DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS–SERIES B
Volume **6**, Number **2**, March **2006**

Website: http://AIMsciences.org

pp. **357–371**

# ON ELECTRO-KINETIC FLUIDS: ONE DIMENSIONAL CONFIGURATIONS

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Abstract. Electro-kinetic fluids can be modeled by hydrodynamic systems describing the coupling between fluids and electric charges. The system consists of a momentum equation together with transport equations of charges. In the dynamics, the special coupling between the Lorentz force in the velocity equation and the material transport in the charge equation gives an energy dissipation law. In stationary situations, the system reduces to a Poisson-Boltzmann type of equation. In particular, under the no flux boundary conditions, the conservation of the total charge densities gives nonlocal integral terms in the equation. In this paper, we analyze the qualitative properties of solutions to such an equation, especially when the Debye constant $\epsilon$ approaches zero. Explicit properties can be derived for the one dimensional case while some may be generalized to higher dimensions. We also present some numerical simulation results of the system.

1. **Introduction.** Electro-kinetics describes the dynamic coupling between incompressible flows and diffuse charge systems. A small physical parameter, $\epsilon$, known as the Debye constant, leads to boundary layers in the charge densities and electrostatic potential, and consequently the flow. At small length scales, this layer is responsible for a variety of geometrically dependent flows, which, in turn, find application in microfluidic devices in bio-applications [8, 10, 13, 17, 19]. Analyzing the dependence of the flow on the domain geometry and $\epsilon$, is one of the focal points in the study of electro-kinetics.

In this paper we largely discuss the qualitative properties of limiting stationary solutions. The stationary solution solves a Poisson-Boltzmann type equation. We derive explicit bounds as $\epsilon$ approaches zero for solution extrema and boundary gradients in one space dimension. A major feature of the system is a nonlocal integral term in the nonlinearity which arises from the charge density conservation. In contrast to other Poisson-Boltzmann equations, this nonlocal dependence is responsible for the fact that solutions blow up like $\log(\epsilon^{-2})$ in certain cases. We present some

2000 *Mathematics Subject Classification.* Primary: 35Q35 35Q99; Secondary: 35J60.
*Key words and phrases.* electrokinetics, Poisson-Boltzmann, Debye length, Nernst-Plank.
The research of C. Liu and R. Ryham is supported in part by NSF-DMS 0405850.

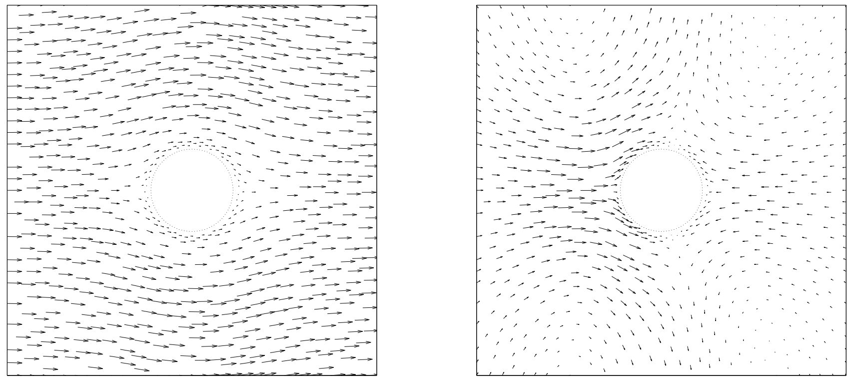

FIGURE 1. Plot of $\mathbf{u}$ of finite element simulation of (1) for flow past an insulated, grounded, conducting cylinder. $\Omega = [-L, L] \times [-1, 1] \setminus \mathbb{B}_r(0,0) \subset \mathbb{R}^2$ for $L \gg 1$ and $r = 0.2$; only $[-1,1] \times [-1,1]$ is shown. Farfield boundary conditions are given by $\partial_y \phi = 0, \partial_y \mathbf{u} = \mathbf{0}$ for $y = 1, -1$ and $\phi = \text{sign}(x)V$ and $\mathbf{u} = (U, 0)$ for $x = 1, -1$. Grounding and no slip boundary conditions on the cylinder are given by $\phi = V/10$, $\mathbf{u} = \mathbf{0}$ for $|\mathbf{x}| = r$. Left, $U = V$: flow dominates charge convection. Right, $U = 0$: Lorentz force dominates the flow.

numerical results to exemplify the three important cases and provide evidence for an upper growth bound.

1.1. **Electro-Kinetic Model.** The (normalized) equations governing hydrodynamic transport of binary diffuse charge densities are [2, 4, 14],

$$\begin{cases} \rho(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}) + \nabla \pi = \lambda \Delta \mathbf{u} + \epsilon^2 \Delta \phi \nabla \phi, \\ \nabla \cdot \mathbf{u} = 0, \\ n_t + \mathbf{u} \cdot \nabla n = \nabla \cdot (\nabla n - n \nabla \phi), \\ p_t + \mathbf{u} \cdot \nabla p = \nabla \cdot (\nabla p + p \nabla \phi), \\ \epsilon^2 \Delta \phi = n - p, \end{cases} \tag{1}$$

with boundary conditions

$$\mathbf{u}\big|_{\partial\Omega} = 0, \tag{2}$$

$$\phi\big|_{\partial\Omega} = \phi_0, \tag{3}$$

$$(\nabla n - n \nabla \phi) \cdot \mathbf{n}\big|_{\partial\Omega} = 0, \tag{4}$$

$$(\nabla p + p \nabla \phi) \cdot \mathbf{n}\big|_{\partial\Omega} = 0. \tag{5}$$

