Exercises 65–68  Express as a quotient of two integers in reduced form.

65. (a) $1.24$, (b) $\frac{124}{100}$
66. (a) 1.45, (b) $\frac{145}{100}$
67. (a) $1.125$, (b) $\frac{1125}{1000}$
68. (a) 0.72, (b) $\frac{72}{100}$

69. Evaluate the sum $\frac{\pi}{2} + \frac{3\pi}{2} + 2\pi + \frac{5\pi}{2} + \cdots + \frac{31\pi}{2} + 16\pi$.

70. (a) How many integers between 200 and 1000 are divisible by 11?
(b) What is their sum?

71. Find the sum of all odd positive integers less than 200.

72. Find the sum of all positive integers between 400 and 500 that are divisible by 3.

73. If $1 < a < b < c$ and $a$, $b$, and $c$ are the first three terms of a geometric sequence, show that the numbers $1, \log_b a, \log_c b$, and $\log_a c$ are three consecutive terms of an arithmetic sequence. (Hint: Use the change of the base formula from page 228.)

74. In a geometric sequence $\{a_n\}$ of positive terms, $a_1 = 12$ and $a_3 - a_1 = 324$. Find the first five terms of the sequence.

75. If the sum of the first 60 odd positive integers is subtracted from the sum of the first 60 even positive integers, what is the result?

76. The measures of the four interior angles of a quadrilateral form four terms of an arithmetic sequence. If the smallest angle is $72^\circ$, what is the largest angle?

77. In a right triangle with legs $a$, $b$, and hypotenuse $c$, suppose that $a$, $b$, and $c$ are three consecutive terms of a geometric sequence. Find the common ratio $r$.

78. The seats in a theater are arranged in 31 rows with 40 seats in the first row, 42 in the second, 44 in the third, and so on.

8.4 Patterns, Guesses, and Formulas

What humans do with the language of mathematics is to describe patterns. Mathematics is an exploratory science that seeks to understand every kind of pattern—patterns that occur in nature, patterns invented by the human mind, and even patterns created by other patterns.

Lynn Arthur Steen

Arithmetic and geometric sequences are highly structured, and it is precisely because we can analyze the regularity of their patterns that we can do so much with them. The formulas developed in the preceding section are examples of what can be done when patterns are recognized and used appropriately.
One of the strongest urges of the human mind is to discover, seek out, or impose some kind of order in the world around us. If we can organize new information into some kind of recognizable pattern, we can learn more efficiently and remember more accurately.

Patterns and mathematical formulas to describe patterns permeate mathematics. Where did all these formulas come from? All too often people have the impression that mathematics just is, that it has always been around in precisely its current form. Students frequently get the feeling that their main responsibility is just to learn the wisdom that has passed down through the ages.

In the 1970s Penrose’s lifelong passion for geometric puzzles yielded a bonus. He found that as few as two geometric shapes, put together in jigsaw-puzzle fashion, can cover a surface in patterns that never repeat themselves. “To a small extent I was thinking about how simple structures can force complicated arrangements, but mainly I was doing it for fun.”

Roger Penrose

Mathematics should be seen as an experimental, growing, changing science. It has never been limited to professional mathematicians. Some very important discoveries have been made by amateurs, ordinary people who became involved in the fascinating questions that are always at the heart of mathematics. Mathematics has grown from discoveries that excited those who found answers in patterns they were investigating. It has been strengthened by vigorous disagreements and arguments between different investigators. It has grown in much the same way as other scientific disciplines and it continues to develop today as much as ever before.

Our goal in this section is not to make a mathematician of every reader, but we do want to involve you in the discovery process, to provide some opportunity to experience the feeling of creation that drives mathematics. Someone who sees what appears to be a relationship and then can work through to an understanding of why it is valid is truly doing mathematics, whether or not someone else may have made a similar discovery before.

