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HARDY INEQUALITIES WITH BOUNDARY TERMS

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ABSTRACT. In this note, we present some Hardy type inequalities for functions which do not vanish on the boundary of a given domain. We establish these inequalities for both bounded and unbounded domains and also obtain the best embedding constants in these inequalities for special domains. Our results are motivated by and building upon some recent work in [5, 6, 9, 12].

1. INTRODUCTION

The standard Hardy inequality states that for $N \geq 3$,

$$\frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} dx \leq \int_{\mathbb{R}^N} |\nabla u|^2 dx \tag{1.1}$$

for any $u \in C_0^\infty(\mathbb{R}^N)$. Here $(N-2)^2/4$ is the best possible constant. This inequality can be extended to functions in the space $\mathcal{D}^{1,2}(\mathbb{R}^N)$ which is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|^2 = \int_{\mathbb{R}} |\nabla u|^2 dx.$$

There are many generalizations of this inequality, see for example, [2, 3, 4, 6, 7, 8, 10, 11] and references therein. The weighted version of this inequality was given in [4]. In this paper we will consider another type of generalizations (motivated by recent work of Li and Zhu [9] and Zhu [12]). Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $u \in \mathcal{D}_0^{1,2}(\Omega)$. Since we can trivially extend u to a new function in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ which vanishes outside Ω, we obtain the following Hardy inequality on a bounded domain:

$$\frac{(N-2)^2}{4} \int_{\Omega} \frac{u^2}{|x|^2} dx \leq \int_{\Omega} |\nabla u|^2 dx. \tag{1.2}$$

Naturally, one may ask whether there are some analogous inequalities that hold for function $u \in H^1(\Omega)$ (Notice that $u(x)$ may not vanish on the boundary of Ω). Since (1.2) does not hold for any constant function, we shall expect, like in the case of Sobolev inequality (see, for example, [9]), the right hand side may include some lower order terms.

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We shall consider a more general version of the Hardy inequality – the weighted version ([4]): for $a < \frac{N-2}{2}$, it holds for all $u \in C_0^\infty(\mathbb{R}^N)$

$$\frac{(N-2-2a)^2}{4}\int_{\mathbb{R}^N} |x|^{-2(a+1)}u^2dx \le \int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2dx.$$

Recently, in [5, 6] a new formulation of this inequality has been given by using a conformal transformation. Based on this conformal transformation we first establish weighted Hardy type inequalities with boundary terms in two specific domains. Denote $B_1(0) = \{x \in \mathbb{R}^N \ : \ |x| < 1\}$, and $B_1^c(0) = \mathbb{R}^N \setminus \overline{B}_1(0)$. We assume below $N \ge 2$ when we treat the weighted version of the Hardy inequality and $N \ge 3$ when we treat the classical Hardy inequality. Let us define the weighted Sobolev space $\mathcal{D}_a^{1,2}(\mathbb{R}^N)$ to be the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the following norm

$$\|u\|_a^2 = \int_{\mathbb{R}} |x|^{-2a}|\nabla u|^2dx.$$

In the following, for simplicity of notations we omit the integration variables when the situation is clear.

Theorem 1.1. *Let $a < \frac{N-2}{2}$. Then for all $u \in \mathcal{D}_a^{1,2}(\mathbb{R}^N)$*

$$\frac{(N-2-2a)^2}{4}\int_{B_1(0)} \frac{u^2}{|x|^{2(a+1)}} < \int_{B_1(0)} |x|^{-2a}|\nabla u|^2 + \frac{N-2-2a}{2}\int_{\partial B_1(0)} u^2, \tag{1.3}$$

and

$$\frac{(N-2-2a)^2}{4}\int_{B_1^c(0)} \frac{u^2}{|x|^{2(a+1)}} < \int_{B_1^c(0)} |x|^{-2a}|\nabla u|^2 - \frac{N-2-2a}{2}\int_{\partial B_1(0)} u^2. \tag{1.4}$$

Remark 1.2. The strict inequalities are due to the non-existence of extremal functions in (1.3) and (1.4). The constants involved in the above inequalities are sharp in the sense that

$$\frac{(N-2-2a)^2}{4} = \inf_{u\in\mathcal{D}_a^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{B_1(0)} |x|^{-2a}|\nabla u|^2 + \frac{N-2-2a}{2}\int_{\partial B_1(0)} u^2}{\int_{B_1(0)} \frac{u^2}{|x|^{2(a+1)}}},$$

and a similar statement holds for (1.4).