The first two equations of (1) are the linear momentum equations of incompressible flow. $\mathbf{u}$ is the velocity field, $\rho$ is the fluid density (which we take as a constant), and $\pi$ is the pressure, not to be confused with $p$, the positive ion density. The rightmost term in the momentum equation is the Lorentz force[1]. This can be seen by noting that $\mathbf{E} = -\nabla \phi$ is the electric field and $\epsilon^2 \Delta \phi = n - p$ is the net charge density. Moreover, the Lorentz force represents the balance between kinetic energy

[1] We assume that the average ion velocities is small (compared to the speed of light), so that the contribution to the Lorentz force from the magnetic field is negligable.

and electric energy as seen through the Least Action Principle applied to the action functional

$$\mathcal{A}[x] = \int_0^T \int_\Omega \frac{\rho}{2}|x_t(X,t)|^2 - \frac{\epsilon^2}{2}|\nabla\phi(x(X,t))|dXdt,$$

under the pure transport of charge. See [16]. (2) is the no slip boundary condition.

The third and fourth equations of (1) model the balance between diffusion and convective transport of charge densities by flow and electric fields. $n$ and $p$ are the charge densities of a negatively and positively charged species respectively, hence the sign difference in front of the convective term in either equation. The assumption that the charge densities are transported by the electric field (up to a scalar multiple and the valence of the charge) is equivalent to assuming that the material in question is Ohmic, on average, charges migrate according to Coloumb's law, see [14]. We have set the diffusivity and mobility tensors which appear in front of the density diffusion term and convective term to unity. Boundary conditions (4) and (5) model non-reactive boundaries and along with (2) guarantee zero flux of $n$ and $p$ at the boundary.

The fifth equation of (1) is the Poisson equation for the electrostatic potential $\phi$, where the right hand side is the net charge density. $\epsilon$ is a small parameter, known as the Debye length[2], related to vacuum permittivity and characteristic charge density (molarity).

We study qualitative properties of solutions to (1-5) as $\epsilon$ goes to zero.

## 1.2. **Other Electro-kinetic and Diffuse Charge Models.**

**Remark 1.** The diffuse charge system, has been studied in the context of electrochemical cells [1, 3, 6, 9, 15] and electrorheological systems [2, 4, 11, 14, 18, 19, 20]. The fundamental difference between these approaches and that presented in this paper are the boundary conditions (3) and (4). In the study of electrochemical cells, one introduces the ionic flux $j$ in place of 0 in (5). $j, \phi\big|_{\partial\Omega}$ then satisfy an additional (usually nonlinear) constraint to model reaction dynamics as in [9]. They seek an (formal) asymptotic relationship $j(\phi)$ (or $\phi(j)$) which captures the limiting behavior of the system as $\epsilon \to 0$.

In contrast, we take $j = 0$. A benefit from this simplification is that the stationary equations have a variational structure from which we make rigorous assertions about the existence and limiting qualitative properties of solutions. We point out that in this setting, only the electroneutral case leads to a finite asymptotic solution. See theorem 4 and theorem 6. The reader may note, and as will be seen below, the third and fourth equations of (1) along with boundary conditions (4) and (5) have a one dimensional kernel. Traditionaly, the charge densities are then uniquely determined by an extra Dirichlet, or bulk, boundary condition, giving rise to a Boltzmann distributions of the form $c_0 e^{\pm\phi}$ where $c_0$ is a constant (w.r.t. $\epsilon$) given by the boundary data. One may check that in the stationary setting such a boundary condition leads to a finite limiting electrostatic potential, which is in contradiction with our findings and with the notion of charge accumulation at boundaries.

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[2]Exactly, $\epsilon^2 = \epsilon_0\epsilon_r kT/C_\infty e^2$, where $\epsilon_0$ is the permittivity of vacuum, $\epsilon_r$ is the relative permittivity, $kT$ is thermal energy, $C_\infty$ is the characteristic charge density and $e$ is the elementary charge. Typically, $\epsilon$ ranges from $10^{-4}$ to $10^{-7}$.

In [9], the authors incorporate an effective capacitance $C_{\mathrm{S}}$ of the "Stern layer", through the boundary condition

$$\phi + C_{\mathrm{S}}^{-1}\epsilon\nabla\phi\cdot\mathbf{n} = \phi_0. \tag{6}$$

For $\Omega = (0,1)$, and $\phi_0(0) = 0$, $\phi_0(1) = v$, $p$ and $\phi$ satisfy a nonlinear boundary condition

$$k_c p(0)e^{\alpha_c\phi(0)} - j_r e^{-\alpha_a\phi(0)} = j$$
$$-k_c p(1)e^{\alpha_c(\phi(1)-v)} + j_r e^{\alpha_a(v-\phi(0))} = j$$

for fixed, positive constants $k_c, \alpha_c, \alpha_a, j_r$ (see for example [1],[9]). A corollary of our results is the following consistency requirement for the Butler-Volmer boundary condition: if $C_{\mathrm{S}} = \infty$, then one recovers the boundary condition $\phi(0) = 0$ and $\phi(1) = v$. For $j = 0$, it follows that $p(0)$ and $p(1)$ are determined, and in particular, bounded, and nonzero. However, we have shown that $p(0)$ and $p(1)$ remain finite and bounded away from zero if and only if $\int_0^1 p\,dx - \int_0^1 n\,dx = O(\epsilon^2)$. Thus for an electrochemical cell with zero current and infinite effective capacitance, the Butler-Volmer condition is valid only if the total charge densities are equal.

Electrorheological models decouple the momentum equation of (1) from $\phi$ by introducing an effective slip velocity; $\mathbf{u}_s \propto \epsilon^2\mathbf{E}_t(l)(\phi(l,j)-\phi_\infty)/\lambda$ in place of (2). $l$ is the thickness of the diffuse charge layer. It is, however, difficult to capture the coupling between $l$, $\phi$, $\mathbf{u}$ and the domain geometry, rigorously. Typically, this coupling is ignored [4], while it is unclear whether the equilibrium distribution of charges on the boundary is independent of the velocity field, see figure 1.

