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Journal of Functional Analysis 203 (2003) 550–568
JOURNAL OF Functional Analysis
http://www.elsevier.com/locate/jfa

Caffarelli–Kohn–Nirenberg inequalities with remainder terms ☆

Zhi-Qiang Wang[a] and Michel Willem[b,*]

[a] Department of Mathematics, Utah State University, Logan, UT 84322-3900, USA
[b] Département Mathématique pures et Appliquees, Université Catholique de Louvain La-Neuve, 1348 Louvain-la-Neuve, Belgium

Received 15 July 2002; revised 1 November 2002; accepted 27 November 2002

Communicated by H. Brezis

1. Introduction

This paper is concerned with Hardy and Hardy–Sobolev type inequalities with remainder terms. In particular, we shall focus on the following Hardy–Sobolev type inequalities due to [7]. For all $u \in C_0^\infty(\mathbb{R}^N)$ it holds

$$\int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2\,dx \geqslant C_{a,b}\left(\int_{\mathbb{R}^N} |x|^{-bp}|u|^p\,dx\right)^{\frac{2}{p}}, \tag{1}$$

where

$$\left.\begin{array}{lll} \text{for } N\geqslant 3: a<\frac{N-2}{2}, & a\leqslant b\leqslant a+1, & p=\frac{2N}{N-2+2(b-a)};\\ \text{for } N=2: a<0, & a<b\leqslant a+1, & p=\frac{2}{b-a};\\ \text{for } N=1: a<-\frac{1}{2}, & a+\frac{1}{2}<b\leqslant a+1, & p=\frac{2}{-1+2(b-a)}. \end{array}\right\} \tag{2}$$

Let $D_a^{1,2}(\mathbb{R}^N)$ be the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$||u||^2 = \int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2\,dx, \tag{3}$$

☆ Research supported by NATO grant PST.CLG.978736.

*Corresponding author. Fax: +32-10-472530.
E-mail address: willem@math.ucl.ac.be (M. Willem).

0022-1236/03/$ - see front matter
doi:10.1016/S0022-1236(03)00017-X

which is given by the inner product $(u,v) = \int_{\mathbb{R}^N} |x|^{-2a} \nabla u \cdot \nabla v \, dx$. Then (1) holds for $u \in D_a^{1,2}(\mathbb{R}^N)$. Define the best constant

$$S(a,b) = \inf_{D_a^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |x|^{-2a} |\nabla u|^2 \, dx}{\left(\int_{\mathbb{R}^N} |x|^{-bp} |u|^p \, dx \right)^{\frac{2}{p}}}. \tag{4}$$

Then it is known that $S(a,a+1) = (\frac{N-2-2a}{2})^2$ is never achieved and that for $N \geqslant 3$, $0 \leqslant a < \frac{N-2}{2}$, $a \leqslant b < a+1$, $S(a,b)$ is achieved only by radial functions (in the case of $a = b = 0$, up to a translation in $\mathbb{R}^N$), which are given by

$$CU_\lambda(x) = C\lambda^{\frac{N-2}{2}} U(\lambda x), \tag{5}$$

where $C \in \mathbb{R}$, $\lambda > 0$ and

$$U(x) = k_0 (1 + |x|^\alpha)^{-\beta}, \quad \alpha = \frac{2(N-2-2a)(1+a-b)}{N-2+2(b-a)}, \quad \beta = \frac{N-2+2(b-a)}{2(1+a-b)} \tag{6}$$

with k_0 being chosen such that $||U||_a^2 = S(a,b)$ (see [9]).

To motivate our discussion, let us start with the Hardy inequality for the special case $a = 0$, $b = 1$. In this case (1) gives for $N \geqslant 3$, $u \in D^{1,2}(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \geqslant \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} \, dx.$$

This inequality still holds for $u \in H_0^1(\Omega)$ for any bounded domain Ω. Using a very delicate argument, Brezis and Vazquez first discovered the following improved version of the inequality in bounded domains.

Theorem A (Brezis and Vazquez [5]). *Let $N \geqslant 3$, $\Omega \subset \mathbb{R}^N$ bounded. Then there exists $C = C(\Omega) > 0$ such that for all $u \in H_0^1(\Omega)$,*

$$||\nabla u||_2^2 - \left(\frac{N-2}{2}\right)^2 \left|\left| |x|^{-1} u \right|\right|_2^2 \geqslant C ||u||_2^2. \tag{7}$$

From this result, they deduced that for any $2 \leqslant q < \frac{2N}{N-2}$,

$$||\nabla u||_2^2 - \left(\frac{N-2}{2}\right)^2 \left|\left| |x|^{-1} u \right|\right|_2^2 \geqslant C ||u||_q^2 \tag{8}$$

for some $C = C(q, \Omega) > 0$, and that q cannot be replaced by $\frac{2N}{N-2}$. Very recently, Vazquez and Zuazua obtained an improved version of this result.

Theorem B (Vazquez and Zuazua [14]). *Let $N \geqslant 3$, and $1 \leqslant q < 2$. Assume Ω is bounded. Then there exists $C = C(q, \Omega) > 0$ such that, for all $u \in H_0^1(\Omega)$,*

$$||\nabla u||_2^2 - \left(\frac{N-2}{2}\right)^2 \left|\left||x|^{-1}u\right|\right|_2^2 \geqslant C||\nabla u||_q^2. \tag{9}$$

Here q cannot be replaced by 2.

Motivated by and related to the above results, our first result here improves the above Theorems A and B, and covers the weighted version as well. To avoid confusion of notations, we define

$$D_a^{1,2}(\Omega) = \overline{C_0^\infty(\Omega)}^{||.||}, \tag{10}$$

where $||.||$ is given in (3). Here Ω is a domain in $\mathbb{R}^N$ (not necessarily bounded). Note that when Ω is bounded, $D_0^{1,2}(\Omega) = H_0^1(\Omega)$. Whenever without confusion, we shall use $||.||$ for the norm in (3) with a relevant $a < \frac{N-2}{2}$ in place and a domain $\Omega \subset \mathbb{R}^N$ in the context. The symbol $||.||_p$ will be used to denote $L^p(\Omega)$ norm when Ω is clear in the context.