Using similar arguments we obtain a Hardy inequality on any C^1 smooth domains with bounded boundary and $0 \notin \partial\Omega$.

Theorem 1.3. *Let $a < \frac{N-2}{2}$. If $\Omega \subset \mathbb{R}^N$ is a smooth domain with $\partial\Omega$ being bounded and $0 \notin \partial\Omega$, then there is a constant C_h (depending on Ω), such that for all $u \in \mathcal{D}_a^{1,2}(\mathbb{R}^N)$*

$$\frac{(N-2-2a)^2}{4}\int_{\Omega} \frac{u^2}{|x|^{2(a+1)}} \le \int_{\Omega} |x|^{-2a}|\nabla u|^2 + C_h\int_{\partial\Omega} u^2. \tag{1.5}$$

If in addition $\Omega \subset \mathbb{R}^N$ is bounded and star-shaped with respect to the origin, then there is a constant $C_h' > 0$ (depending on Ω), such that for all $u \in \mathcal{D}_a^{1,2}(\mathbb{R}^N)$

$$\frac{(N-2-2a)^2}{4}\int_{\Omega^c} \frac{u^2}{|x|^{2(a+1)}} \le \int_{\Omega^c} |x|^{-2a}|\nabla u|^2 - C_h'\int_{\partial\Omega} u^2. \tag{1.6}$$

A natural question following the theorem is what may happen if Ω is convex (thus star-shaped with respect to any interior point) but the origin lies outside Ω? Based on a new integral inequality established in [12], we have the following result.

Theorem 1.4. *Let $\Omega \subset \mathbb{R}^N$ be a bounded piecewise smooth domain which contains the origin. Assume that $\partial\Omega$ consists of two smooth hyper-surfaces Γ_1 and Γ_2. If Γ_2 is concave with respect to the domain Ω and is part of the boundary of a rotationally symmetric convex domain, then*

$$2^{-2/N}\frac{(N-2)^2}{4}\int_\Omega \frac{u^2}{|x|^2} \le \int_\Omega |\nabla u|^2 \tag{1.7}$$

holds for any $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ with $u = 0$ on Γ_1.

Remark 1.5. *Let $\Omega \subset \mathbb{R}^N$ be a smooth convex revolution solid which does not contain the origin. As a simple corollary of Theorem 1.4, we see that for any $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$,*

$$2^{-2/N}\frac{(N-2)^2}{4}\int_{\Omega^c} \frac{u^2}{|x|^2} \le \int_{\Omega^c} |\nabla u|^2. \tag{1.8}$$

For any domain Ω and any $u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}$, let

$$I(u,\Omega) := \frac{\int_\Omega |\nabla u|^2}{\int_\Omega \frac{u^2}{|x|^2}}\,.$$

We will give an example of a domain that Ω satisfies the conditions in Theorem 1.4,

$$\inf_{u\in\mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\},u=0 \text{ on } \Gamma_1} I(u,\Omega) < \frac{(N-2)^2}{4}. \tag{1.9}$$

Quite similar to the case of Sobolev inequality, we have the following theorem.

