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ASYMPTOTIC UNIQUENESS AND EXACT SYMMETRY OF k-BUMP SOLUTIONS FOR A CLASS OF DEGENERATE ELLIPTIC PROBLEMS

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Abstract. In this article, we prove that the problem

$$-\Delta v + v = v^{p-1},\ \ v \in H^1(\mathcal{C}_\lambda),\ \ v > 0 \text{ in } \mathcal{C}_\lambda,$$

with $2 < p < \frac{2N}{N-2}$, $\mathcal{C}_\lambda = \mathbb{S}^{N-1}_\lambda \times \mathbb{R}$, has a unique ground state solution provided λ is sufficiently large. We also prove uniqueness results in symmetric subspaces, and determine the exact symmetry of solutions. These results have direct implications regarding to the uniqueness and symmetry of the extremal functions in certain Sobolev type inequalities with weights due to Caffarelli, Kohn, and Nirenberg.

1. **Introduction.** In two earlier papers ([5], [6]), the authors studied the problem

$$-div(|x|^{-2a}\nabla u) = |x|^{-bp}u^{p-1}, u > 0 \text{ in } \mathbb{R}^N, \tag{1}$$

where for $N \geq 2$:

$$\begin{gathered} -\infty < a < \frac{N-2}{2},\quad a \leq b \leq a+1\ \ (a < b \leq a+1,\ \text{if } N = 2), \\ \text{and } p = \frac{2N}{N-2+2(b-a)}. \end{gathered} \tag{2}$$

Our primary concern was the study of the ground states solutions which are extremal functions of and correspond to the best constants in a family of Sobolev type inequalities with weights due to Caffarelli, Kohn, and Nirenberg. They established in a more general context (see [2], and for higher order versions [16]), the following inequalities

$$\left(\int_{\mathbb{R}^N} |x|^{-bp}|u|^p\ dx\right)^{2/p} \leq C_{a,b}\int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2\ dx \tag{3}$$

hold for all $u \in C_0^\infty(\mathbb{R}^N)$, if and only if conditions (2) are satisfied.

Let $\mathcal{D}^{1,2}_a(\mathbb{R}^N)$ be the completion of $C_0^\infty(\mathbb{R}^N)$, with respect to the inner product

$$(u, v) = \int_{\mathbb{R}^N} |x|^{-2a}\nabla u\cdot\nabla v\ dx.$$

Then we see that (3) holds for $u \in \mathcal{D}^{1,2}_a(\mathbb{R}^N)$. We define

$$S(a, b) = \inf_{u\in\mathcal{D}^{1,2}_a(\mathbb{R}^N)\setminus\{0\}} E_{a,b}(u), \tag{4}$$

to be the best embedding constants, where

$$E_{a,b}(u) = \frac{\int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2\ dx}{\left(\int_{\mathbb{R}^N} |x|^{-bp}|u|^p\ dx\right)^{2/p}}. \tag{5}$$

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The extremal functions for $S(a,b)$ are ground state solutions of equation (1).

Regard to the existence and uniqueness of solutions for (1), work was done as follows: in [1], [20], the solution was found explicitly and unique up to translations and dilations in the case $a=b=0$, by Aubin, and Talenti. For $0 \le a < \frac{N-2}{2}$, and $a \le b < a+1$, problem (1) has *only* radially symmetric solutions which are unique up to dilations and known explicitly. This was established by Lieb (in [15]) for $a=0$, using rearrangement methods, and by Chou and Chu (in [7]), using a generalization of the moving plane method. On the other hand, for $a<0$, little has been known until the recent work in [3], [5], [6], [13], [22]. In [5] and [6], we have given some new results which reveal interesting new phenomena. Let $S_p(\mathbb{R}^N)$ be the best embedding constant from $H^1(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$, i.e.

$$S_p(\mathbb{R}^N) = \inf_{u \in H^1(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 + u^2 \, dx}{\left(\int_{\mathbb{R}^N} |u|^p \, dx\right)^{\frac{2}{p}}}.$$

Among other things, in [5] the authors have proved the following theorem.