Theorem 1. *Let $N \geqslant 1$, $a < \frac{N-2}{2}$. Assume $\Omega \subset\subset B_R(0)$ for some $R > 0$. Then there exists $C = C(a, \Omega) > 0$ such that for all $u \in D_a^{1,2}(\Omega)$,*

$$\begin{aligned} &|||x|^{-a}\nabla u||_2^2 - \left(\frac{N-2-2a}{2}\right)^2 \left|\left||x|^{-(a+1)}u\right|\right|_2^2 \\ &\geqslant C\left|\left|\left(\ln\frac{R}{|x|}\right)^{-1}|x|^{-a}\nabla u\right|\right|_2^2. \end{aligned} \tag{11}$$

Moreover, when $0 \in \Omega$ the inequality is sharp in the sense that $\left(\ln\frac{R}{|x|}\right)^{-1}$ cannot be replaced by $g(x)\ln\left(\frac{R}{|x|}\right)^{-1}$ with g satisfying $|g(x)| \to \infty$ as $|x| \to 0$.

In the case $a = 0$, by using Hölder inequality, we see (11) implies Theorem B. Our approach is quite different from that in [5,14], in some sense simpler and easier to be adapted for the weighted versions. Following the idea used in [8], we convert the problem from $\mathbb{R}^N$ to one defined on a cylinder $\mathscr{C} = \mathbb{R} \times S^{N-1}$. From there an inequality similar to the classical one-dimensional Hardy inequality on $(0, \infty)$ is used to tackle the technical part of the proof. We also note that while the sharpness of Theorems A and B is open-ended (for $q < \frac{2N}{N-2}$ and $q < 2$, respectively), the sharpness in Theorem 1 is close-ended in the sense $\left(\ln\frac{R}{|x|}\right)^{-1}$ cannot be replaced by $\left(\ln\frac{R}{|x|}\right)^{-d}$ for $d < 1$.

We take $\Omega \subset\subset B_R(0)$ just to avoid the singularity of $\left(\ln \frac{R}{|x|}\right)^{-1}$ at $|x| = R$. Here we are interested in the singularity at zero. In fact, if we take $\delta > 0$ such that $B_\delta(0) \subset \Omega$, then it holds for all $u \in D_a^{1,2}(\Omega)$,

$$\||x|^{-a}\nabla u\|^2_{L^2(\Omega)} - \left(\frac{N-2-2a}{2}\right)^2 \left\||x|^{-(a+1)}u\right\|^2_{L^2(\Omega)} \geqslant C\||x|^{-a}\nabla u\|^2_{L^2(\Omega \setminus B_\delta(0))},$$

for some $C > 0$ (see the second remark in Section 2).

Using similar ideas, we give another result of the same spirit, which works for bounded domains as well as exterior domains. It also takes into account the singularity of $\ln \frac{R}{|x|}$ at $|x| = R$.

Theorem 2. *Let $N \geqslant 1$, $a \leqslant \frac{N-2}{2}$. Assume $\Omega \subset B_R(0)$ or $\Omega \subset B_R^C(0) = \mathbb{R}^N \setminus B_R(0)$. Then for all $u \in D_a^{1,2}(\Omega)$,*

$$\||x|^{-a}\nabla u\|^2_2 - \left(\frac{N-2-2a}{2}\right)^2 \left\||x|^{-(a+1)}u\right\|^2_2 \geqslant \frac{1}{4}\left\|\left(\ln \frac{R}{|x|}\right)^{-1}|x|^{-(a+1)}u\right\|^2_2. \tag{12}$$

This inequality is sharp in the sense that $\left(\ln \frac{R}{|x|}\right)^{-1}$ cannot be replaced by $g(x)\left(\ln \frac{R}{|x|}\right)^{-1}$ with $|g(x)| \to \infty$ as $|x| \to 0$ when $0 \in \Omega$ (by $g(x)(\ln \frac{R}{|x|})^{-1}$ with $|g(x)| \to \infty$ as $|x| \to \infty$ when $B_\rho^C(0) \subset \Omega$). The best constant $\frac{1}{4}$ is then also sharp.

For $a = 0$, this was proved recently in [1] (see also [6]) under condition $\Omega \subset B_{e^{-1}R}(0)$ and no estimate on the best constant is given there except for $a = 0$, $N = 2$.

Next, we turn to Hardy–Sobolev type inequalities which correspond to $a \leqslant b < a + 1$ in CKN inequality (1). Recall the norm on $L_w^q(\Omega)$ is defined by

$$\|u\|_{q,w} = \sup_S \frac{\int_S |u|\, dx}{|S|^{\frac{1}{q'}}},$$

where q' is the conjugate exponent of q, i.e. $\frac{1}{q} + \frac{1}{q'} = 1$ and $S \subset \Omega$ has a finite measure.

Theorem 3. *Let $N \geqslant 3$, $0 \leqslant a < \frac{N-2}{2}$, $a \leqslant b < a + 1$, $p = \frac{2N}{N-2+2(b-a)}$. Assume $\Omega \subset \mathbb{R}^N$ is bounded. Then there exists $C = C(a, b, \Omega)$ such that for all $u \in D_a^{1,2}(\Omega)$,*

$$|||x|^{-a}\nabla u||_2^2 - S(a,b)\left|\left||x|^{-b}u\right|\right|_p^2 \geqslant C|||x|^{-a}u||^2_{\frac{N}{N-2-a},w} \tag{13}$$

and

$$|||x|^{-a}\nabla u||_2^2 - S(a,b)\left|\left||x|^{-b}u\right|\right|_p^2 \geqslant C|||x|^{-a}\nabla u||^2_{\frac{N}{N-1-a},w}. \tag{14}$$

Moreover, the weak norm on the right-hand side cannot be replaced by the strong norm.

For $a = b = 0$, (13) was proved by Brezis and Lieb [3] (see also [4], and also by Bianchi and Egnell with a different proof [2]). For $a = 0$, $0 < b < 1$, (13) was proved by Radulescu et al. [12]. For $a = b = 0$, (14) was proved in [3].

Our approach to prove Theorem 3, though follows the idea in [12,13], but improves theirs. Without using Schwarz symmetrization, our approach is easily adapted for the weighted versions. Moreover, our method can be used to establish results like (13) in unbounded domains. This partially addresses a question raised by Brezis and Lieb [3].

In order to state our results for unbounded domains, let us define for a domain $\Omega \subset \mathbb{R}^N$,

$$\lambda_1(\Omega) = \inf_{D_0^{1,2}(\Omega)} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2}.$$

We say Ω satisfies (Ω_0) condition if there exists an open cone with its vertex at 0, V_0, such that for some $R > 0$, $\Omega^C \supset (V_0 \backslash B_R(0))$. We say Ω satisfies (Ω_1) condition if there exists an open cone at 0, V_0, such that for some $R > 0$, for all $y \in \Omega$, $\Omega^C \supset (y + V_0) \backslash B_R(y)$.