Theorem 1.6. *Let $\Omega \subset \mathbb{R}^N$ be a smooth domain such that $\partial\Omega$ is bounded and $0 \notin \partial\Omega$. If $0 < \inf_{u\in\mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}} I(u,\Omega) < (N-2)^2/4$, then the infimum is achieved by a function $\bar{u} \in \mathcal{D}^{1,2}(\mathbb{R}^N)$.*

Remark 1.7. Assume that Ω satisfies the conditions in Theorem 1.4. Following the proof of Theorem 1.6, we easily prove that if

$$\inf_{u\in\mathcal{D}^{1,2}(\mathbb{R}^N)H^1(\Omega)\setminus\{0\},\ u=0 \text{ on } \Gamma_1} I(u,\Omega) < (N-2)^2/4,$$

then $\inf_{u\in\mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\},\ u=0 \text{ on } \Gamma_1} I(u,\Omega)$ is achieved by some functions. This indicates that if $\Omega \subset \mathbb{R}^N$ is a convex domain which does not contain the origin, then there might be no uniform lower bound for $\inf_{u\in\mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}} I(u,\Omega^c)$.

2. Proofs of Theorems

The proofs of Theorem 1.1–1.3 are based on the following conformal transformation which was used in [5, 6] to give a new formulation of a family of weighted Sobolev inequalities due to Caffarelli-Kohn-Nirenberg in [4]. This family of inequalities include the weighted version of the Hardy inequalities.

We define $\varphi : \mathbb{R}^N \to \mathcal{C} := \mathbb{R} \times S^{N-1}$ as the conformal transformation

$$\varphi(x) = (-\ln|x|, \frac{x}{|x|}). \tag{2.1}$$

Here we use $(t,\theta) \in \mathbb{R} \times S^{N-1}$. And we define

$$u(x) = |x|^{-\frac{N-2-2a}{2}} v(-\ln|x|, \frac{x}{|x|}), \quad \forall x \in \mathbb{R}^N. \tag{2.2}$$

Due to the density lemma in [6, Lemma 2.1], we need to prove Theorem 1.1–1.3 only for functions in $C_0^1(\mathbb{R}^N)$.

Proof of Theorem 1.1. Let $u \in C^1(B_1(0))$, and v be given by (2.2). By [6, Proposition 2.2], we know that $v \in H^1(\mathcal{C}_+)$, where $\mathcal{C}_+ = \{(t,\theta) \in \mathbb{R} \times S^{N-1} : t > 0\}$. Denote $\mathcal{S} := 0 \times S^{N-1}$, we have

$$\begin{aligned}\int_{B_1} |x|^{-2a}|\nabla u|^2 &= \int_{\mathcal{C}_+} (|\nabla_\theta v|^2 + (v_t + \frac{N-2-2a}{2} v)^2) d\mu \\ &= \int_{\mathcal{C}_+} (|\nabla v|^2 + (N-2)v_t v + (\frac{N-2-2a}{2})^2 v^2) d\mu \\ &= \int_{\mathcal{C}_+} (|\nabla v|^2 + (\frac{N-2-2a}{2})^2 v^2) d\mu + \int_{\mathcal{C}_+} \frac{N-2-2a}{2} (v^2)_t d\mu,\end{aligned} \tag{2.3}$$

and

$$\int_{\mathcal{C}_+} \frac{N-2-2a}{2}(v^2)_t d\mu = \int_0^\infty \int_{\mathcal{S}_t} \frac{N-2-2a}{2}(v^2)_t d\theta dt = -\frac{N-2-2a}{2}\int_{\mathcal{S}} v^2 d\theta,$$

where $\mathcal{S}_t := t \times S^{N-1}$ and $d\mu = d\theta dt$. Also, it is easy to check that

$$\int_{\partial B_1} u^2 d\theta = \int_{\partial B_1} |x|^{-N+2+2a} v^2 d\theta = \int_{\mathcal{S}} v^2 d\theta.$$

Therefore, we have

$$\int_{B_1} |x|^{-2a}|\nabla u|^2 + \frac{N-2-2a}{2}\int_{\partial B_1} u^2 d\theta = \int_{\mathcal{C}_+} (|\nabla v|^2 + (\frac{N-2-2a}{2})^2 v^2) d\mu.$$