Theorem 1. *([5]) (Existence and symmetry breaking of ground state solutions)*

(i) For $a<b<a+1$, $S(a,b)$ is always achieved. I.e., there is always a ground state solution.

(ii) There is a function $h(a)$ defined for $a \le 0$, satisfying $h(0)=0$, $a<h(a)<a+1$ for $a<0$, and $a+1-h(a) \to 0$ as $-a \to \infty$, such that for any (a,b) satisfying $a<0$ and $a<b<h(a)$, the grounds state solution is nonradial.

In this paper we are interested in the uniqueness of ground state solutions as well as their exact symmetry. A consequence of our main theorem in this paper is the following

Theorem 2. *(Asymptotic uniqueness and symmetry of the ground states)*

For $b-a \in (0,1)$ fixed, provided that $-a$ is sufficiently large, the problem (1) has a unique (up to rotations and dilations) ground state solution u_a. Moreover, this solution has an $\mathbf{O}(N-1)$ symmetry, i.e. there is a direction in $\mathbb{R}^N$ about which u_a is axially symmetric.

Since the problem is radially symmetric we are also interested in the uniqueness and exact symmetry of G-ground state solutions when we consider the extremal functions of $S(a,b)$ in $\mathcal{D}^{1,2}_{a,G}(\mathbb{R}^N)$ for any closed subgroup of $\mathbf{O}(N)$. Here $\mathcal{D}^{1,2}_{a,G}(\mathbb{R}^N)$ consists of G invariant functions of $\mathcal{D}^{1,2}_{a}(\mathbb{R}^N)$. Results in this direction will be stated in the next section as corollaries of our main results which are set up in a more general frame of work.

2. **Main results and Proofs.** In order to state our main results we first give an equivalent problem which we first introduced in [5]. Our approach was new in the sense that we transformed problem (1) to an equivalent autonomous problem which is set up on a cylinder. Following [5], [6], we use the transformation

$$u(x) = \lambda^{\frac{2}{p-2}} |x|^{-\frac{N-2-2a}{2}} v\left(\lambda \frac{x}{|x|}, -\ln(|x|^{\lambda})\right), \tag{6}$$

where $\lambda = \frac{N-2-2a}{2}$. Problem (1) is equivalent under this transformation to

$$-\Delta v + v = v^{p-1}, \; v \in H^1(\mathcal{C}_\lambda), \; v>0 \text{ in } \mathcal{C}_\lambda. \tag{7}$$

Here, $\mathcal{C}_\lambda$ is the cylinder $\mathbb{S}^{N-1}_\lambda \times \mathbb{R}$, with $\mathbb{S}^{N-1}_\lambda$ being the sphere with radius λ in $\mathbb{R}^N$, and Δ is the Laplace-Beltrami operator corresponding to the metric induced from $\mathbb{R}^{N+1}$. On the cylinder $\mathcal{C}_\lambda$ we have the energy functional

$$F_\lambda(v) = \frac{\int_{\mathcal{C}_\lambda} |\nabla v|^2 + v^2 \, d\mu}{\left(\int_{\mathcal{C}_\lambda} v^p \, d\mu\right)^{\frac{2}{p}}}. \tag{8}$$

On $\mathcal{C}_\lambda$, we use either coordinates $(\theta,t) \in \mathbb{S}^{N-1}_\lambda \times \mathbb{R}$ or $y \in \mathcal{C}_\lambda \subset \mathbb{R}^{N+1}$.

In [6], we proved by the moving plane method the following

Theorem 3. *([5]) Any solution of (7), possibly after a translation in t, is even in t, i.e. $v(\theta,-t) = v(\theta,t)$, and $\frac{\partial v}{\partial t}(\theta,t) < 0$, for all θ and $t > 0$.*

Due to this result, in the following, we shall always assume that solutions of (7) are even in t, so that we are concerned only with symmetries in $\mathbf{O}(N)$.