Theorem 4. *Let $N = 3, 4$, $\Omega \subset \mathbb{R}^N$ satisfy (Ω_1) and $\lambda_1(\Omega) > 0$. Then there exists $C = C(\Omega) > 0$ such that for all $u \in D_0^{1,2}(\Omega)$,*

$$||\nabla u||_2^2 - S(0,0)||u||_{2^*}^2 \geqslant C||u||^2_{\frac{N}{N-2},w}$$

and

$$||\nabla u||_2^2 - S(0,0)||u||_{2^*}^2 \geqslant C||\nabla u||^2_{\frac{N}{N-1},w}.$$

Theorem 5. *Let $N \geqslant 3$, $\max\{0, \frac{N-4}{2}\} \leqslant a < \frac{N-2}{2}$, $a \leqslant b < a + 1$, $a + b \neq 0$, $p = \frac{2N}{N-2+2(b-a)}$. Assume $\Omega \subset \mathbb{R}^N$ satisfies $\lambda_1(\Omega) > 0$ and condition (Ω_0). Then there exists $C =$*

$C(a,b,\Omega)$ such that for all $u \in D_a^{1,2}(\Omega)$,

$$|||x|^{-a}\nabla u||_2^2 - S(a,b)\left|\left||x|^{-b}u\right|\right|_p^2 \geqslant C|||x|^{-a}u||^2_{\frac{N}{N-2-a},w}$$

and

$$|||x|^{-a}\nabla u||_2^2 - S(a,b)\left|\left||x|^{-b}u\right|\right|_p^2 \geqslant C|||x|^{-a}\nabla u||^2_{\frac{N}{N-1-a},w}.$$

Typical domains that satisfy $\lambda_1(\Omega) > 0$ and (Ω_0) or (Ω_1) are strips or sub-domains of strips. Here by strip we mean domains that are bounded in at least one direction. We shall discuss more on this in Section 4.

Due to the translation invariance in Theorem 4, we need the stronger condition (Ω_1).

2. Hardy inequalities with remainder terms

We prove Theorems 1 and 2 in this section. The idea is to use a conformal transformation to convert the problem to an equivalent one defined in a domain on a cyliner $\mathscr{C} = \mathbb{R} \times S^{N-1}$. This idea has been used in [8] to study the symmetry property of extremal functions for the Caffarelli–Kohn–Nirenberg inequalities (1). More precisely, to a function $u \in C_0^\infty(\Omega \backslash \{0\})$ we associate $v \in C_0^\infty(\tilde{\Omega})$ by the transformation

$$u(x) = |x|^{-\frac{N-2-2a}{2}} v\left(-\ln|x|, \frac{x}{|x|}\right), \tag{15}$$

where $\tilde{\Omega}$ is a domain on $\mathscr{C}$ defined by

$$(t,\theta) = \left(-\ln|x|, \frac{x}{|x|}\right) \in \tilde{\Omega} \iff x \in \Omega. \tag{16}$$

In [8], it was proved that when $\Omega = \mathbb{R}^N$, the above transformation defines a Hilbert space isomorphism between $D_a^{1,2}(\mathbb{R}^N)$ and $H^1(\mathscr{C})$ whose norm is given by $||v||^2_{H^1(\mathscr{C})} = \int_{\mathscr{C}} (|\nabla v|^2 + (\frac{N-2-2a}{2})v^2)\, d\mu$.

Using the same computation, we have

Lemma 1. *Let $N \geqslant 1$, $a < \frac{N-2}{2}$, $\Omega \subset \mathbb{R}^N$ a domain. Then under the transformation* (15)

$$\int_\Omega |x|^{-2a}|\nabla u|^2\, dx = \int_{\tilde{\Omega}} \left[|\nabla v|^2 + \left(\frac{N-2-2a}{2}\right)^2 v^2\right] d\mu$$

and

$$\int_\Omega |x|^{-2(a+1)} u^2\,dx = \int_{\tilde\Omega} |v|^2\,d\mu.$$

Let $\mathscr{C}_+$ ($\mathscr{C}_-$, resp.) denote the domain on $\mathscr{C}$ with t component positive (negative, resp.).

Lemma 2. *Let $N \geqslant 1$ and $\tilde\Omega \subset \mathscr{C}_+$ or $\tilde\Omega \subset \mathscr{C}_-$ be a domain. Then for all $v \in C_0^\infty(\tilde\Omega)$,*

$$\int_{\tilde\Omega} |\nabla v|^2\,d\mu \geqslant \frac{1}{4}\int_{\tilde\Omega} \frac{v^2}{t^2}\,d\mu. \tag{17}$$

Moreover $\frac{1}{4}$ is the best constant if $[L,\infty) \times S^{N-1} \subset \tilde\Omega$ or $(-\infty,-L] \times S^{N-1} \subset \tilde\Omega$ for $L>0$.

Proof. This is a version of the classical Hardy inequality adapted for the cylinder case. For $v \in C_0^\infty(\tilde\Omega)$,

$$\int_0^\infty \frac{v^2(t,\theta)}{t^2}\,dt = -2\int_0^\infty \frac{v v_t}{t}\,dt \leqslant 2\left(\int_0^\infty \frac{v^2}{t^2}\,dt\right)^{\frac{1}{2}}\left(\int_0^\infty v_t^2\,dt\right)^{\frac{1}{2}}.$$

Thus

$$\int_0^\infty \frac{v^2(t,\theta)}{t^2}\,dt \leqslant 4\int_0^\infty v_t^2(t,\theta)\,dt.$$

Integrating on S^{N-1} gives the result. Since $\frac{1}{4}$ is the best constant for the classical one-dimensional Hardy inequality (see [10]), the optimality is proved by considering functions depending only on t. □

Lemma 2 implies that if $\tilde\Omega \subset \mathscr{C}_+$ or $\tilde\Omega \subset \mathscr{C}_-$, the completion of $C_0^\infty(\tilde\Omega)$ under the norm $\sqrt{\int_{\tilde\Omega} |\nabla v|^2\,d\mu}$ is well defined, even for $N=1$, and 2. We denote this space by $D_0^{1,2}(\tilde\Omega)$.