On the other hand

$$\int_{B_1} \frac{u^2}{|x|^{2(a+1)}} dx = \int_{\mathcal{C}_+} |v|^2 d\mu.$$

It follows that

$$\begin{aligned}\frac{\int_{B_1} |x|^{-2a}|\nabla u|^2 dx + \frac{N-2-2a}{2}\int_{\partial B_1} u^2 d\theta}{\int_{B_1} \frac{u^2}{|x|^{2(a+1)}} dx} &= \frac{\int_{\mathcal{C}_+} (|\nabla v|^2 + (\frac{N-2-2a}{2})^2 v^2) d\mu}{\int_{\mathcal{C}_+} |v|^2 d\mu} \\ &> (\frac{N-2-2a}{2})^2\end{aligned}$$

which yields (1.3). The last inequality in the above expression follows from v being in $H^1(\mathcal{C}_+)$. The inequality (1.4) can be proved in the same spirit, and we shall omit the details. □

Proof of Theorem 1.3. Without loss of generality, we can assume that $\partial\Omega \subset B_1(0)$. For any $u(x) \in C^1(\Omega)$, let φ be the transformation given by (2.1), and $v(x)$ be given by (2.2). Denote $\mathcal{C}_\omega = \varphi(\Omega)$. Thus $\partial\mathcal{C}_\omega \subset \mathcal{C}_+$. Similar to (2.3), we have

$$\begin{aligned}\int_\Omega |x|^{-2a}|\nabla u|^2 &= \int_{\mathcal{C}_\omega} (|\nabla_\theta v|^2 + (v_t + \frac{N-2-2a}{2} v)^2) d\mu \\ &= \int_{\mathcal{C}_\omega} (|\nabla v|^2 + (\frac{N-2-2a}{2})^2 v^2) d\mu + \int_{\mathcal{C}_\omega} \frac{N-2-2a}{2}(v^2)_t d\mu.\end{aligned}$$

But due to Green's formula

$$\int_{\mathcal{C}_\omega} \frac{N-2-2a}{2}(v^2)_t d\mu = \frac{N-2-2a}{2} \int_{\partial C_\omega} (v^2, 0)\bar{\eta} dS_\omega,$$

where $\bar{\eta}$ is the unit out norm vector of ∂C_ω and dS_ω is the volume element on ∂C_ω. Then

$$\begin{aligned}
\Big|\frac{N-2-2a}{2} \int_{\partial C_\omega} (v^2, 0)\bar{\eta} dS_\omega\Big| &\le \frac{N-2-2a}{2} \int_{\partial C_\omega} v^2 dS_\omega \\
&= \frac{N-2-2a}{2} \int_{\partial C_\omega} |x|^{(N-2-2a)} u^2(-\ln|x|, x/|x|) dS_\omega \\
&= \frac{N-2-2a}{2} \int_{\partial\Omega} |x|^{-1-2a} u^2 \\
&\le \frac{N-2-2a}{2} C_0 \int_{\partial\Omega} u^2,
\end{aligned}$$

where

$$C_0 = \begin{cases} (\max\{|x| : x \in \partial\Omega\})^{-1-2a} & \text{if } -1-2a \ge 0, \\ (\min\{|x| : x \in \partial\Omega\})^{-1-2a} & \text{if } -1-2a < 0. \end{cases}$$

Therefore,

$$\int_\Omega |x|^{-2a} |\nabla u|^2 + \frac{N-2-2a}{2} C_0 \int_{\partial\Omega} u^2 \ge \int_{\mathcal{C}_\omega} (|\nabla v|^2 + (\frac{N-2-2a}{2})^2 v^2) d\mu.$$

On the other hand,

$$\int_\Omega \frac{u^2}{|x|^{2(a+1)}} dx = \int_{\mathcal{C}_\omega} |v|^2 d\mu.$$

Above two inequalities yield (1.5). (1.6) can be proved in the same spirit, and we shall omit details here. □