To avoid ambiguities, we make the following conventions. For a subgroup G of $\mathbf{O}(N)$, we *fix an action* on $\mathbb{S}^{N-1}$. By $H^1_G(\mathcal{C})$ we understand the subspace of functions $u \in H^1(\mathcal{C})$ with the properties $u(g^{-1}\theta,t) = u(\theta,t)$, and $u(\theta,-t) = u(\theta,t)$ for all $g \in G$, $\theta \in \mathbb{S}^{N-1}$, and $t \in \mathbb{R}$. We remark that choosing a different action of the same group G on $\mathbb{S}^{N-1}$, will produce a different subspace $H^1_G(\mathcal{C})$ in $H^1(\mathcal{C})$.

In order to present the main theorem in [6], we make the following definitions

Definition 1. Let $k \in \mathbb{N}$, and $G \subset \mathbf{O}(N)$ be a closed subgroup, so that G naturally acts on $\mathbb{S}^{N-1}$. We say that the action of G has a locally minimal k-orbit set $\Omega \subset \mathbb{S}^{N-1}$, if there exists $\delta > 0$ such that
(a) Ω is G-invariant,
(b) $\#Gy = k$ for any $y \in \Omega$,
(c) $\#Gy > k$ for any $y \in \mathbb{S}^{N-1}$ with $0 < \mathrm{dist}(y,\Omega) < \delta$.

Definition 2. We say G is maximal with respect to a locally minimal k-orbit set Ω if for any closed subgroup H, with $G \le H \le \mathbf{O}(N)$, $H \neq G$, we have $\#Hy > k$ for any $y \in \Omega$.

Definition 3. For $v \in H^1_G(\mathcal{C}_\lambda)$, the symmetry group of v is defined to be

$$\Sigma_v = \{g \in \mathbf{O}(N) \ : \ gv = v \text{ a.e. }\},$$

where by gv we understand the function $(gv)(y) = v(g^{-1}y)$.

We shall consider k-bump type solutions in a more general context which include the ground state solutions as special cases.

Theorem 4. *([6]) (Existence of bound states) Let G be a closed subgroup of $\mathbf{O}(N)$, and $\Omega \subset \mathbb{S}^{N-1}\times\{0\} \subset \mathcal{C}$ a locally minimal k-orbit set of G. Then for λ sufficiently large, problem (7) has a solution w_λ satisfying: (i) w_λ is G-invariant; (ii) w_λ is of k-bump type in the sense that w_λ has exactly k maximum points which form a k-orbit Gy_λ for some $y_\lambda \in \Omega_\lambda := \lambda\Omega$ and*

$$\lim_{\lambda\to\infty} \| w_\lambda - \sum_{y\in Gy_\lambda} \bar{T}_{\lambda,\sqrt{\lambda}y}(S_p(\mathbb{R}^N)^{\frac{1}{p-2}}U) \|_{H^1(\mathcal{C}_\lambda)} = 0;$$

(iii)

$$\lim_{\lambda\to\infty} I_\lambda(w_\lambda) = k^{\frac{p-2}{p}} S_p(\mathbb{R}^N);$$

(iv) If in addition, G is maximal with respect to Ω, then $\Sigma_{w_\lambda} = G$.

Our main result in this paper is the following.

Theorem 5. *For λ sufficiently large, the solution w_λ obtained in Theorem 4 with maximum points $\{y_{\lambda,1},...,y_{\lambda,k}\}$ is unique in $H^1_G(\mathcal{C}_\lambda)$.*

Theorem 5 has some very important corollaries, which we state below.