In many situations there may be several ways to write formulas, and there are almost always different ways to verify their correctness. In general, there is no single correct response. As you look for patterns, guess freely. Examine possible solutions. Try to understand what is happening. A guess remains just a guess until it is proven to be correct or shown to be incorrect. Proofs generally are much more difficult, and there is never any guarantee that a proof even exists. Some guesses that appear to be sound have not yet been established, even years after they were made. We illustrate some typical procedures in the following examples.

**EXAMPLE 1** Divisibility

For what positive integers $n$ is $2^n - 1$ divisible by 3?

**Solution**

Follow the strategy. Substituting $n = 1, 2, 3, \ldots$, calculate some numbers.

\[
2^1 - 1 = 1 \quad 2^2 - 1 = 3 \quad 2^3 - 1 = 7 \quad 2^4 - 1 = 15
\]

This indicates that $2^n - 1$ is divisible by 3 when $n$ is 2 or 4. On the basis of this very
small sample, we should probably hesitate to make a guess with much confidence, but it appears that every even value of \( n \) may give a number that is divisible by 3.

**GUESS:** \( 2^n - 1 \) is divisible by 3 for every even positive integer \( n \).

Now test the guess. The next even number for \( n \) is 6, and \( 2^6 - 1 = 63 \), which does have a factor of 3, reinforcing confidence in the guess. Also check what happens when \( n = 5 \). If \( 2^5 - 1 = 31 \) is not divisible by 3. The next even values for \( n \) yield \( 2^7 - 1 = 255 \) and \( 2^{10} - 1 = 1023 \), both of which are divisible by 3.

To prove that the guess is correct, we can use mathematical induction, which is discussed in the next section.

Often one guess about a pattern leads to recognition of a related pattern. After evaluating \( 2^n - 1 \) for several values of \( n \), other patterns may emerge. For convenience, let us use function notation \( f(n) = 2^n - 1 \):

\[
\begin{align*}
  f(1) &= 1 \\
  f(2) &= 3 \\
  f(3) &= 7 \\
  f(4) &= 15 \\
  f(5) &= 31 \\
  f(6) &= 63 \\
  f(7) &= 127
\end{align*}
\]

A natural question is: For what \( n \) is \( f(n) \) a prime number? If function \( g \) is given by \( g(n) = 2^n + 1 \), then ask when is \( g(n) \) divisible by 3, or when is \( g(n) \) a prime? For what values of \( n \) are \( f(n) \) or \( g(n) \) divisible by other numbers?

### Pascal’s Triangle

One marvelous source for pattern observation, called **Pascal’s triangle**, is a triangular array of numbers named after Blaise Pascal (1623–1687). Pascal may be considered the father of modern probability theory, in which these numbers play an important role. The numbers in Pascal’s triangle are also called **binomial coefficients**, a name we will justify in Section 8.6. Pascal was not the first, or only, discoverer of some of the properties of binomial coefficients. A beautiful representation of the triangle appeared in China as early as 1303, but Pascal did a great deal of work with the numbers we now associate with his name.

We shall examine binomial coefficients in greater detail in Section 8.6. At the moment, we are concerned primarily with the way the triangle is generated, one row at a time. Figure 3 shows only the first six rows, but the triangle can be continued as needed. The first and last entries on each row are always ones, and every other entry is obtained by adding the two adjacent entries immediately above.

Figure 3 shows the numbers themselves. In order to refer to specific entries in the triangle, we need to identify entries by location. The rows are numbered in obvious fashion; the columns are numbered diagonally, starting with column 0, not column 1. The entry in the \( n \)th row and the \( c \)th column is denoted by \( \binom{n}{c} \).