Proof of Theorem 1.4. We need to prove the inequality only for non-negative smooth functions. Suppose that $u \in C^1(\Omega)$ is a non-negative function satisfying $u = 0$ on Γ_1. Let Ω^* be the ball centered at the origin which has the same volume as Ω. Let u^* be the Schwartz symmetrization of u. Namely, we define

$$u^*(x) = \sup\{t : \mu(t) > \omega_N |x|^N\},$$

where ω_N is the volume of the unit ball in $\mathbb{R}^N$, and $\mu(t)$ is the Lebesgue measure of the set $\{x \in \Omega : u(x) > t\}$. Then, it is well-known (see, e.g., Bandle [1]) that

$$\int_\Omega \frac{u^2}{|x|^2} \le \int_{\Omega^*} \frac{(u^*)^2}{|x|^2}.$$

On the other hand, from Zhu [12] (this is the place where we use the assumption on Γ_2) we know that

$$\int_{\Omega^*} |\nabla u^*|^2 \le 2^{2/N} \int_\Omega |\nabla u|^2.$$

Since $u^* = 0$ on $\partial\Omega^*$, we know from the standard Hardy inequality that

$$(\frac{N-2}{2})^2 \int_{\Omega^*} \frac{(u^*)^2}{|x|^2} \le \int_{\Omega^*} |\nabla u^*|^2.$$

These three inequalities yield Theorem 1.4. □

There might be a guess that for Ω satisfying the condition in Theorem 1.4,

$$\inf_{u\in\mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\},u=0 \text{ on } \Gamma_1} I(u,\Omega)$$

will always be larger than or equal to $(N-2)^2/4$. However, we present an example to show that this is not the case.

An example. Let $\mathbb{R}^N_{-1}=\{(x',x_N)\in\mathbb{R}^N : x_N>-1\}$. We are going to show that

$$\inf_{u\in\mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N_{-1}}|\nabla u|^2}{\int_{\mathbb{R}^N_{-1}}\frac{u^2}{|x|^2}} < \frac{(N-2)^2}{4}.$$

Let φ be the transformation given by (2.1), and $v(x)$ be given by (2.2) for $x\in\mathbb{R}^N_{-1}$. Denote $\mathcal{C}_{-1}=\varphi(\mathbb{R}^N_{-1})$. Then

$$\frac{\int_{\mathbb{R}^N_{-1}}|\nabla u|^2}{\int_{\mathbb{R}^N_{-1}}\frac{u^2}{|x|^2}} = \frac{\int_{\mathcal{C}_{-1}}(|\nabla v|^2+(\frac{N-2}{2})^2v^2)d\mu+\frac{N-2}{2}\int_{\mathcal{C}_{-1}}(v^2)_t d\mu}{\int_{\mathcal{C}_{-1}}v^2 d\mu}.$$

Now, we choose

$$\tilde v(t,\theta)=\begin{cases}0, & t\le -R-R_0\\ (t+R+R_0)/R_0, & -R-R_0\le t\le -R\\ 1, & -R\le t\le R\\ (R+R_0-t)/R_0, & R\le t\le R+R_0\\ 0, & t\ge R+R_0,\end{cases}$$

where $R_0>4/(N-2)$, and R will be chosen sufficiently large. A simple calculation shows that

$$\begin{aligned}&\int_{\mathcal{C}_{-1}}(|\nabla\tilde v|^2+(\frac{N-2}{2})^2\tilde v^2)d\mu+\frac{N-2}{2}\int_{\mathcal{C}_{-1}}(\tilde v^2)_t d\mu\\ &=\int_{\mathcal{C}_{-1}}(\frac{N-2}{2})^2\tilde v^2 d\mu+(\frac{3}{2}+o(1))\frac{|S^{N-1}|}{R_0}-(\frac{1}{2}+o(1))\frac{N-2}{2}|S^{N-1}|,\end{aligned}$$

where $o(1)\to 0$ as $R\to\infty$. Let $\tilde u(x)=|x|^{-\frac{N-2}{2}}\tilde v(-\ln|x|,\frac{x}{|x|})$. It is easy to see that $\tilde u\in\mathcal{D}^{1,2}(\mathbb{R}^N)$. Thus for sufficiently large R

$$\inf_{u\in\mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N_{-1}}|\nabla u|^2}{\int_{\mathbb{R}^N_{-1}}\frac{u^2}{|x|^2}} \le \frac{\int_{\mathbb{R}^N_{-1}}|\nabla \tilde u|^2}{\int_{\mathbb{R}^N_{-1}}\frac{\tilde u^2}{|x|^2}} < (\frac{N-2}{2})^2.$$