Theorem 6. *Let G be a closed subgroup of $\mathbf{O}(N)$ with an action on $\mathbb{S}^{N-1}$ admitting a locally minimal k-orbit set Ω. Let $\{y_1,...,y_k\} \subset \Omega$ be fixed. Let K be the maximal subgroup of $\mathbf{O}(N)$ whose action on $\mathbb{S}^{N-1}$ extends the action of G and has $\{y_1,...,y_k\}$ as an orbit. Assume that as $\lambda \to \infty$, there is $w_\lambda \in H^1_G(\mathcal{C}_\lambda)$ given by Theorem 4 with maximum points $\{\lambda y_1,...,\lambda y_k\}$. Then for λ sufficiently large, $\Sigma_{w_\lambda} = K$.*

In order to study the exact symmetry of the ground state solutions in G-invariant subspaces (i.e., G-ground states), we need another definition.

Definition 4. For $k \in \mathbb{N}$, and a closed subgroup G of $\mathbf{O}(N)$, we fix an action on $\mathbb{S}^{N-1}$. We say this action admits a homogeneous minimal k-orbit set if there is an orbit containing k points, any orbit contains at ground$k$ points, and all orbits with k elements are congruent (i.e. if Ω and Λ are two orbits with k elements, there is $h \in \mathbf{O}(N)$ such that $h\Omega = \Lambda$).

Corollary 1. *If the action of G admits a homogeneous minimal k-orbit set, then for λ sufficiently large, the ground state solution of (7) in $H^1_G(\mathcal{C}_\lambda)$ is unique up to a congruence transformation.*

Corollary 2. *For λ sufficiently large, problem (7) has the ground state solution w_λ, unique up to rotations. Moreover, $\Sigma_{w_\lambda} = \mathbf{O}(N-1)$ the isotropy subgroup of $(\theta, 0)$, where $(\theta, 0)$ is the maximum point of w_λ.*

We first prove Theorem 5. We need the following lemma.

Lemma 1. *Any solution w_λ given by Theorem4 has the property that for any $\epsilon > 0$, there is $R = R_\epsilon$ such that $w_\lambda(y) < \epsilon$ for any y with dist$(y, \{y_{\lambda,1}, ..., y_{\lambda,k}\}) \geq R$.*

Proof. Following the construction of the solutions in [6] (see the proof of Theorem 4.1 there), We have that for any $\delta > 0$, there is $R = R_\delta$ such that

$$\int_{\mathcal{C}_\lambda \setminus \cup_{j=1}^k B_{\lambda,R}(y_{\lambda,j})} |\nabla w_\lambda|^2 + w_\lambda^2 \, d\mu < \delta.$$

The Lemma then follows by the elliptic estimates (e.g, [10]) and bootstrap arguments. □

Proof of Theorem 5. To prove that the solution $w_\lambda \in H^1_G(\mathcal{C}_\lambda)$ with maximum points $\{y_{\lambda,1}, ..., y_{\lambda,k}\}$ is unique as $\lambda \to \infty$, we use a argument similar to that from and [8] and [11]. Assume there is a sequence $\lambda_n \to \infty$, for which problem (7) has two ground state solutions $w_{1,n}$ and $w_{2,n}$. According to Theorem 3, we assume that the solutions $w_{i,n}$ are even in t, and both $w_{i,n}$ have their global maxima $M_n = \{y_{n,1}, ..., y_{n,k}\}$ on the sphere $t = 0$.

For each n, we define on $\mathcal{C}_{\lambda_n}$ the function

$$z_n = \frac{w_{1,n} - w_{2,n}}{||w_{1,n} - w_{2,n}||_{L^\infty}}. \tag{9}$$

Each z_n satisfies the following PDE

$$-\Delta z_n + z_n = c_n(\theta, t) z_n, \text{ on } \mathcal{C}_{a\lambda_n}, \tag{10}$$

where

$$c_n(\theta, t) = (p-1) \int_0^1 (s w_{1,n}(\theta, t) + (1-s) w_{2,n}(\theta, t))^{p-2} \, ds. \tag{11}$$

We have that z_n are G-invariant, smooth functions on $\mathcal{C}_{\lambda_n}$, with global maximum value $\max_{\mathcal{C}_{\lambda_n}} z_n = 1$ for all n (if $\max_{\mathcal{C}_{\lambda_n}} z_n < 1$, then $\min_{\mathcal{C}_{\lambda_n}} z_n = -1$ and we can interchange the labels of the two functions $w_{1,n}$ and $w_{2,n}$). Denote by (ξ_n, t_n) one maximum point for z_n, i.e. $z_n(\xi_n, t_n) = 1$ for all n.