Proof of Theorem 1. A simple scaling argument shows it suffices to take $R=1$. Let $\gamma_0 = \max_{x\in\Omega}|x|$. Then $\gamma_0 < 1$. By Lemma 1, under transformation (15), it suffices to show that there exists $C>0$ such that for all $v \in D_0^{1,2}(\tilde\Omega)$,

$$\int_{\tilde\Omega} |\nabla v|^2\,d\mu \geqslant C\int_{\tilde\Omega} \frac{1}{t^2}\left[|\nabla_\theta v|^2 + \left(v_t + \frac{N-2-2a}{2}v\right)^2\right]d\mu.$$

But by Lemma 2

$$\int_{\tilde{\Omega}} \frac{1}{t^2}\left[|\nabla_\theta v|^2 + \left(v_t + \frac{N-2-2a}{2} v\right)^2\right] d\mu$$
$$\leqslant 2\frac{1}{(\ln \gamma_0)^2}\int_{\tilde{\Omega}} |\nabla v|^2\, d\mu + 2\left(\frac{N-2-2a}{2}\right)^2 4\int_{\tilde{\Omega}} v_t^2\, d\mu$$
$$\leqslant \left(\frac{2}{(\ln \gamma_0)^2} + 2(N-2-2a)^2\right)\int_{\tilde{\Omega}} |\nabla v|^2\, d\mu.$$

To show the sharpness part of the theorem, assume $g(x)$ satisfies $|g(x)| \to +\infty$ as $|x| \to 0$. We may assume

$$\lim_{|x|\to 0} \frac{|g(x)|}{|\ln |x||} = 0.$$

Now it suffices to construct $v_n \in D_0^{1,2}(\tilde{\Omega})$ such that

$$\frac{\int_{\tilde{\Omega}} |\nabla v_n|^2\, d\mu}{\int_{\tilde{\Omega}} \frac{|g(e^{-t})|^2}{t^2}\left(|\nabla_\theta v_n|^2 + \left(\frac{\partial v_n}{\partial t} + \frac{N-2-2a}{2} v_n\right)^2\right) d\mu} \to 0, \quad \text{as } n \to \infty.$$

Let $R_n \to \infty$, and η be a function defined on $[0, \infty)$ such that $\eta(t) = 1,\ 0 \leqslant t \leqslant 1$, $\eta(t) = 0,\ t \geqslant 2,\ |\eta'(t)| \leqslant 2$. Define

$$v_n(t,\theta) = \eta\left(\frac{|t-R_n|}{R_n}\right).$$

Then for n large, $v_n \in D_0^{1,2}(\tilde{\Omega})$ since $\tilde{\Omega}$ contains $[L, \infty) \times S^{N-1}$ for some L large. Then,

$$A_n := \int_{\tilde{\Omega}} |\nabla v_n|^2\, d\mu \leqslant C\int_{2R_n}^{6R_n} \frac{1}{R_n^2}(\eta')^2\, dt \leqslant \frac{C}{R_n}$$
$$B_n := \int_{\tilde{\Omega}} \frac{|g(e^{-t})|^2}{t^2}\left[|\nabla_\theta v_n|^2 + \left(\frac{\partial v_n}{\partial t} + \frac{N-2-2a}{2} v_n\right)^2\right] d\mu$$
$$\geqslant C\int_{3R_n}^{5R_n} \frac{|g(e^{-t})|^2}{t^2}\left(\left(\frac{\partial v_n}{\partial t}\right)^2 + \left(\frac{N-2-2a}{2}\right)^2 v_n^2 + (N-2-2a) v_n \frac{\partial v_n}{\partial t}\right) dt.$$

Then

$$\int_{3R_n}^{6R_n} \frac{|g(e^{-t})|^2}{t^2}\left(\frac{\partial v_n}{\partial t}\right)^2 dt = o\left(\frac{1}{R_n}\right), \quad \text{as } n \to \infty,$$

and choosing $0<\beta<1$,

$$\left|\int_{3R_n}^{5R_n} \frac{|g(e^{-t})|^2}{t^2} v_n \frac{\partial v_n}{\partial t}\, dt\right|$$
$$\leqslant \int_{3R_n}^{5R_n} \frac{|g(e^{-t})|^{2\beta}}{t^{2\beta}} \left(\frac{\partial v_n}{\partial t}\right)^2 dt + \int_{3R_n}^{5R_n} \frac{|g(e^{-t})|^{2(2-\beta)}}{t^{2(2-\beta)}} v_n^2\, dt$$
$$= o\left(\frac{1}{R_n}\right) + o(1) \int_{3R_n}^{5R_n} \frac{|g(e^{-t})|^2}{t^2} v_n^2\, dt.$$

Then

$$B_n \geqslant C \int_{3R_n}^{5R_n} \frac{|g(e^{-t})|^2}{t^2} v_n^2\, dt - o\left(\frac{1}{R_n}\right)$$
$$\geqslant C\left(\inf_{t\geqslant 3R_n} |g(e^{-t})|^2\right) \cdot \frac{1}{R_n} - o\left(\frac{1}{R_n}\right).$$

Therefore,

$$\frac{A_n}{B_n} \leqslant \frac{CR_n^{-1}}{C(\inf_{t\geqslant 3R_n} |g(e^{-t})|^2)R_n^{-1} + o(1)R_n^{-1}} \to 0, \quad n\to\infty.$$

The proof of Theorem 1 is complete. □

Remark. From the proof, $C = C(a, \Omega)$ can be taken as

$$\left(\frac{2}{(\ln \gamma_0)^2} + 2(N-2-2a)^2\right)^{-1}.$$

Remark. If we take $\Omega\backslash B_\delta(0)$ on the right-hand side for some $\delta>0$, $\widetilde{\Omega\backslash B_\delta(0)}$ is a bounded domain in $\mathscr{C}_+$ so the t-component has positive upper and lower bounds. Thus we get for some $C = C(a, \Omega, \delta)>0$,

$$|||x|^{-a}\nabla u||^2_{L^2(\Omega)} - \left(\frac{N-2-2a}{2}\right)^2 \left|\left||x|^{-(a+1)}u\right|\right|^2_{L^2(\Omega)} \geqslant C|||x|^{-a}\nabla u||^2_{L^2(\Omega\backslash B_\delta(0))}.$$

Proof of Theorem 2. Again we may assume $R = 1$. Let us assume $a<\frac{N-2}{2}$ first. It suffices then to use Lemmas 1 and 2.

Since the constant on the right-hand side is $\frac{1}{4}$, which is independent of $a<\frac{N-2}{2}$, we may send $a\to\frac{N-2}{2}$ in the inequality. This can be done first for smooth functions, i.e. for all $u\in C_0^\infty(\Omega)$, with $a = \frac{N-2}{2}$, (12) is satisfied. This implies $D_a^{1,2}(\Omega)$ with $a = \frac{N-2}{2}$ is well defined and $|||x|^{-a}\nabla u||_2$ can be taken as its norm. Now a density argument finishes the proof for the case $a = \frac{N-2}{2}$.