Figure 3b shows addresses of the corresponding entries in Figure 3a. For example, in the sixth row, the figure shows the following:

\[
\begin{align*}
  \binom{6}{0} &= 1 \\
  \binom{6}{1} &= 6 \\
  \binom{6}{2} &= 15 \\
  \binom{6}{3} &= 20 \\
  \binom{6}{4} &= 15
\end{align*}
\]
The rule for generating each row of Pascal’s triangle from the one just above it is indicated by the arrows in Figure 3a, showing how the fifth row generates the sixth row. In address notation,

\[
\begin{align*}
S_5^0 & \quad D_5^1 S_6^1 \quad D_5^1 S_6^2 \quad D_5^1 S_6^3 \quad D_5^1 S_6^4 \\
S_5^1 & \quad D_5^1 S_6^1 \quad D_5^1 S_6^2 \quad D_5^1 S_6^3 \quad D_5^1 S_6^4 \\
S_5^2 & \quad D_5^1 S_6^1 \quad D_5^1 S_6^2 \quad D_5^1 S_6^3 \quad D_5^1 S_6^4 \\
S_5^3 & \quad D_5^1 S_6^1 \quad D_5^1 S_6^2 \quad D_5^1 S_6^3 \quad D_5^1 S_6^4 \\
S_5^4 & \quad D_5^1 S_6^1 \quad D_5^1 S_6^2 \quad D_5^1 S_6^3 \quad D_5^1 S_6^4 \\
S_6^0 & \quad D_6^1 S_6^1 \quad D_6^1 S_6^2 \quad D_6^1 S_6^3 \quad D_6^1 S_6^4 \quad D_6^1 S_6^5 \\
\end{align*}
\]

This rule may be stated recursively.

**Binomial coefficients (recursive form)**

The symbol \( \binom{n}{c} \) denotes the entry in the \( n \)th row and the \( c \)th column of Pascal’s triangle. The end entries (where \( c \) is 0 or \( n \)) are 1 on each row. If \( 0 < c < n \), then adding adjacent entries in the \( n \)th row gives the entry between them in the next row.

\[
\binom{n}{0} + \binom{n}{1} = \binom{n+1}{1} \quad \text{and} \quad \binom{n}{c} + \binom{n}{c+1} = \binom{n+1}{c+1}
\]

**EXAMPLE 2  Sums in Pascal’s triangle**  Guess a formula for the sum of the entries on the \( n \)th row of Pascal’s triangle.

**Solution**

We first begin by looking at some specific cases that will help us understand the problem, and from which we may be able to recognize patterns. From Figure 3a, we add the numbers across each row and get the following sums.

<table>
<thead>
<tr>
<th>Row</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1 + 1 = 2</td>
</tr>
<tr>
<td>2</td>
<td>1 + 2 + 1 = 4</td>
</tr>
<tr>
<td>3</td>
<td>1 + 3 + 3 + 1 = 8</td>
</tr>
<tr>
<td>4</td>
<td>1 + 4 + 6 + 4 + 1 = 16</td>
</tr>
</tbody>
</table>

The sums appear to be doubling at each stage, suggesting an obvious guess.
8.4 Patterns, Guesses, and Formulas

HISTORICAL NOTE

Researchers in the area of artificial intelligence marvel at the capacity of the human mind to see patterns and discover relationships. As Douglas Hofstadter has said, “An inherent property of intelligence [is] that it is always looking for, and often finding, patterns.”

How can a machine be instructed to recognize that a set of data points lie along some line, that they are essentially linear? Such judgments require the ability to ignore exceptional cases and to consider others in clusters. Recent applications of pattern recognition routines are as diverse as space probes that can make midcourse corrections using star patterns for guidance, computer programs in medicine that can make probable diagnoses from patient responses to programmed questions, and programs to identify insect pests by analyzing the patterns in which leaves are chewed.

GUESS: The sum of the entries on the \( n \)th row of Pascal’s triangle is \( 2^n \).

The sum of the entries on the fifth row is 32, which is \( 2^5 \), and for the sixth row, the sum is 64, which is \( 2^6 \). The guess still looks good.

The key to understanding why the guess is correct is the way each row is derived from the row above it. Look at the arrows from the fifth to the sixth row in Figure 3a. The first 6 comes from adding the 1 and 5 above, and the same 5 with the 10 gives 15. Similarly, the same 10 is used again for the next entry. Thus each entry on the fifth row is added twice to get the sixth row, including the outside 1s to get the outside 1s on the sixth row. It follows that the sum of the entries on Row 6 is twice the sum of the entries on Row 5. Since \( 2^5 \) is the sum for Row 5, the sum of Row 6 must equal \( 2(2^5) = 2^6 \).