Lemma 2. *There is $R > 0$ independent of n, such that dist$((\xi_n, t_n), M_n) \leq R$, for all $|z_n(\xi_n, t_n)| = 1$.*

Proof. If the lemma is false, then there are (ξ_n, t_n) with $z_n(\xi_n, t_n) = 1$, and $\text{dist}((\xi_n, t_n), M_n) \to \infty$. By Lemma 1, for $\epsilon > 0$, there is R_ϵ sufficiently large such that $w_{i,n}(\xi_n, t_n) < \left(\frac{\epsilon}{p-1}\right)^{\frac{1}{p-2}}$.

Therefore $c_n(\xi_n, t_n) < \epsilon$. On the other hand, from $-\Delta z_n(\xi_n, t_n) \geq 0$ and equation (10) we obtain $1 \leq c_n(t_n, \xi_n)$. This provides the necessary contradiction. The case $z_n(\xi_n, t_n) = -1$ is similar. □

Now we come back to the proof of Theorem 5. We define a diffeomorphism between the ball of radius r centered at the origin in $\mathbb{R}^N$ and a subset of $\mathcal{C}_\lambda$ as follows. We identify $\mathbb{R}^N$ with the tangent space to $\mathcal{C}_\lambda$ at y and consider the projection in $\mathbb{R}^{N+1}$ in the direction of the normal to $\mathcal{C}_\lambda$ at y.

To be exact, let

$$\mathcal{C}_\lambda = \{(x_0, ..., x_N) \in \mathbb{R}^{N+1} : x_0^2 + ... + x_{N-1}^2 = \lambda^2\}. \tag{12}$$

Assume $y = (\lambda, 0, ..., 0, y_N) \in \mathcal{C}_\lambda$ and for $0 < r < \lambda$, define a map from

$$B_{\lambda,r}(y) := \{x \in \mathcal{C}_\lambda : x_1^2 + ... + x_{N-1}^2 + (x_n - y_N)^2 < r^2\},$$

onto $B_r(0) \subset \mathbb{R}^N$, by

$$\phi_{\lambda,r,y}(x) = (x_1, ..., x_{N-1}, x_N - y_N) \in \mathbb{R}^N.$$

For any $y \in \mathcal{C}_\lambda$ let R a rotation in $\mathbb{R}^{N+1}$ that leaves the x_N-axis fixed and such that

$$Ry = (\lambda, 0, ..., 0, y_N) \in \mathbb{R}^N.$$

We then define

$$\phi_{\lambda,r,y}(x) = \phi_{\lambda,r,Ry}(Rx).$$

Therefore for all $y \in \mathcal{C}_\lambda$, $\phi_{\lambda,r,y}$ is defined for $x \in B_{\lambda,r}(y)$.

Conversely, let $y = (\lambda, 0, ..., 0, y_N) \in \mathcal{C}$ and for $x \in \mathbb{R}^N$ with $|x| < r$ let

$$\phi^{-1}_{\lambda,r,y}(x) = (\sqrt{\lambda^2 - (x_1^2 + ... + x_{N-1}^2)}, x_1, ..., x_{N-1}, x_N + y_N) \in B_{\lambda,r}(y).$$

Again, for arbitrary $y \in \mathcal{C}$ let R a rotation in $\mathbb{R}^{N+1}$ that leaves the x_N-axis (in $\mathbb{R}^{N+1}$) fixed and such that

$$Ry = (\lambda, 0, ..., 0, y_N).$$

For $x \in B_r(0) \subset \mathbb{R}^N$, define

$$\phi^{-1}_{\lambda,r,y}(x) = R^{-1}\phi^{-1}_{\lambda,r,Ry}(x) \in \mathcal{C}_\lambda.$$

We note here that the Jacobians $J_{\phi_{\lambda,r,y}}(x)$, and $J_{\phi^{-1}_{\lambda,r,y}}(x)$, tend to 1 uniformly on $B_{\lambda,r}(y)$, respectively $B_r(0)$, as $r/\lambda \to 0$.