For the sharpness of the weight, we use the same test functions v_n as in the proof of Theorem 1. Then it is easy to see

$$\frac{\int_{\tilde{\Omega}} |\nabla v_n|^2 \, d\mu}{\int_{\tilde{\Omega}} \frac{|g(e^{-t})|^2}{t^2} v_n^2 \, d\mu} \leqslant \frac{C}{\inf_{t \geqslant R_n} |g(e^{-t})|^2} \to 0, \quad \text{as } n \to \infty.$$

Finally, the constant $\frac{1}{4}$ on the right-hand side is the best constant by Lemma 2. □

3. Hardy–Sobolev inequalities with remainder terms

In this section we consider the weighted Hardy–Sobolev inequality (1) on $D_a^{1,2}(\mathbb{R}^N)$,

$$|||x|^{-a}\nabla u||_2^2 - S(a,b)\left|\left||x|^{-b}u\right|\right|_p^2 \geqslant 0$$

for the parameter range: $N \geqslant 3,\ 0 < a < \frac{N-2}{2},\ a \leqslant b < a+1,\ p = \frac{2N}{N-2+2(b-a)} \in (2, 2^*]$, where $2^* = \frac{2N}{N-2}$. Recall from introduction that the best constant $S(a,b)$ is achieved by the functions given in (5) and (6). Thus the minimizers for $S(a,b)$ consist of a two-dimensional manifold $\mathcal{M} \subset D_a^{1,2}(\mathbb{R}^N)$. Let us define

$$d(u, \mathcal{M}) = \inf\{|||x|^{-a}\nabla(u - CU_\lambda)||_2 \colon C \in \mathbb{R}, \lambda > 0\}.$$

We need the following result first which generalizes the results in [2,3] for the case $a = 0$ to the case $a > 0$.

Theorem 6. *For $N \geqslant 3,\ 0 < a < \frac{N-2}{2},\ a \leqslant b < a+1,\ p = \frac{2N}{N-2+2(b-a)}$, there exists $C = C(N,a,b)$ such that for all $u \in D_a^{1,2}(\mathbb{R}^N)$,*

$$|||x|^{-a}\nabla u||_2^2 - S(a,b)\left|\left||x|^{-b}u\right|\right|_p^2 \geqslant C d(u, \mathcal{M})^2. \tag{18}$$

We first consider the eigenvalue problem

$$\begin{cases} -\mathrm{div}(|x|^{-2a}\nabla u) = \lambda |x|^{-bp} U^{p-2} u \\ u \in D_a^{1,2}(\mathbb{R}^N). \end{cases} \tag{19}$$

Lemma 3. *Let $a > 0,\ a \leqslant b < a+1$. The first two eigenvalues of* (19) *are given by $\lambda_1 = S(a,b)$ and $\lambda_2 = (p-1)S(a,b)$. The eigenspaces are spanned by U and $\frac{d}{d\lambda}|_{\lambda=1} U_\lambda$, respectively.*

Proof. It is easy to check that U and $\frac{d}{d\lambda}|_{\lambda=1}U_\lambda$ are eigenfunctions corresponding to $S(a,b)$ and $(p-1)S(a,b)$, respectively. Then it suffices to show that any eigenfunction corresponding to an eigenvalue $\lambda \leqslant (p-1)S(a,b)$ has to be radial. Let Ψ_i, $i=0,1,\dots$ the sequence of spherical harmonics, which are eigenfunctions of the Laplace–Beltrami operator on $S^{N-1}: -\Delta_{S^{N-1}}\Psi_i = \sigma_i\Psi_i$, $\sigma_0 = 0$, $\sigma_1 = \cdots = \sigma_N = N-1$, $\sigma_{N+1} > \sigma_N$. Let u be an eigenfunction corresponding to an eigenvalue $\lambda \leqslant (p-1)S(a,b)$. We shall show for all $i \geqslant 1$,

$$\int_{S^{N-1}} u(r,\theta)\Psi_i(\theta)\, d\theta \equiv 0.$$

Let $\varphi_i = \int_{S^{N-1}} u(r,\theta)\Psi_i(\theta)\, d\theta$. Then we can check

$$\begin{aligned}
\operatorname{div}(|x|^{-2a}\nabla\varphi_i) &= -2a|x|^{-2a-1}\frac{\partial}{\partial r}\varphi_i + |x|^{-2a}\Delta_r\varphi_i \\
&= \int_{S^{N-1}} \left[|x|^{-2a}\Delta_r u(r,\theta) - 2a|x|^{-2a-1}\frac{\partial u}{\partial r}(r,\theta)\right]\Psi_i(\theta)\, d\theta \\
&= \int_{S^{N-1}} \left[\operatorname{div}(|x|^{-2a}\nabla u) - \frac{|x|^{-2a}\Delta_\theta u}{r^2}\right]\Psi_i(\theta)\, d\theta \\
&= \int_{S^{N-1}} -\lambda|x|^{-bp}U^{p-2}u\Psi_i(\theta)\, d\theta + \frac{r^{-2a}\sigma_i}{r^2}\int_{S^{N-1}} u\Psi_i(\theta)\, d\theta \\
&= (r^{-2a-2}\sigma_i - \lambda r^{-bp}U^{p-2})\varphi_i.
\end{aligned}$$

Then for any $R > 0$,

$$0 = \int_{B_R(0)} \left[\operatorname{div}(|x|^{-2a}\nabla\varphi_i)\frac{\partial U}{\partial r} + (\lambda r^{-bp}U^{p-2} - r^{-2a-2}\sigma_i)\varphi_i\frac{\partial U}{\partial r}\right] dx.$$

The first term can be calculated as follows:

$$\begin{aligned}
&\int_{B_R(0)} \operatorname{div}(|x|^{-2a}\nabla\varphi_i)U_r\, dx \\
&\quad = \int_{B_R(0)} \varphi_i \operatorname{div}(|x|^{-2a}\nabla(U_r))\, dx - \int_{\partial B_R(0)} |x|^{-2a}\varphi_i\left\langle\nabla(U_r), \frac{x}{R}\right\rangle d\mu \\
&\qquad + \int_{\partial B_R(0)} U_r\left\langle |x|^{-2a}\nabla\varphi_i, \frac{x}{R}\right\rangle d\mu \\
&\quad = \int_{B_R(0)} \varphi_i \operatorname{div}(|x|^{-2a}\nabla(U_r))\, dx + \int_{\partial B_R(0)} R^{-2a}\left(U_r\frac{d\varphi_i}{dr} - U_{rr}\varphi_i\right) d\mu.
\end{aligned}$$