This argument is essentially a proof by mathematical induction, which we will discuss formally in the next section.

COMPUTERS AND PATTERN RECOGNITION

Now investigators are exploring ways to connect large numbers of computer chips into neural networks to more closely simulate the way they think the brain may work. Some results are very promising. Neural nets are proving to be remarkably adept at predicting the biological activity of comparatively short fragments of DNA. They can find patterns in sequences that appear to be random. While we do not clearly understand just how these networks operate, they seem to take functions that describe the given numbers and combine the functions to predict the next terms. Scientists still cannot approach the capabilities of the human mind with machines, but, in some instances, the neural nets can now recognize patterns more complicated than human brains can handle.
EXAMPLE 3  Using Pascal’s triangle  Find a formula for the sum of the first \( n \) positive integers in terms of the entries in Pascal’s triangle.

**Solution**

Let \( f(n) \) denote the sum of the first \( n \) positive integers,

\[
 f(n) = 1 + 2 + 3 + \cdots + n.
\]

Express \( f(n) \) in terms of entries in Pascal’s triangle. First get some data:

\[
 f(1) = 1, \quad f(2) = 1 + 2 = 3, \quad f(3) = 1 + 2 + 3 = 6, \quad f(4) = 1 + 2 + 3 + 4 = 10.
\]

The numbers 1, 3, 6, and 10 are successive entries in Column 2 of Pascal’s triangle in Figure 8.1a. In address notation,

\[
 f(1) = \binom{2}{2}, \quad f(2) = \binom{3}{2}, \quad f(3) = \binom{4}{2}, \quad f(4) = \binom{5}{2}.
\]

Based on the data we gathered, we arrive at the following guess.

**GUESS:** \( f(n) = \binom{n+1}{2} \) for every positive integer \( n \).

If we look at the way each entry is obtained from the two immediately above it, we may see that we are adding precisely what is needed for the pattern to continue. Therefore, our guess must be correct.

EXAMPLE 4  Dividing a circle  Given \( n \) points on a circle, consider two functions:

- \( C(n) \) is the number of chords determined by connecting each pair of these points.
- \( R(n) \) is the number of regions into which the chords divide the interior of the circle, where no three chords have a common point of intersection inside the circle.

Guess a formula for (a) \( C(n) \) and (b) \( R(n) \).

**Solution**

For each value of \( n \), make a sketch from which we can get the information needed for the table (see Figure 4).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( C(n) )</th>
<th>( R(n) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>8</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>16</td>
</tr>
</tbody>
</table>

**Strategy:** From a table showing the first few values of \( C(n) \) and \( R(n) \), look for numbers that may be related to obvious powers of numbers or to entries in Pascal’s triangle.
(a) For \( n \geq 2 \), the values of \( C(n) \) are numbers in Column 2 of Pascal’s triangle. In address notation

\[
C(2) = \binom{2}{2}, \quad C(3) = \binom{3}{2}, \quad C(4) = \binom{4}{2}, \quad C(5) = \binom{5}{2}
\]

A pattern emerges on which to base a reasonable guess.

GUESS: \( C(n) = \binom{n}{2} \) for \( n \geq 2 \).

To see why the guess for \( C(n) \) continues to give the correct values, see what happens when you add one more point (going from \( n \) to \( n + 1 \)). Check to see that you get \( n \) new chords. Compare the resulting values with the values predicted by the formulas.

(b) From the table of values of \( R(n) \), the number of pieces appears to be a power of 2, doubling with each new point:

\[
R(1) = 1 = 2^0, \quad R(2) = 2 = 2^1, \quad R(3) = 4 = 2^2, \quad R(4) = 8 = 2^3, \quad \text{and} \quad R(5) = 16 = 2^4
\]

Make the obvious guess: \( R(n) = 2^{n-1} \) for every positive integer \( n \).