For $r < \lambda$ and $y \in \mathcal{C}_\lambda$ we construct the operators

$$T_{\lambda,r,y} : H^1(B_{\lambda,r}(y)) \to H^1(B_r(0)), \quad \text{and} \quad \bar{T}_{\lambda,r,y} : H^1(B_r(0)) \to H^1(B_{\lambda,r}(y))$$

as follows

$$T_{\lambda,r,y}(v)(x) = v(\phi^{-1}_{\lambda,r,y}(x)),$$

and

$$\bar{T}_{\lambda,r,y}(u)(x) = u(\phi_{\lambda,r,y}(x)).$$

Denote $v_{1,n} = T_{\lambda_n,\sqrt{\lambda_n},(y_{n,1})}(w_{1,n})$, $v_{2,n} = T_{\lambda_n,\sqrt{\lambda_n},y_{n,1}}(w_{2,n})$, and $\zeta_n = T_{\lambda_n,\sqrt{\lambda_n},y_{n,1}}(z_n)$.

By standard elliptic theory, both $v_{i,n}$ tend in $C^2(B_{\sqrt{\lambda_n}}(0))$ to a solution of

$$-\Delta u + u = u^{p-1}. \tag{13}$$

It is well known (see [14]), that equation (13) has a unique positive solution in $H^1(\mathbb{R}^N)$, with maximum point at the origin. We denote this solution by $U(x)$.

Due to the fact that $y_{n,1}$ are maximum points of $w_{i,n}$, from the equation (7), we have $w_{i,n}(y_{n,1}) \geq 1$. Since $v_{i,n}(0) = w_{i,n}(y_{n,1}) \geq 1$, neither $v_{1,n}$ nor $v_{2,n}$ can tend to the trivial solution of (13). Therefore,

$$v_{i,n} \to U, \text{in } C^2_{loc}, \text{ as } n \to \infty. \tag{14}$$

From (11) and (14), we obtain $c_n \to (p-1)U^{p-2}$ pointwise. Again, by elliptic theory, we get that ζ_n tends in $Cloc^2$ to a solution ϕ, of the linear equation

$$-\Delta\phi + \phi = (p-1)U^{p-2}(x)\phi, \text{ in } \mathbb{R}^N. \tag{15}$$

According to Oh [19], we have

$$\phi = \sum_{j=1}^{N} \alpha_j U_j, \tag{16}$$

with $\alpha_j \in \mathbb{R}$, and $U_j = \frac{\partial U}{\partial x_j}$.

From Lemma 2, we have that ϕ cannot be identically zero, which means not all α_j are zero. As it is shown in [11], the function $U(x)$ satisfies $U_{ij}(0) = 0$ for $i \neq j$, and $U_{ii}(0) < 0$. For a j fixed so that $\alpha_j \neq 0$, we have

$$w_j(0) = \alpha_j U_{jj}(0) \neq 0. \tag{17}$$

On the other hand, since $y_{n,1}$ are maximum points for $w_{i,n}$, we have $\nabla z_n(\theta_n, 0) = 0$ for all n. Hence $\nabla\zeta_n(0) = 0$ for all n. This, and (17) contradict the fact that $\zeta_n \to \phi$ in C^2_{loc}. □

Proof of Theorem 6. Since the action of K is an extension of the action of G, it follows that $\{y_1, ..., y_k\}$ is part of a locally minimal k-orbit set $\Lambda \subset \Omega$ for the action of K. Let v_λ a solution in $H^1_K(\mathcal{C}_\lambda)$, with maximum points $\{\lambda y_1, ..., \lambda y_k\}$. Theorem 5, shows that $w_\lambda = v_\lambda$. □