And using equation $-\mathrm{div}(|x|^{-2a}\nabla U) = S(a,b)|x|^{-bp}U^{p-1}$, we have

$$\begin{aligned}
&\int_{B_R(0)} \varphi_i \,\mathrm{div}(|x|^{-2a}\nabla(U_r))\,dx \\
&= \int_{B_R(0)} \varphi_i \,\mathrm{div}\Big(|x|^{-2a}U_{rr}\frac{x}{r}\Big)\,dx \\
&= \int_{B_R(0)} \varphi_i \Big[Nr^{-2a-1}U_{rr} + |x|^{-2a}U_{rrr} - (2a+1)r^{-2a-1}U_{rr}\Big]\,dx \\
&= \int_{B_R(0)} \varphi_i \Big[(N-2a-1)r^{-2a-1}U_{rr} + r^{-2a}\frac{d}{dr}\Big(\frac{2aU_r}{r} - \frac{N-1}{r}U_r \\
&\quad - S(a,b)r^{-bp+2a}U^{p-1}\Big)\Big]\,dx \\
&= \int_{B_R(0)} \varphi_i \Big[(N-2a-1)r^{-2a-1}U_{rr} + r^{-2a}\Big(2a\frac{rU_{rr}-U_r}{r^2} - \frac{(N-1)(rU_{rr}-U_r)}{r^2} \\
&\quad + (bp-2a)S(a,b)r^{-bp+2a-1}U^{p-1} - r^{-bp+2a}(p-1)S(a,b)U^{p-2}U_r\Big)\Big]\,dx \\
&= \int_{B_R(0)} \varphi_i r^{-2a}\frac{N-1-2a}{r^2}U_r + \int_{B_R(0)} (bp-2a)S(a,b)r^{-bp-1}U^{p-1}\varphi_i \\
&\quad - (p-1)S(a,b)\int_{B_R(0)} r^{-bp}U^{p-2}U_r\varphi_i.
\end{aligned}$$

Putting all these together, we get

$$\begin{aligned}
0 &= \int_{\partial B_R(0)} R^{-2a}\Big(U_r\frac{d\varphi_i}{dr} - U_{rr}\varphi_i\Big)\,d\mu + \int_{B_R(0)} \varphi_i r^{-2a-2}(N-1-\sigma_i-2a)U_r\,dx \\
&\quad + \int_{B_R(0)} (bp-2a)S(a,b)r^{-bp-1}U^{p-1}\varphi_i\,dx \\
&\quad + (\lambda-(p-1)S(a,b))\int_{B_R(0)} r^{-bp}U^{p-2}U_r\varphi_i\,dx.
\end{aligned}$$

Let R be the first zero of φ_i with $R=+\infty$ if φ_i is not zero anywhere. Without loss of generality assume $\varphi_i(r)>0$, $r\in(0,R)$. Then $\frac{d\varphi_i}{dr}(R)\leqslant 0$. Thus the first and the forth terms are non-negative and the second and the third are positive unless $\varphi_i\equiv 0$ since $bp-2a>0$ for $a>0$. The proof is finished. □

Lemma 4. *For any sequence* $(u_n)\subset D_a^{1,2}(\mathbb{R}^N)\backslash\mathscr{M}$ *such that* $\inf_n \||x|^{-a}\nabla u_n\|_2^2>0$ *and* $d(u_n,\mathscr{M})\to 0$, *it holds*

$$\varliminf_{n\to\infty} \frac{\||x|^{-a}\nabla u_n\|_2^2 - S(a,b)\left\||x|^{-b}u_n\right\|_p^2}{d(u_n,\mathscr{M})} \geqslant 1-\frac{\lambda_2}{\lambda_3}.$$

Proof. First we assume $d(u_n, \mathscr{M}) = |||x|^{-a}\nabla(u_n - U)||_2$. Since $v_n = u_n - U$ is orthogonal to the tangent space of $\mathscr{M}$,

$$T_U\mathscr{M} = \operatorname{span}\left\{U, \frac{d}{d\lambda}\bigg|_{\lambda=1} U_\lambda\right\},$$

we have by Lemma 3,

$$\lambda_3 \int |x|^{-bp} U^{p-2} v_n^2 \, dx \leqslant |||x|^{-a}\nabla v_n||_2^2 = d^2(u_n, \mathscr{M}).$$

Let $d_n = d(u_n, \mathscr{M})$. Using the equation $-\operatorname{div}(|x|^{-2a}\nabla U) = S(a,b)|x|^{-bp}U^{p-1}$, we get

$$\begin{aligned}\int |x|^{-bp}|u_n|^p \, dx &= \int |x|^{-bp} U^p \, dx + p\int |x|^{-bp} U^{p-1} v_n \, dx \\ &\quad + \frac{p(p-1)}{2}\int |x|^{-bp} U^{p-2} v_n^2 \, dx + o(d_n^2) \\ &= 1 + \frac{p}{2}\frac{\lambda_2}{S(a,b)\lambda_3} d_n^2 + o(d_n^2).\end{aligned}$$

Then,

$$|||x|^{-b}u_n||_p^2 \leqslant 1 + \frac{\lambda_2}{\lambda_3}\frac{d_n^2}{S(a,b)} + o(d_n^2).$$

By $|||x|^{-a}\nabla u_n||_2^2 = S(a,b) + d_n^2$, we have

$$|||x|^{-a}\nabla u_n||_2^2 - S(a,b)\left|\left||x|^{-b}u_n\right|\right|_p^2 \geqslant \left(1 - \frac{\lambda_2}{\lambda_3}\right) d_n^2 + o(d_n^2).$$

For the general case, $d(u_n, \mathscr{M}) = |||x|^{-a}\nabla(u_n - C_n U_{\lambda_n})||_2$ for some $C_n \in \mathbb{R}$, $\lambda_n > 0$. We can use the invariance of the inequality by dilations to reduce it to the special case above. We omit it here. □

Proof of Theorem 6. If the theorem is false, we find $(u_n) \subset D_a^{1,2}(\mathbb{R}^N) \setminus \mathscr{M}$ such that

$$\frac{|||x|^{-a}\nabla u_n||_2^2 - S(a,b)\left|\left||x|^{-b}u_n\right|\right|_p^2}{d(u_n, \mathscr{M})^2} \to 0.$$

We may assume $|||x|^{-a}\nabla u_n||_2^2 = 1$ and thus $L = \lim_{n\to\infty} d(u_n, \mathscr{M}) \in [0,1]$. Then

$$\left|\left||x|^{-b}u_n\right|\right|_p^2 \to S(a,b)^{-1}.$$

By a concentration-compactness argument [11,15] we can find $\lambda_n > 0$,

$$\lambda_n^{\frac{N-2-2a}{2}} u_n(\lambda_n x) \to V \in \mathscr{M} \text{ in } D_a^{1,2}(\mathbb{R}^N).$$