Draw circles with 6 and 7 points and carefully count the number of regions. According to the guess \( R(n) = 2^{n-1} \), we should get \( R(6) = 32 \) and \( R(7) = 64 \). What numbers do you get?

This is an excellent guess based on a beautiful pattern that simply happens to be wrong. Sometimes people speak of a pattern “breaking down.” The pattern does not break down; we have failed to find the right pattern. The correct formula is more complicated and may be expressed in terms of the entries in Pascal’s triangle.

\[
R(n) = \binom{n}{0} + \binom{n}{2} + \binom{n}{4}.
\]

For instance, when \( n = 5 \), from Figure 8.1

\[
\binom{5}{0} + \binom{5}{2} + \binom{5}{4} = 1 + 10 + 5 = 16
\]

which is the value of \( R(5) \). Evaluate

\[
\binom{n}{0} + \binom{n}{2} + \binom{n}{4}
\]

when \( n = 6 \) and when \( n = 7 \). (Extend Pascal’s triangle to include row seven.) Compare your results with the actual count of the number of regions, \( R(6) \) and \( R(7) \).

In Section 8.6 we will see how to evaluate binomial coefficients to get

\[
R(n) = \frac{n^4 - 6n^3 + 23n^2 - 18n + 24}{24}.
\]
**EXERCISES 8.4**

**Check Your Understanding**

**Exercises 1–10. True or False. Give reasons.**

1. When $n$ is 1, 2, 3, 4, or 5, the sum of the first $n$ odd positive integers is equal to $n^2$.
2. When $n$ is 1, 2, 3, 4, or 5, the sum of the first $n$ even positive integers is equal to $2n(n + 1)$.
3. If $f(n) = n^2 - n + 17$, then $f(n)$ is a prime number for $n = 1, 2, 4, 8$, and 17.
4. If $f(n) = n^2 + n$, then $f(n)$ is an even number for every positive integer $n$.
5. Evaluating the expressions $(n + 1)^2$ and $2^n$ for $n = 1, 2, 3, 4, 5, 6$, it is reasonable to conclude that $(n + 1)^2 > 2^n$ for every positive integer $n$.
6. For every positive integer $n$, $3^n + 1$ is an even number.
7. For every positive integer $n$, the units digit of $5^n - 1$ is 4.
8. When $n$ is 1, 2, 3, or 4, $5^n + 1$ is not divisible by 4.
9. For every positive integer $n$, the units digit of $2^n$ is 2, 4, or 8.
10. For every positive integer $n$, the units digit of $4^n - 1$ is 3 or 5.

**Develop Mastery**

*In these exercises, $n$ always denotes a natural number.*

**Exercises 1–5 Recognize Patterns** As a first step for each exercise, complete the following table by entering the values of $f(n)$ for the given function. In order to see patterns, it is very important that computations be correct. As a check, one of the values of $f(n)$ is given.

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(n)$</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>11</td>
<td>11</td>
<td>11</td>
</tr>
</tbody>
</table>

1. $f(n) = 5^n - 1$. Check: $f(4) = 5^4 - 1 = 624$.
   (a) For what values of $n$ in your table is $f(n)$ a multiple of 3? Of 4? Of 12?
   (b) Based on these observations, make a guess that describes all natural numbers $n$ for which $f(n)$ is a multiple of 3, of 4, and of 12. State your guess in complete sentences.

2. $f(n) = 5^n + 1$. Check: $f(5) = 5^5 + 1 = 3126$.
   (a) For what values of $n$ in your table is $f(n)$ a multiple of 6? Of 12?
   (b) Make a guess that describes all natural numbers $n$ for which $f(n)$ is a multiple of 6. Express your guess in complete sentences.
   (c) For additional information: Is $f(9)$ a multiple of 7? Is $f(15)$ a multiple of 7? Is $f(18)$ a multiple of 7? For which $n$ is $f(n)$ a multiple of 7?