Proof of Corollary 1. Assume w_λ and v_λ are two solutions as $\lambda \to \infty$, with maximum points $\{\bar{y}_{\lambda,1}, ..., \bar{y}_{\lambda_k}\}$, and $\{\tilde{y}_{\lambda,1}, ..., \tilde{y}_{\lambda_k}\}$, respectively. By hypothesis, there are $\bar{h}_\lambda$, $\tilde{h}_\lambda \in \mathbf{O}(N)$ such that $\bar{h}_\lambda\{\bar{y}_{\lambda,1}, ..., \bar{y}_{\lambda_k}\} = \tilde{h}_\lambda\{\tilde{y}_{\lambda,1}, ..., \tilde{y}_{\lambda_k}\} = \{\lambda y_1, ..., \lambda y_k\}$, for a fixed k-orbit $\{y_1, ..., y_k\}$. By Theorem 5, we have that for λ sufficiently large $w_\lambda = v_\lambda$. □

Proof of Corollary 2. The proof is immediate from Theorem 1 where $G = \{I\}$ is the trivial subgroup of $\mathbf{O}(N)$. The exact symmetry follows from Theorem 6. □

Proof of Theorem 2. It follows from Theorem 5 and Corollary 2. □

We finish the paper with some remarks.

Remark 1. *Without the homogeneous condition on the group actions we do not know uniqueness and exact symmetry of the G-ground states. These remain to be interesting questions. Below we give an example of a group action which has a non-homogeneous minimal 4-orbit set.*

Example 1. *Denote by T_3 the group which leaves invariant a regular tetrahedron in $\mathbb{R}^3$, and D_4 the group which leaves invariant a square in $\mathbb{R}^2$. Let $G = T_3 \times D_4$ acting on $\mathbb{S}^4$ as follows: $\mathbb{S}^4$ is included in $\mathbb{R}^5$ in the standard way, and points in $\mathbb{R}^5$ are labeled by coordinates $(x_1, ..., x_5)$. The T_3 part acts on (x_1, x_2, x_3), while the D_4 part acts in the x_4x_5-plane. The minimal 4-orbit set Ω, consists in the vertices of the tetrahedron in the x_1, x_2, x_3 dimensions, the reflections about the origin of these vertices, and the vertices of a square in dimensions x_4 and x_5. Therefore Ω is formed out of three orbits, of which only two are congruent.*

Remark 2. *Next we want to mention some work for the exact symmetry of ground state solutions for elliptic problems. Using the moving plane method, Gidas, Ni, and Nirenberg ([9]) showed for some elliptic Dirichlet problems in ball domains or in the whole space all positive solutions are radially symmetric. The issue of exact symmetry of positive solutions has not been studied very much for elliptic problems which have both radial and nonradial positive solutions. Using the rearrangement method Kawohl ([12]) studied the exact symmetry of positive solutions in some symmetric subspaces for an elliptic Dirichlet problem on annular domains. Using similar method Ni and Takagi gave the exact symmetry of the ground state solutions for a Neumann problem in ball domains. The method we use in this paper is different from the above mentioned ones and* *follow closely to some of our earlier work in* [4], [17], *and* [21], *in which we have constructed k-bump type symmetric positive solutions having prescribed symmetry for Dirichlet problems (*[4]*), Neumann problems (*[17]*) and nonlinear Schrödinger problems (*[21]*). Our arguments in this paper can be used to obtain similar results for these problems and we leave the details to interested reader.*

Remark 3. *When an elliptic problem has both radial and nonradial positive solutions we expect to obtain solutions having prescribed symmetry as we work on (1) in this paper. Conversely, we would like to know given a symmetry G how many solutions with exact symmetry G the problem can have. Note that the radial solution is unique. It would be interesting to know in general under what conditions on G the solution with exact symmetry G is unique.*

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