This implies $L = 0$, a contradiction to Lemma 4. □

Proof of Theorem 3. Assume that (13) is not true. Then there exist $(u_n) \subset H_0^1(\Omega)$ such that

$$\frac{|||x|^{-a}\nabla u_n||_2^2 - S(a,b)\left|\left||x|^{-b}u_n\right|\right|_p^2}{|||x|^{-a}u_n||^2_{\frac{N}{N-2-a},w}} \to 0.$$

We assume $|||x|^{-a}\nabla u_n||_2^2 = 1$ and $|||x|^{-a}u_n||^2_{\frac{N}{N-2-a},w}$ is bounded by Sobolev's inequality. Then $|||x|^{-a}u_n||_p^2 \to S(a,b)^{-1}$. By Theorem 6, there exist $(C_n, \lambda_n) \to (1, \infty)$ such that

$$d(u_n, \mathscr{M}) = |||x|^{-a}\nabla(u_n - C_n U_{\lambda_n})||_2 \to 0.$$

A direct computation shows

$$\begin{aligned} d(u_n, \mathscr{M})^2 &\geqslant C_n^2 \int_{|x|\geqslant 1} |x|^{-2a} |\nabla U_{\lambda_n}|^2 \, dx \\ &= C\lambda_n^{N-2-2a} \int_1^\infty r^{-2a}(1 + (\lambda_n r)^2)^{-2(\beta+1)} (\lambda_n r)^{2(\alpha-1)} \lambda_n^2 r^{N-1} \, dr \\ &= C \int_{\lambda_n}^\infty S^{-2a}(1 + S^\alpha)^{-2(\beta+1)} S^{2(\alpha-1)} S^{N-1} \, dS \\ &\geqslant C\lambda_n^{2a-(N-2)}, \end{aligned}$$

where $C > 0$ is a constant independent of n.

Therefore,

$$\begin{aligned} &|||x|^{-a}u_n||_{L_w^{\frac{N}{N-2-a}}(\Omega)} \\ &\quad \leqslant |||x|^{-a}(u_n - C_n U_{\lambda_n})||_{L_w^{\frac{N}{N-2-a}}(\Omega)} + |||x|^{-a} C_n U_{\lambda_n}||_{L_w^{\frac{N}{N-2-a}}(\Omega)} \end{aligned}$$

$$\leqslant C|||x|^{-a}(u_n - C_n U_{\lambda_n})||_{L^{\frac{2N}{N-2}}(\Omega)} + ||C_n|x|^{-a} U_{\lambda_n}||_{L_w^{\frac{N}{N-2-a}}(\mathbb{R}^N)}$$

$$\leqslant C|||x|^{-a}(u_n - C_n U_{\lambda_n})||_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)} + C_n\lambda_n \left|\left| |x|^{\frac{2a-(N-2)}{2-a}} U \right|\right|_{\frac{N}{N-2-2a},w}$$

$$\leqslant Cd(u_n, \mathscr{M}) + C_n \lambda_n^{\frac{2a-(N-2)}{2}} |||x|^{-a} U||_{\frac{N}{N-2-a},w}$$

$$\leqslant Cd(u_n, \mathscr{M}).$$

This is a contradiction with Theorem 6.

Since, by a direct computation

$$|||x|^{-a} C_n \nabla U_{\lambda_n}||_{\frac{N}{N-1-a},w} = C_n \lambda_n^{\frac{2a-(N-2)}{2}} |||x|^{-a} \nabla U||_{\frac{N}{N-1-a},w},$$

we obtain (14) by a similar argument. □

4. Hardy–Sobolev inequalities with remainder terms on unbounded domains

This section is devoted to proving Theorems 4 and 5. We need a few preliminary results.

When $a = b = 0$, the manifold of minimizers for $S(0,0)$ is a $N+2$ dimensional, given by

$$\mathscr{M}(0,0) = \{CU_\lambda(. + y) \mid C \in \mathbb{R}, \lambda > 0, y \in \mathbb{R}^N\}$$

U is given in (6) with $a = b = 0$.

Lemma 5. *Let $N \geqslant 3$, $a = b = 0$. Assume Ω satisfies condition (Ω_1). Then there exists $C = C(\Omega) > 0$, such that as $\lambda \to \infty$,*

$$\inf_{y \in \Omega} ||\nabla U_\lambda(x+y)||^2_{L^2(\Omega^C)} \geqslant C\lambda^{2-N}.$$

Proof. Just note that $|\nabla U_\lambda(x+y)|$ is radial in $|x+y|$ and there exists $C > 0$ such that as $\lambda \to \infty$,

$$||\nabla U_\lambda(x)||^2_{L^2(B_R^C(0))} \geqslant C\lambda^{2-N}. \qquad \square$$

Similarly, we have

Lemma 6. *Let $N \geqslant 3$, $0 \leqslant a < \frac{N-2}{2}$, $a \leqslant b < a+1$, $a+b \neq 0$. Assume Ω satisfies condition (Ω_0). Then there exists $C = C(\Omega) > 0$ such that for $U_\lambda \in \mathscr{M}(a,b)$ as $\lambda \to \infty$,*

$$|||x|^{-a} \nabla U_\lambda||^2_{L^2(\Omega^C)} \geqslant C\lambda^{2a+2-N}.$$

Lemma 7. *Let $N \geqslant 3$, $0 \leqslant a < \frac{N-2}{2}$, $a \leqslant b < a+1$. Let $\Omega \subset \mathbb{R}^N$ and $P : D_a^{1,2}(\mathbb{R}^N) \to D_a^{1,2}(\Omega)$ be the projection operator. Then for any $U \in \mathscr{M}(a,b)$, $0 \leqslant PU \leqslant U$ in $\mathbb{R}^N$.*

Proof. PU is given by $PU = U - v$ where v is the solution of

$$\begin{cases} -\operatorname{div}(|x|^{-2a} \nabla v) = 0 & \text{in } \Omega, \\ v = U & \text{on } \partial\Omega. \end{cases}$$

Then PU satisfies

$$\begin{cases} -\operatorname{div}(|x|^{-2a} \nabla (PU)) = S(a,b)|x|^{-bp} U^{p-1} & \text{in } \Omega, \\ PU = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $P(U) \geqslant 0$ in Ω for otherwise, assume $P(U) < 0$ in $\Omega_- \subset \Omega$. Multiplying the equation by PU and integrating on Ω_-, we get

$$\int_{\Omega_-} |x|^{-2a} |\nabla (PU)|^2 = S(a,b) \int_{\Omega_-} |x|^{-bp} U^{p-1} P(U) \leqslant 0,$$

which says $PU \equiv$ constant in Ω_-. Then $PU \equiv 0$ in Ω_- a contradiction.
Also v satisfies $v \geqslant 0$ in Ω. Then $PU \leqslant U$. $\square$