3. $f(n) = 9 + 9^2 + 9^3 + \ldots + 9^n$.
   Check: $f(3) = 9 + 9^2 + 9^3 = 819$.
   Based on the data in your table, make a guess that describes all natural numbers $n$ for which the units digit of $f(n)$ is a zero, a one, a nine. Convince your teacher that your guess is correct.

4. $f(n) = n^2 - n + 11$.
   Check: $f(5) = 5^2 - 5 + 11 = 31$.
   (a) For what values of $n$ in your table is $f(n)$ a prime number? Is $f(n)$ a prime number for every natural number $n$?
   (b) Make a guess concerning the units digit of $f(n)$.
   (c) To lead to a recursive formula, enter appropriate numbers in each of the blank spaces:

   
   \[ f(2) = f(1) + \underline{\hspace{2cm}} \]
   
   \[ f(3) = f(2) + \underline{\hspace{2cm}} \]
   
   \[ f(4) = f(3) + \underline{\hspace{2cm}} \ldots \]

   Now guess the quantity that should be entered in the general case:

   \[ f(n + 1) = f(n) + \underline{\hspace{2cm}}. \]

   Prove that your guess is valid (or not valid) by actually evaluating $f(n + 1)$ and $f(n) + \underline{\hspace{2cm}}$ to see if they are equal.

   (d) Evaluate $f(11), f(22)$, and $f(33)$. Now check your conclusion in (a).

5. $f(n) = n^2 - n + 41$.
   Check: $f(5) = 5^2 - 5 + 41 = 61$.
   (a)-(c) Same as in Exercise 4.
   (d) Evaluate $f(41)$ and $f(82)$. Are these primes?

**Exercises 6–14 Guess a Formula** Function $f$ is defined as a sum. (a) Make a table that shows the values of $f(n)$ for $n = 1, 2, 3, 4, 5$, and 6. (b) Based on the data in the table, guess a simpler formula for $f$. The value of $f(4)$ is also given, which should serve as a check on your computations and also possibly as a hint to help you recognize patterns.

6. $f(n) = 1 + 3 + 5 + \ldots + (2n - 1)$
   \[ = \sum_{k=1}^{n} (2k - 1); \quad f(4) = 16. \]

7. $f(n) = 2 + 4 + 6 + \ldots + 2n = \sum_{j=1}^{n} 2j$
   \[ f(4) = 4 \cdot 5. \]

8. $f(n) = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{n(n + 1)}$
   \[ = \sum_{i=1}^{n} \frac{1}{i(i + 1)}; \quad f(4) = \frac{4}{5}. \]
9. Let \( f(n) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} \).
\[ f(n) = \sum_{i=1}^{n} \frac{1}{2^i}; \quad f(4) = \frac{16 - 1}{16}. \]

10. Let \( f(n) = 2\left(\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots + \frac{1}{3^n}\right) \).
\[ f(n) = 2 \cdot \left(\sum_{i=1}^{n} \frac{1}{3^i}\right); \quad f(4) = \frac{80}{81}. \]

11. For \( f(n) = 1 + 2(3^0 + 3^1 + 3^2 + \ldots + 3^{n-1}) \)
\[ f(n) = 1 + 2 \sum_{i=1}^{n} 3^{i-1}; \quad f(4) = 3^4. \]

(c) Use the result in (b) to get a formula for \( \Sigma_i 2^i, 3^{i-1} \).

12. \( f(n) = \frac{1}{2} + \frac{2}{3!} + \frac{3}{4!} + \ldots + \frac{n}{(n + 1)!} \).
\[ f(n) = \sum_{k=1}^{n} \frac{k}{(k + 1)!}; \quad f(4) = \frac{119}{120}. \]

(Hint: Keep in mind the values of factorials, such as \(6! = 362880, 24! = 6.20444896E+23, 120 = 5! \), \ldots.)

13. \( f(n) = 1 + (1 + 2 + 4 + \ldots + 2^{n-1}) \)
\[ f(n) = 1 + \sum_{i=1}^{n} 2^{i-1}; \quad f(4) = 2^4. \]

(c) Use your results in (b) to get a simpler formula for \( g(n) = 1 + 2 + 4 + \ldots + 2^{n-1} \).