When $\Omega=\mathbb{R}^N_{-1}\cap(\operatorname{supp}\tilde u)^o$, where $(\operatorname{supp}\tilde u)^o$ is the set of interior points of $\operatorname{supp}\tilde u$, we easily see that

$$\inf_{u\in\mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\},u=0 \text{ on } \Gamma_1} I(u,\Omega)<(N-2)^2/4.$$

Proof of Theorem 1.6. Let $u_m\in\mathcal{D}^{1,2}(\mathbb{R}^N)$ be a minimizing sequence such that $\int_\Omega u_m^2/|x|^2=1$. Then

$$\int_\Omega|\nabla u_m|^2\to\inf_{u\in\mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}} I(u,\Omega):=\xi<(\frac{N-2}{2})^2.$$

If Ω is bounded, $\int_\Omega u_m^2 \le C\int_\Omega u_m^2/|x|^2 = C$. Thus u_m is uniformly bounded in $H^1(\Omega)$. It follows that $u_m \to \bar u$ weakly in $H^1(\Omega)$, thus

$$\xi + o_m(1) = \int_\Omega |\nabla u_m|^2 = \int_\Omega |\nabla u_m - \nabla \bar u|^2 + \int_\Omega |\nabla \bar u|^2 + o_m(1), \tag{2.4}$$

where $o_m(1) \to 0$ as $m \to \infty$. If Ω is unbounded, since $\partial\Omega$ is bounded, Ω contains the exterior of some ball domain. Then we may check that $X = \{u|_\Omega \mid u \in \mathcal{D}^{1,2}(\mathbb{R}^N)\}$ is a Hilbert space with a norm $\|u\|_X^2 = \int_\Omega |\nabla u|^2 dx$, this is due to the Sobolev inequality. Then u_m is bounded in X and has a weak limit $\bar u$ and we again have (2.4). By the weak convergence of $\frac{u_m}{|x|}$ to $\frac{\bar u}{|x|}$ in $L^2(\Omega)$, we also have

$$\int_\Omega \frac{\bar u^2}{|x|^2} = \int_\Omega \frac{u_m^2}{|x|^2} - \int_\Omega \frac{(u_m - \bar u)^2}{|x|^2} + o_m(1). \tag{2.5}$$

Therefore, we have

$$\begin{aligned}
\xi + o_m(1) &= \int_\Omega |\nabla u_m - \nabla \bar u|^2 + \int_\Omega |\nabla \bar u|^2 + o_m(1) \quad \text{by (2.4)}\\
&\ge (\frac{N-2}{2})^2 \int_\Omega \frac{|u_m - \bar u|^2}{|x|^2} - C_h \int_{\partial\Omega} |u_m - \bar u|^2 + \xi \int_\Omega \frac{\bar u^2}{|x|^2} + o_m(1)\\
&\quad \text{(by Theorem 1.3 and the definition of } \xi)\\
&\ge (\frac{N-2}{2})^2 \int_\Omega \frac{|u_m - \bar u|^2}{|x|^2} + \xi \int_\Omega \frac{\bar u^2}{|x|^2} + o_m(1)\\
&\quad \text{(by Sobolev embedding)}\\
&= [(\frac{N-2}{2})^2 - \xi] \int_\Omega \frac{|u_m - \bar u|^2}{|x|^2} + \xi + o_m(1) \quad \text{(by (2.5))},
\end{aligned}$$

which implies $\int_\Omega \frac{|u_m - \bar u|^2}{|x|^2} \to 0$ as $m \to \infty$. It follows that $\int_\Omega |\bar u|^2/|x|^2 = 1$, thus $\bar u$ is the minimizer of $I(u, \Omega)$. □

Notes Added in Proof. After this paper was accepted we found a paper by Adimurthi: Hardy-Sobolev inequality in $H^1(\Omega)$ and its applications, Comm. Contem. Math., 4 (2002), 409-434, which contains related results and uses different methods.

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