Other related works include the coupling of charge densities, electric potential, and elastic deformations which has been modeled in [22], or, a system of equations similar to (1.3-1.5) found in the modeling the overdamped gravitational interaction of a cloud of particles or chemotaxis in bacteria [7]. The latter system represents an attractive force between particles, and thus differs in the sign of the convective term in the third and fourth equations of (1). In contrast to (1) and theorem 2, these equations exhibit finite time singularities.

## 2. **Energy Laws and Global Weak Existence.**

We briefly discuss the existence of a Leray type of solution of (1-5) on the unbounded domain. Smooth solutions of (1-5) satisfy the energy laws developed below. The energy laws extend to a modified-Galerkin method [12], from which we then derive a weak limit. It is worth noting that the dissipation of kinetic energy due to the Lorentz force is realized in the transport of net charge densities.

First, we present a maximum principle argument which guarantees the positivity of the charge denisities $n$ and $p$.

**Theorem 1.** *Let $u \in C([0,T];C^2(\Omega))$, $u_t \in C([0,T];C(\bar{\Omega}))$ and a convective term $\mathbf{b} \in C([0,T];C^1(\Omega))$ satisfy*

$$\begin{cases} u_t - \nabla\cdot(\nabla u + u\mathbf{b}) \geq 0, \\ (\nabla u + u\mathbf{b})\cdot\mathbf{n}\big|_{\partial\Omega} \geq 0. \end{cases}$$

*If $u(0,x) > 0$ for all $x \in \Omega$, then $u > 0$ in $\Omega$ and $u \geq 0$ in $\bar{\Omega}$ for all $t \in [0,T]$.*

*Proof.* We begin by assuming that $u_t - \nabla\cdot(\nabla u + u\mathbf{b}) > 0$. We show first that $u(t,x) \neq 0$ for any $x \in \Omega$ and $t \in [0,T]$ and then use this result to determine nonnegativity on the boundary.

Suppose that $u$ is zero in $\Omega$ or takes negative values in $\partial\Omega$. Let $s = \min\{t \in [0,T] : \min_{x\in\bar{\Omega}} u(x,t) \leq 0\}$. Choose $z \in \bar{\Omega}$ such that $u(z,s) = 0$. By definition, $u(z,t) > u(z,s)$ for $0 \leq t < s$ and $u(x,s) \geq u(z,s)$ for all $x \in \Omega$. If $z \in \Omega$, $\Delta u(z,s) \geq 0$, $\nabla u(z,s) = 0$, and $u_t(z,s) \leq 0$, leading to the contradiction,

$$0 \geq u_t(z,s) > \Delta u(z,s) + \nabla u(z,s) \cdot \mathbf{b}(z,s) + u(z,s)\nabla \cdot \mathbf{b}(z,s). \geq 0$$

Now suppose that $u(x,t) < 0$ for some $x \in \partial\Omega$. Let $Z = \{x \in \bar{\Omega} : u(x,s) = 0\}$. By the above argument, $Z \in \partial\Omega$. Let $U_t = \{x \in \Omega : u(x,t) < 0\}$ and $D_r = \{x \in \bar{\Omega} : \mathrm{dist}(x,Z) \leq r\}$. We make an assertion; there exist a continuous scalar function $\eta : [s,T] \to \mathbb{R}$ with $\eta(s) = 0$ and $U_t \subset D_{\eta(t)}$. Suppose to the contrary that no such $\eta$ exist. Then, for some $\delta > 0$, there exists $x_t \in U_t$ with $\mathrm{dist}(x_t, Z) > \delta$. By compactness, some subsequence of $\{x_t\}$ converges to $x^* \in \bar{\Omega}$ with $u(x^*,s) = \lim_{t\to s} u(x_t,t) = 0$. Hence $x^* \in Z$ and $\mathrm{dist}(x^*,Z) > \delta$, a contradiction. Consider the following;

$$\begin{aligned}\int_{U_t} u_t \, dx &> \int_{U_t} \nabla \cdot (\nabla u + u\mathbf{b}) \, dx \\ &= \int_{\partial U_t \cap \partial\Omega} (\nabla u + u\mathbf{b}) \cdot \mathbf{n} \, dS + \int_{\partial U_t \setminus \partial\Omega} (\nabla u + u\mathbf{b}) \cdot \mathbf{n} \, dS \\ &\geq \int_{\partial U_t \setminus \partial\Omega} \nabla u \cdot \mathbf{n} \, dS \geq 0.\end{aligned}$$

The last inequality follows since since $\partial U_t \setminus \partial\Omega$ is the zero level set of $u$. The inequality $\int_{U_t} u_t \, dx > 0$ implies there exist $x_t \in U_t$ with $u_t(x_t,t) > 0$. Since $x_t \in U_t \subset D_{\eta(t)}$ and $\lim_{t\to s} \eta(t) = 0$, we find that some subsequence of $\{x_t\}$ converges to an element $z^*$ in $Z$. Consequently, $0 < \lim_{t\to s} u_t(x_t,t) = u_t(z^*,s) \leq 0$, a contradiction.