14. \( f(n) = 1 + 2 + 3 + \ldots + (n - 1) + n + (n - 1) + \ldots + 3 + 2 + 1 \)
\[ f(n) = 1 + 2 + 3 + 4 + 3 + 2 + 1 = 16. \]

15. For \( f(n) = 2 \left(1 - \frac{1}{2^2}\right) \left(1 - \frac{1}{3^2}\right) \left(1 - \frac{1}{4^2}\right) \ldots \left(1 - \frac{1}{(n + 1)^2}\right) \).

Evaluate \( f(n) \) for \( n = 1, 2, 3, 4, 5 \), and 6. For instance, \( f(4) = \frac{6}{5} = \frac{4 + 2}{4 + 1} \). Based on your data, guess a simpler formula for \( f(n) \).

For \( f(n) = 2 \left(1 - \frac{1}{2^2}\right) \left(1 - \frac{1}{3^2}\right) \left(1 - \frac{1}{4^2}\right) \ldots \left(1 - \frac{1}{(n + 1)^2}\right) \).

16. \( f(1) = 2 \) and \( f(n) = 2(f(n - 1)) \) for \( n \geq 2; f(5) = 2^5 \).

17. \( f(1) = 2 \) and \( f(n) = f(n - 1) + 2n \) for \( n \geq 2; f(5) = 5 \).

18. \( f(1) = 3 \) and \( f(n) = f(n - 1) + (2n + 1) \) for \( n \geq 2; f(4) = 4 \).

19. Number of Handshakes

Suppose that each of the \( n \) people at a party shakes hands with every other person exactly once. Let \( f(n) \) denote the total number of handshakes, so that \( f(1) = 0 \) (one person, no handshakes), \( f(2) = 1 \) (two people, one handshake), \( f(3) = f(2) + 2 \) (adding a person adds two more handshakes). A newcomer shakes hands with each of the \( k \) people present and so \( f(k + 1) = f(k) + k \). Find a formula for \( f \).

20. In Example 4 let \( D(n) \) be the number of diagonals of the polygon obtained by connecting the points on the circle. For example, \( D(1) = D(2) = D(3) = 0 \), \( D(4) = 2 \), and \( D(5) = 5 \). Guess a formula for \( D(n) \). (Hint: Count the number of new diagonals when adding a point.)

**Exercises 21–24: Pascal’s Triangle**

Extend Pascal’s triangle, shown in Figure 3, for a few more rows. When you are asked to find a number in the triangle, express it in address notation, \( \binom{n}{k} \).

21. How many entries appear on row \( n \)?

22. On what rows of the triangle are the two middle entries the same?

23. Each row in Figure 3 is symmetrical. (Each reads the same forward and backward.) Explain how you can know that Row 7 is symmetrical without computing any entries in Row 7. How about Row 8? Row \( n \)?

24. In Example 2, we showed that the sum of all the entries on Row \( n \) is \( 2^n \) (symbolically, \( \sum_{k=0}^{n} \binom{n}{k} = 2^n \)). Let \( f(n) \) be the sum of the entries in the even-numbered columns of Row \( n \). That is,
\[ f(n) = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots + \binom{n}{m} \]
where \( m \) is the last even-numbered column on Row \( n \). For instance,
\[ f(4) = \binom{4}{0} + \binom{4}{2} + \binom{4}{4} = 1 + 6 + 1 = 8 \]
\[ f(5) = \binom{5}{0} + \binom{5}{2} + \binom{5}{4} = 1 + 10 + 5 = 16. \]
Evaluate \( f(n) \) for several other values of \( n \) and then use the information to help you guess a simpler formula for \( f \).