Lemma 8. *Let $\lambda_1(\Omega) > 0$. Then $\exists C > 0$, for all $u \in D_a^{1,2}(\Omega)$,*

$$|||x|^{-a} u||_{L^2(\Omega)} \leqslant C|||x|^{-a} \nabla u||_{L^2(\Omega)}.$$

Proof. Since $D_a^{1,2}(\Omega) = \overline{C_0^\infty(\Omega \backslash \{0\})}^{|| \, ||_a}$, we need only consider $u \in C_0^\infty(\Omega \backslash \{0\})$. Then $|x|^{-a} u \in C_0^\infty(\Omega \backslash \{0\})$. But for all $v \in C_0^\infty(\Omega \backslash \{0\})$,

$$\int_\Omega v^2 \leqslant \lambda_1 \int_\Omega |\nabla v|^2.$$

Therefore, using Hardy inequality,

$$\begin{aligned}\int_\Omega |x|^{-2a}u^2 &\leqslant \lambda_1 \int_\Omega |\nabla(|x|^{-a}u)|^2 \\ &= 2\lambda_1 \int_\Omega a^2|x|^{-2(a+1)}u^2 + |x|^{-2a}|\nabla u|^2 \\ &\leqslant C\int_\Omega |x|^{-2a}|\nabla u|^2. \qquad \square\end{aligned}$$

Proof of Theorem 4. Assume that Theorem 4 is not true. Then there exist $(u_n)\subset\mathscr{D}_0^{1,2}(\Omega)$ such that

$$\frac{||\nabla u_n||_2^2 - S(0,0)||u_n||_{2^*}^2}{||u_n||^2_{\frac{N}{N-2},w}} \to 0, \quad n\to\infty.$$

We assume $||\nabla u_n||_2 = 1$. If $N = 4$, we have, by assumption,

$$||u_n||_{\frac{N}{N-2},w} \leqslant ||u_n||_{\frac{N}{N-2}} \leqslant C||\nabla u_n||_2 = C.$$

If $N = 3$, by Hölder inequality and Sobolev inequality, we have

$$\begin{aligned}||u_n||_{\frac{N}{N-2},w} \leqslant ||u_n||_{\frac{N}{N-2}} &\leqslant ||u_n||_2^{\lambda}||u_n||_{2^*}^{1-\lambda} \\ &\leqslant C||\nabla u_n||_2 = C.\end{aligned}$$

Then $||u_n||_{2^*}^2 \to S^{-1}(0,0)$. By the proof of Lemma 1 in [2], there exists $(C_n,\lambda_n)\to(1,\infty)$ and $(y_n)\subset\Omega$ such that

$$d(u_n,\mathscr{M}) = ||\nabla(u_n - U_n)||_{L^2(\mathbb{R}^N)} \to 0, \quad n\to\infty,$$

where $U_n = C_n U(\lambda_n(.-y_n))$. By Lemma 5,

$$d(u_n,\mathscr{M})^2 \geqslant \int_{\Omega^C} |\nabla U_n|^2\,dx \geqslant C\ C_n^2\lambda^{2-N}.$$

Using $P: D_0^{1,2}(\mathbb{R}^N)\to D_0^{1,2}(\Omega)$ as the projection operator, we have

$$\begin{aligned}||u_n||_{\frac{N}{N-2},w} &\leqslant ||u_n - PU_n||_{\frac{N}{N-2}} + ||PU_n||_{\frac{N}{N-2},w} \\ &\leqslant C||\nabla(u_n - PU_n)||_{L^2(\Omega)} + ||PU_n||_{\frac{N}{N-2},w} \\ &\leqslant C||\nabla(u_n - U_n)||_{L^2(\mathbb{R}^N)} + ||PU_n||_{\frac{N}{N-2},w}.\end{aligned}$$

It follows from Lemma 7 that

$$||PU_n||_{\frac{N}{N-2},w} \leqslant ||U_n||_{L_w^{\frac{N}{N-2}}(\mathbb{R}^N)} \leqslant C_n \lambda_n^{\frac{2-N}{2}} ||U||_{L_w^{\frac{N}{N-2}}(\mathbb{R}^N)}.$$

Hence

$$||u_n||_{\frac{N}{N-2},w} \leqslant Cd(u_n, \mathscr{M}).$$

This is a contradiction with the Theorem in [2]. The proof of the second part of Theorem 4 is similar. $\square$

Proof of Theorem 5. Assume that Theorem 5 is not true. Then there exist $(u_n) \subset \mathscr{D}_a^{1,2}(\Omega)$ such that

$$\frac{|||x|^{-a}\nabla u_n||_2^2 - S(a,b)\left|\left||x|^{-b}u_n\right|\right|_p^2}{|||x|^{-a}u_n||^2_{\frac{N}{N-2-a},w}} \to 0, \quad n \to \infty.$$

We assume $|||x|^{-a}\nabla u_n||_2 = 1$. Using (1) and Lemma 8, we obtain

$$\begin{aligned}|||x|^{-a}u_n||_{\frac{N}{N-2-2a},w} &\leqslant |||x|^{-a}u_n||_{\frac{N}{N-2-2a}} \\ &\leqslant |||x|^{-a}u_n||_2^{\lambda} |||x|^{-a}u_n||_{2^*}^{1-\lambda} \\ &\leqslant C|||x|^{-a}\nabla u_n||_2^2 = C.\end{aligned}$$

Then $\left|\left||x|^{-b}u_n\right|\right|_p^2 \to S^{-1}(a,b)$. By Theorem 6, there exists $(C_n, \lambda_n) \to (1, \infty)$ such that

$$d(u_n, \mathscr{M}) = |||x|^{-a}\nabla(u_n - C_n U_{\lambda_n})||_{L^2(\mathbb{R}^N)} \to 0, \quad n \to \infty.$$

By Lemma 6,

$$d(u_n, \mathscr{M})^2 \geqslant C_n^2 \int_{\Omega^C} |x|^{-a}|\nabla U_{\lambda_n}|^2\, dx \geqslant C \ C_n^2 \lambda_n^{2a+2-N}.$$

As in the proof of the preceding theorem, we obtain a contradiction with Theorem 6. $\square$

Remark. It is easy to verify that unions of a finite number of strips satisfy conditions $\lambda_1(\Omega) > 0$ and (Ω_1).

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