Returning to the nonstrict inequality, let $u$ satisfy $u_t - \nabla \cdot (\nabla u + u\mathbf{b}) \geq 0$. Then $v = u + \epsilon t$ satisfies $v_t - \nabla \cdot (\nabla v + v\mathbf{b}) \geq \epsilon > 0$ for all $\epsilon > 0$. $v > 0$ in $\Omega \times [0,T]$ and $v \geq 0$ in $\bar{\Omega} \times [0,T]$. Since this holds for all $\epsilon > 0$, $u > 0$ in $\Omega \times [0,T]$ and $u \geq 0$ in $\bar{\Omega} \times [0,T]$ for all $\epsilon$. □

Suppose that $n$, $p$ and $\phi$ are smooth solutions of (1-5). Multiplying equations three and four of (1) by $n$ and $p$ respectively, and integrating over $\Omega$;

$$\frac{1}{2}\frac{d}{dt}\|n\|^2_{L^2(\Omega)} = -\|\nabla n\|^2_{L^2(\Omega)} - \frac{1}{2}(n^2, \Delta\phi) + \frac{1}{2}\int_{\partial\Omega} n^2 \nabla\phi \cdot \mathbf{n} \, dS.$$

$$\frac{1}{2}\frac{d}{dt}\|p\|^2_{L^2(\Omega)} = -\|\nabla p\|^2_{L^2(\Omega)} + \frac{1}{2}(p^2, \Delta\phi) - \frac{1}{2}\int_{\partial\Omega} p^2 \nabla\phi \cdot \mathbf{n} \, dS.$$

Summing the above equations, we find

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|n\|^2_{L^2(\Omega)} + \|p\|^2_{L^2(\Omega)}) + (\|\nabla n\|^2_{L^2(\Omega)} + \|\nabla p\|^2_{L^2(\Omega)}) \\ &= -\frac{1}{2\epsilon^2}(n+p, (n-p)^2) + \frac{1}{2}\int_{\partial\Omega} ((n)^2 - (p)^2)\nabla\phi \cdot \mathbf{n} \, dS.\end{aligned} \tag{EL1}$$

$(\cdot,\cdot)$ denotes the $L^2(\Omega)$ inner product. Note that the second to last term above is positive if one can guarantee that $n(t,x), p(t,x) \geq 0$ for all $(x,t) \in \Omega \times [0,T]$. Choosing $\mathbf{b} = \pm\nabla\phi$ and $j = 0$ in theorem 1, it follows that $n, p > 0$ in $\Omega \times [0,T]$ and $n, p \geq 0$ in $\bar{\Omega} \times [0,T]$ if $n_0, p_0 > 0$ in $\Omega$.

Multiplying the first equation of (1) by $\mathbf{u}$ and integrating over $\Omega$, we find the kinetic energy dissipation;

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|^2_{L^2(\Omega)} = -\lambda\|\nabla\mathbf{u}\|^2_{L^2(\Omega)} + \epsilon^2\int_\Omega \Delta\phi\nabla\phi\cdot\mathbf{u}\,dx. \tag{7}$$

Subtracting the fourth from the third equation of (1), one finds that $\phi$ satisfies

$$\epsilon^2\Delta\phi_t + \epsilon^2\mathbf{u}\cdot\nabla\Delta\phi = \nabla\cdot(\epsilon^2\nabla\Delta\phi - (n+p)\nabla\phi).$$

Multiplying this expression by $\phi$ and integrating over $\Omega$ we find

$$\begin{aligned}\epsilon^2\int_{\partial\Omega}\phi\nabla\phi_t\cdot\mathbf{n}\,dS - \frac{\epsilon^2}{2}\frac{d}{dt}\|\nabla\phi\|^2_{L^2(\Omega)} - \epsilon^2\int_\Omega\Delta\phi\nabla\phi\cdot\mathbf{u}\,dx\\ = -\epsilon^2\int_{\partial\Omega}\Delta\phi\nabla\phi\cdot\mathbf{n}\,dS + \epsilon^2\|\Delta\phi\|^2_{L^2(\Omega)} + \int_\Omega(n+p)|\nabla\phi|^2\,dx\end{aligned} \tag{8}$$

Equating $\epsilon^2\int_\Omega\Delta\phi\nabla\phi\cdot\mathbf{u}\,dx$ in (7) and (8), we find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\mathbf{u}\|^2_{L^2(\Omega)} + \epsilon^2\|\nabla\phi\|^2_{L^2(\Omega)}) + \lambda\|\nabla\mathbf{u}\|^2_{L^2(\Omega)} + \epsilon^2\|\Delta\phi\|^2_{L^2(\Omega)}\\ = -\int_\Omega(n+p)|\nabla\phi|^2\,dx + \epsilon^2\int_{\partial\Omega}(\phi\nabla\phi_t + \Delta\phi\nabla\phi)\cdot\mathbf{n}\,dS\end{aligned} \tag{EL2}$$

We specify a tuple $(\mathbf{u}_N, n_N, p_N, \phi_N)$, where $\mathbf{u}_N$ is solution of the finite dimensional Galerkin formulation of equation one and two of (1) and $(n_N, p_N, \phi_N)$ solve equations three through five of (1) identically with $\mathbf{u}$ replaced by $\mathbf{u}_N$. Immediately, $(\mathbf{u}_N, n_N, p_N, \phi_N)$, also satisfies (EL1) and (EL2). If $\phi$ takes the natural boundary condition $\nabla\phi\cdot\mathbf{n} = 0$ or $\Omega = \mathbb{R}^n$, then all boundary terms vanish and (EL1) and (EL2) are dissipative. We have then

**Theorem 2.** *Let (i) $\nabla\phi\cdot\mathbf{n} = 0$ or (ii) $\Omega = \mathbb{R}^n$. For $\mathbf{u}_0 \in (L^2(\Omega))^2$, $n_0, p_0 \in L^2(\Omega)$, there exists $(\mathbf{u}, n, p, \phi)$, defined on $[0,T]$ satisfying (1) weakly and $\mathbf{u}(0) = \mathbf{u}_0$, $n(0) = n_0$, $p = p_0$, almost everywhere in $\Omega$.*

For a proof of theorem (2), and discussion of (EL1) and (EL2) for bounded domains, see [16].