25. Follow the instructions for Exercise 24, but let \( f(n) \) denote the sum of all the entries in odd-numbered columns on Row \( n \), that is
\[ f(n) = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots + \binom{n}{m} \]
where \( m \) is the last odd-numbered column in Row \( n \).
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Exercises 26–29 Sums in Pascal’s Triangle Let \( f(n) \) be the sum of the first \( n \) entries in the given column.

(a) Evaluate \( f(n) \) for several values of \( n \) (\( f(4) \) is given as a check). (b) Locate these sums in the triangle, and then guess a simpler formula for \( f \) in terms of address notation.

26. Column 0
\[
f(n) = \binom{1}{0} + \binom{2}{0} + \binom{3}{0} + \cdots + \binom{n}{0}
\]
\( f(4) = \frac{4}{1} \).

27. Column 1
\[
f(n) = \binom{1}{1} + \binom{2}{1} + \binom{3}{1} + \cdots + \binom{n}{1}
\]
\( f(4) = \frac{3}{2} \).

28. Column 2
\[
f(n) = \binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{n+1}{2}
\]
\( f(4) = \frac{6}{3} \).

29. Column 3
\[
f(n) = \binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \cdots + \binom{n+2}{3}
\]
\( f(4) = \frac{7}{4} \).

30. Let \( f(n) \) denote the sum of the squares of the first \( n \) natural numbers:
\[
f(n) = 1^2 + 2^2 + 3^2 + \cdots + n^2.
\]
(a) Evaluate \( f(n) \) for \( n = 1, 2, 3, 4, 5, \) and 6. For instance, \( f(4) = 30 \).
(b) Now look in Column 3 of Pascal’s triangle and find two consecutive entries whose sum is \( f(n) \). For instance,
\[
f(4) = 30 = 10 + 20 = \binom{5}{3} + \binom{6}{3}.
\]
Use this information to help you guess a formula that gives \( f(n) \) as the sum of two consecutive entries in Column 3. Use address notation in your answer.

31. Let \( P(n) \) denote the number of pieces (regions) into which \( n \) lines divide the plane. Assume that no two lines are parallel, and that no three lines contain a common point. Draw figures to illustrate the cases for \( n = 1, 2, 3, 4, \) and 5. By actually counting the pieces in each case, evaluate \( P(n) \) for \( n = 1, 2, 3, 4, \) and 5. Guess a formula for \( P(n) \). (Hint: Look for \( P(n) - 1 \) in Pascal’s triangle.)

32. Related Functions Function \( f \) is defined recursively by
\[
f(1) = 1 \quad f(2) = 5 \quad \text{and} \quad f(n) = f(n-1) + 2f(n-2) \quad \text{for } n \geq 3.
\]
Functions \( g \) and \( h \) are given in closed form by
\[
g(n) = 2^n + 1 \quad h(n) = 2^n - 1.
\]
(a) Complete the following table:

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(n) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( g(n) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( h(n) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

As a check, you should have \( f(4) = 17 \), \( g(4) = 17 \), \( h(4) = 15 \).
(b) Based on the data in the table, make a guess about the values of \( n \) for which \( f(n) = g(n) \) and for which \( f(n) = h(n) \).
(c) Using your guess in (b), is \( f(n) = 2^n + (-1)^n \) for every natural number \( n \)?

33. At the end of Example 4, we indicated that \( R(n) \) is given by the formula:
\[
R(n) = \frac{n^4 - 6n^3 + 23n^2 - 18n + 24}{24}.
\]
Evaluate \( R(n) \) for \( n = 4, 5, 6, \) and 7. For each value of \( n \) draw an appropriate diagram and actually count the number of regions to see if there is agreement with the formula prediction.

34. Sequence \( \{b_n\} \) is defined recursively by \( b_1 = 3, b_2 = 5, b_n = b_{n-1} - b_{n-2}, n > 2 \). That is, after the second term, each term is the difference of the two preceding terms.
(a) List the first ten terms.
(b) Based on the results in part (a), describe a general pattern for the sequence.
(c) What is the sum of the first 1996 terms? The first 2000 terms?

35. Repeat Exercise 34 for \( b_1 = x, b_2 = y \).