explain.

54. Towers of Hanoi  There are three pegs on a board.  Start with $n$ disks on one peg, as suggested in the drawing.  Move all disks from the starting peg onto another peg, one at a time, placing no disk atop a smaller one.  You can experiment yourself without pegs; use coins of different sizes, for example, a dime on top of a penny, on top of a nickel, on top of a quarter. It may take some patience to find the minimum number of moves required.

(a) What is the minimum number of moves required if you start with 1 disk? 2 disks? 3 disks? 4?