## 3. Stationary Solutions.

The stationary problem is defined as that where $\mathbf{u} = \mathbf{0}$ and $n_t = p_t = 0$. In this setting, (1) becomes

$$\begin{cases}\nabla\cdot(\nabla n - n\nabla\phi) = 0,\\ \nabla\cdot(\nabla p + p\nabla\phi) = 0,\\ \epsilon^2\Delta\phi = n - p.\end{cases} \tag{9}$$

Due to incompressibility, (2), (4) and (5), the total charge densities from the dynamic problem are conserved (for the moment consider the dynamic variables $n, p$ and $\mathbf{u}$ again);

$$\frac{d}{dt}\int_\Omega n\,dx = \int_\Omega n_t\,dx = -\int_{\partial\Omega}(\nabla n - n\nabla\phi)\cdot\mathbf{n} + n\mathbf{u}\cdot\mathbf{n}\,dS - \int_\Omega n\nabla\cdot\mathbf{u}\,dx = 0.$$

Similarly, the dynamic variable $p$ would satisfy $d/dt\int_\Omega p\,dx = 0$. If the the stationary equations are to be viewed as the equilibrium equations of (1), then specifying $\int_\Omega n\,dx$ and $\int_\Omega p\,dx$ are necessary constraints on (9). Infact, the equations one and two of (9) along with (4) and (5) respectively, are well posed (see [16]) with the additional constraints

$$\int_\Omega n\,dx = \alpha, \quad \int_\Omega p\,dx = \beta. \tag{10}$$

for some $\alpha, \beta > 0$.

Suppose now that $(n, p, \phi)$ satisfy (9) weakly. Then $n$ and $p$ satisfy equations one and two of (9) and (10) respectively for some $\phi$, and by uniqueness of solutions to these equations with boundary conditions (4) and (5) and constraints (10) respectively,

$$n = \alpha \frac{e^\phi}{\int_\Omega e^\phi\,dx}, \quad p = \beta \frac{e^{-\phi}}{\int_\Omega e^{-\phi}\,dx}. \tag{11}$$

When written in the form of exponentials of $\phi$, $n$ and $p$ satisfy a Boltzmann distribution. However, in the case above, the coefficient has a nonlocal correction for the mass of $n$ and $p$ respectively. Entering the relations (11) in the Poisson equation of (9), one finds

$$\epsilon^2 \Delta\phi = \alpha \frac{e^\phi}{\int_\Omega e^\phi\,dx} - \beta \frac{e^{-\phi}}{\int_\Omega e^{-\phi}\,dx}, \quad \phi\big|_{\partial\Omega} = \phi_0. \tag{12}$$

We point out that (12) is the Euler-Lagrange equation of the energy

$$E[u] = \frac{\epsilon^2}{2}\|\nabla u\|^2_{L^2(\Omega)} + \alpha \log\left(\int_\Omega e^u\,dx\right) + \beta \log\left(\int_\Omega e^{-u}\,dx\right).$$

The existence of a unique solution to (12) and consequently the existence of a unique stationary solution of (9) is guaranteed by the direct method of the calculus of variations;

**Theorem 3.** *Let $\Omega$ be a bounded, open subset of $\mathbb{R}^n$ with smooth boundary and $\phi_0 \in C^0(\partial\Omega)$. Then there exists at most one $\phi \in C^\infty(\Omega) \cap C^0(\bar{\Omega})$ satisfying* (12).

Theorem 3 follows from the fact that $E$ is a convex functional bounded from below. See [16] for details.

3.1. **Summary of One Dimensional Results.** Henceforth we restrict ourselves to the one dimensional problem. The three limiting properties for $\epsilon \to 0$ are

1. If $\alpha = \beta$, then solutions stay bounded and converge uniformly in the interior to the average of the boundary values. The boundary layer has an $\epsilon$ thickness and the limiting profile is exponential.
2. If $\alpha < \beta$, then for sufficiently small $\epsilon$, solutions are convex and converge uniformly to a constant (w.r.t. x) in the interior. This constant is asymptotic to $\log(\epsilon^{-2})$; the lower bound is rigorous, while the upper bound is numerical.
3. Analogous results hold for $\alpha > \beta$.

Figures 2 and 3 demonstrate the numerical simulation of these cases.

3.2. **Properties of Poisson-Boltzmann Type Equation.** An interpretation of the one dimensional problem is that of a stationary diffuse charge system, enclosed by two infinite, nonreactive plates with fixed voltage. Let $\Omega = (-1, 1)$ and let $\phi$ be

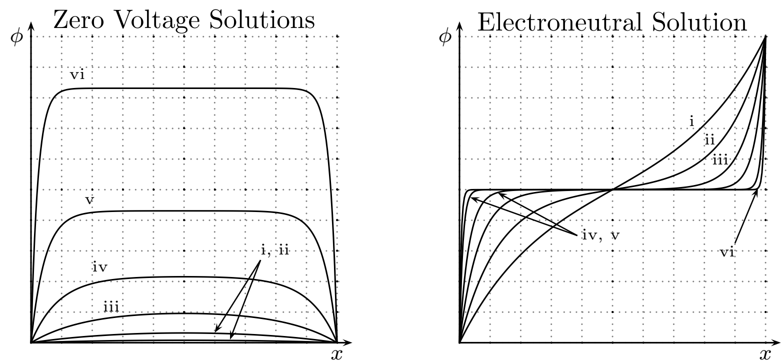


FIGURE 2. Graphs i, ii, . . . , vi of numerical solutions of (13) corresponding to limiting parameter $\epsilon = 1, 2^{-2}, \ldots, 2^{-10}$. Numerical solutions are convergent limit of the iterative procedure; $c_i^{\pm} \mapsto \phi_{i+1}$ by solving the poisson equation with standard finite elements and $\phi_{i+1} \mapsto c_{i+1}^{\pm}$ by solving the convection/diffution equations with finite element scheme laid out in [23] for piecewise linears over 128 equally spaced grid points. Left, $\phi_0(-1) = \phi_0(1)$: $\phi$ diverges as $\epsilon \to 0$. Right, $\alpha = \beta$: $\phi$ converges to $(\phi_0(-1) + \phi_0(1))/2$.

a solution of

$$\epsilon^2 \phi''(x) = \alpha \frac{e^{\phi(x)}}{\int_{-1}^{1} e^{\phi(y)}\, dy} - \beta \frac{e^{-\phi(x)}}{\int_{-1}^{1} e^{-\phi(y)}\, dy}, \quad \forall x \in (-1, 1), \tag{13}$$

$$\phi(-1) = \phi_0(-1), \quad \phi(1) = \phi_0(1). \tag{14}$$

Note that solutions commute with translation of (14), i.e. $\phi + c$ is a solution to (13), (14) if $\phi_0$ is replaced by $\phi_0 + c$ for $c \in \mathbb{R}$. Consequently, in the lemma and theorem statements below, we may without loss of generality shift the boundary data.

The major difficulty of analyzing properties of $\phi$ is the nonlocal terms in (13). The righthand side of (13) does, however, have several properties that can be analyzed. We introduce the following notation related to the right hand side of (13); given a continuous function $\phi$, define $b(\phi) : \mathbb{R} \to \mathbb{R}$ by

$$b(\phi)(t) = \alpha \frac{e^{t}}{\int_{-1}^{1} e^{\phi(y)}\, dy} - \beta \frac{e^{-t}}{\int_{-1}^{1} e^{-\phi(y)}\, dy}. \tag{15}$$

Further, define $a(\phi) : \mathbb{R} \to \mathbb{R}$ by $a(\phi)(t) = b(\phi)'(t)$. Note that $a(\phi)(t) > 0$ for all $t$ and $b(\phi)(t)$ is strictly increasing in $t$. Also, define $B(\phi), A(\phi) : (-1, 1) \to \mathbb{R}$ by $B(\phi)(x) = b(\phi)(\phi(x))$ and $A(\phi)(x) = a(\phi)(\phi(x))$. Equation (13) is then equivalent to

$$\epsilon^2 \phi''(x) = B(\phi)(x), \quad \forall x \in (-1, 1). \tag{16}$$

**Lemma 1.** *Let $\phi \in C^0(\bar{\Omega})$. Then*

*1. $\int_{-1}^{1} B(\phi)\, dx = \alpha - \beta$ and $\int_{-1}^{1} A(\phi)\, dx = \alpha + \beta$.*

*2. $B(\phi)' = A(\phi)\phi'$ and $A(\phi)' = B(\phi)\phi'$.*

*3. $B(\phi)$ is monotone with respect to $\phi$ in the sense that if $\phi(x) < \phi(y)$, then $B(\phi)(x) < B(\phi)(y)$.*

*4. There exists a unique* $\phi_* \in \mathbb{R}$ *such that* $B(\phi)(x) = a(\phi)(\phi_*)\sinh(\phi(x) - \phi_*)/2$ *and* $a(\phi)(\phi_*) \le \min_{x\in(-1,1)} A(\phi)(x)$.

*Proof.* (1) and (2) follow immediately from the definition of $B(\phi)$ and $A(\phi)$. (3) follows from the fact $b(\phi)$ is strictly increasing so that $B(\phi)(x) = b(\phi)(\phi(x)) < b(\phi)(\phi(y)) = B(\phi)(y)$. $\phi_*$, the unique zero of $b(\phi)$ and consequently unique critical point of $a(\phi)$ can be calculated as:

$$e^{2\phi_*} = \frac{\beta \int_{-1}^{1} e^{\phi}\, dy}{\alpha \int_{-1}^{1} e^{-\phi}\, dy}.$$

It is clear that $a(\phi)(\phi_*)$ minimizes $a(\phi)$ and consequently $A(\phi)$. We check the identity in (4) directly;

$$a(\phi)(\phi_*)\sinh(\phi - \phi_*) = \alpha \frac{e^{\phi} - e^{-\phi}e^{2\phi_*}}{\int_{-1}^{1} e^{\phi}\, dy} - \beta \frac{e^{-\phi} - e^{\phi}e^{-2\phi_*}}{\int_{-1}^{1} e^{-\phi}\, dy} = 2B(\phi).$$

□

The above properties hold for all continuous functions $\phi$. If, however, $\phi$ solves (13) and (14), then $B(\phi)$ and $A(\phi)$ have additional structural properties. Lemma 2 shows that $B(\phi)$ solves a second order equation with a negative zeroth order coefficient bound away from zero. Although the boundary values of $B(\phi)$ are not known, the equation it solves, in contrast to (13), is local and linear. Below we develope positivity criterion for extremal values and comparison functions for $B(\phi)$. These become important in proving that all non-electroneutral ($\alpha \ne \beta$) solutions diverge as $\epsilon \to 0$ (see Lemma 3, theorem 6.)

**Lemma 2.** *If* $\phi \in C^\infty(\Omega) \cap C^0(\bar{\Omega})$ *satisfies* (13), *then* $B(\phi)$ *has no positive internal local maxima and no negative internal local minima. Further, there exist* $\mu, \theta > 0$ *independent of* $\epsilon$ *such that* $\theta \le a(\phi)(\phi_*) \le \mu$ *for all* $\epsilon \ne 0$ *where* $\phi_*$ *is given in Lemma 1.*

*Proof.* Consider the following; $B(\phi)'' = (A(\phi)\phi')' = B(\phi)(\phi')^2 + A(\phi)\phi''$. Note that $\epsilon^2\phi'' = B(\phi)$, so that $B(\phi)$ satisfies

$$\epsilon^2 B(\phi)'' = B(\phi)(\epsilon^2(\phi')^2 + A(\phi)). \tag{17}$$

Since $A(\phi) > 0$, it follows that if $B(\phi)(x) > 0$ for some $x \in (-1, 1)$, then $B(\phi)''(x) > 0$ so that $B(\phi)(x)$ cannot be a maximum. Similarly, a negative value cannot be a mimimum.

By Lemma 1 (4),

$$2\sqrt{\alpha\beta}\left(\int_{-1}^{1} e^{\phi}\, dx \int_{-1}^{1} e^{-\phi}\, dx\right)^{-\frac{1}{2}} = a(\phi)(\phi_*) \le \min_{x\in(-1,1)} A(\phi)(x).$$

The lemma is proved if we can show that $\int_{-1}^{1} e^{\phi}\, dx \int_{-1}^{1} e^{-\phi}\, dx$ is bounded above and below independently of $\epsilon \ne 0$. We have shown that $B(\phi)$ has no negative internal minima. Without loss of generality, assume that $\phi_0(-1) \le \phi_0(1)$. Then $\min\{0, B(\phi)(-1)\} \le B(\phi)(x)$ for all $x \in (-1, 1)$. In particular,

$$-\beta \frac{e^{-\phi_0(-1)}}{\int_{-1}^{1} e^{-\phi}\, dy} \le \min\{0, B(\phi)(-1)\}$$

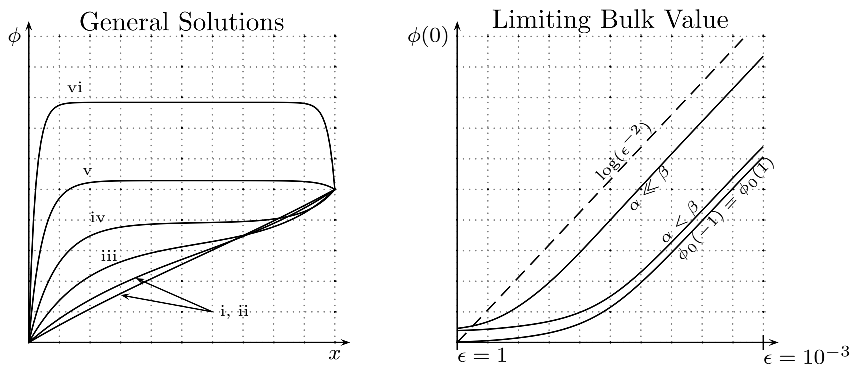


FIGURE 3. Left: numerical solutions i, ii, . . . , vi for $\phi_0(-1) < \phi_0(1)$, $\alpha < \beta$ and $\epsilon = 1, 2^{-2}, \dots, 2^{-10}$. Right: $\log(\epsilon^{-2})$ is plotted by a dashed line for comparison. Note that all solid lines become parallel with the dashed line as $\epsilon \to 0$.

which, after some arithmetic, implies that

$$e^{-2\phi(x)} \le \frac{\alpha \int_{-}^{1} e^{-\phi}\, dy}{\beta \int_{-1}^{1} e^{\phi}\, dy} + e^{-\phi_0(-1)} e^{-\phi(x)} \quad \forall x \in (-1,1).$$

Integrating this inequality over $(-1, 1)$, by Hölder's inequality, we find

$$\frac{1}{2}\left(\int_{-1}^{1} e^{-\phi}\, dx\right)^2 \le \int_{-1}^{1} e^{-2\phi}\, dx \le 2\frac{\alpha \int_{-1}^{1} e^{-\phi}\, dx}{\beta \int_{-1}^{1} e^{\phi}\, dx} + e^{-\phi_0(-1)} \int_{-1}^{1} e^{-\phi}\, dx.$$

Multiplying by $2\int_{-1}^{1} e^{\phi}\, dx / \int_{-1}^{1} e^{-\phi}\, dx$, we find the inequality below;

$$\int_{-1}^{1} e^{-\phi}\, dx \int_{-1}^{1} e^{\phi}\, dx \le 4\frac{\alpha}{\beta} + 2e^{-\phi_0(-1)} \int_{-1}^{1} e^{\phi}\, dx. \tag{18}$$

Similarly, $B(\phi)$ has no positive internal maxima so that $B(\phi)(x) \le \max\{0, B(\phi)(1)\}$ for all $x \in (-1, 1)$. An analogous argument to the one above will give the inequality

$$\int_{-1}^{1} e^{-\phi}\, dx \int_{-1}^{1} e^{\phi}\, dx \le 4\frac{\beta}{\alpha} + 2e^{\phi_0(1)} \int_{-1}^{1} e^{-\phi}\, dx. \tag{19}$$

Finally, we claim that (18) and (19) imply the result

$$\int_{-1}^{1} e^{-\phi}\, dx \int_{-1}^{1} e^{\phi}\, dx \le C$$

for some $C = C(\alpha, \beta, \phi_0)$. To see this, let $x = \int_{-1}^{1} e^{\phi}\, dx$, $y = \int_{-1}^{1} e^{-\phi}\, dx$, $c_1 = 4\alpha/\beta$, $c_2 = 2e^{-\phi_0(-1)}$, $c_3 = 4\beta/\alpha$, and $c_4 = 2e^{\phi_0(1)}$. Then $x$ and $y$ satisfy $xy \le c_1 + c_2 x$ and $xy \le c_3 + c_4 y$. If $x \le y$, then $x^2 \le xy \le c_1 + c_2 x \le c_1 + c_2^2/\eta + \eta x^2$ so that $x \le C(c_1, c_2)$ after choosing $\eta$ sufficiently less than 1. But then $xy \le c_1 + c_2 x \le c_1 + c_2 C(c_1, c_2)$. If $x \ge y$, the same holds true for $C = C(c_3, c_4)$.

To produce a lower bound, Hölder's inequality shows that $4 = (\int_{-1}^{1} e^{\phi/2} e^{-\phi/2}\, dx)^2 \le \int_{-1}^{1} e^{-\phi}\, dx \int_{-1}^{1} e^{\phi}\, dx$. □

3.3. **Limiting Behaviour in Several Classes.** Here we consider solutions of (13) as $\epsilon \to 0$. Two distinct limiting behaviours emerge depending solely on the ratio of $\alpha$ to $\beta$. In the case when $\alpha = \beta$ (called electroneutral), theorem 4 demonstrates that solutions stay bounded. For $\alpha \neq \beta$ (non-electroneutral case), Lemma 3 demonstrates that solutions diverge with the order $\log(\epsilon^{-1})$. By a bootstrap argument, theorem 6 increases the growth order to $\log(\epsilon^{-2})$. Numerical solutions suggest that this is infact an upper bound on growth as well, see figure 3. All results for $\alpha < \beta$ below have an analogous result for $\alpha > \beta$, where convexity is replaced by concavity, etc.

Theorem 4 demonstrates that electroneutral solutions converge exponentially to the average value of the boundary data. These solutions have a boundary layer of thickness $\epsilon$, and exponential boundary layer profile. In particular, the boundary gradients are of order $\epsilon^{-1}$. In constrast, non-electroneutral solutions have boundary gradients of order $\epsilon^{-2}$. See theorem 5. Electroneutral solutions for vanishing $\epsilon$ are graphed in figure 2.

**Theorem 4.** *Let $\phi \in C^\infty(\Omega) \cap C^0(\bar{\Omega})$ satisfy (13) and (14) for $\alpha = \beta$ and $-\phi_0(-1) = \phi_0(1) \geq 0$. Then $\phi$ is odd, monotone, convex for $x \in (0,1)$ and concave for $x \in (-1,0)$, and there exist $\eta_{1,2}$ and $g \in C_c^\infty([0,1))$ independent of $\epsilon$ such that*

$$\phi_0(1)[e^{\eta_1(x-1)/\epsilon} - e^{-\eta_1/\epsilon} g] \leq \phi(x) \leq \phi_0(1) e^{\eta_2(x-1)/\epsilon}, \quad x \in (0,1). \tag{20}$$

*where $g(0) = 1$ and $g'(0) = 0$.*

**Remark 2.** Commutativity of solutions with the boundary data with respect to addition of a constant implies that we may without loss of generality shift the boundary data so that it is odd.

With $g$ compactly supported in $[0,1)$, (20) implies that $\phi'(1) = \phi'(-1)$ is bounded above and below by constant multiples of $\epsilon^{-1}$.

*Proof.* It is immediate to check that if $\phi$ satisfies (13) and (14), then $\psi$ defined by $\psi(x) = -\phi(-x)$ does as well. By the uniqueness of solutions to (13) and (14), $\psi = \phi$ and consequently $\phi$ is odd.

Using the oddness of $\phi$, one may check that $\int_{-1}^1 e^\phi dx = \int_{-1}^1 e^{-\phi} dx$. Along with $\alpha = \beta$, this implies that $b(\phi)(0) = 0$ so that in lemma 1 (4), $\phi_* = 0$. By lemma 1 (4), we may rewrite (13) as

$$\epsilon^2 \phi'' = \rho \sinh(\phi)$$

where $\rho = a(\phi)(\phi_*)/2 > 0$. Note that $\phi(-1) \leq 0$ and $\phi(0) = 0$. Suppose that $\phi$ is positive somewhere in $(-1,0)$. Then $\phi$ has a positive maximum at $x_0 \in (-1,0)$ such that $0 \leq \epsilon^2 \phi''(x_0) = \rho \sinh(\phi(x_0)) > 0$, a contradiction. Thus $\phi(x) \leq 0$, is convex, and consequently monotone for $x \in (-1,0)$. By oddness, $\phi$ is concave and monotone on $(0,1)$ as well. In particular, $\phi_0(-1) \leq \phi(x) \leq \phi_0(1)$ for all $x \in (-1,1)$.

By lemma 2 and the above remark, $\rho$ and $\phi$ are bounded above and below independently of $\epsilon$. Thus there exist $C_{1,2} > 0$ independent of $\epsilon$ such that $C_2 \phi \leq \rho \sinh(\phi) \leq C_1 \phi$ for $x \in (0,1)$. Certainly there exists a $g$ satisfying the hypothesis. (20) is then obtained by ODE comparison from above and below with the right and left hand sides of (20) respectively, with $\eta_2^2 \leq C_2$ and $\eta_1^2 \geq C_1(1+\max_{x\in[0,1]}|g(x)|) + \epsilon^2 \max_{x\in[0,1]}|g''(x)|$. □

**Lemma 3.** *If $\phi \in C^\infty(\Omega) \cap C^0(\bar{\Omega})$ satisfies (13) for $\alpha < \beta$, then*

$$\max_{x\in(-1,1)} \phi(x) \geq \log(\epsilon^{-1}) + C$$

*for some* $C = C(\alpha, \beta, \phi_0)$ *independent of* $\epsilon$.

*Proof.* Without loss of generality, assume that $\phi_0(-1) \leq \phi_0(1)$. Then, by lemma 1 (1) and lemma 2, $B(\phi)$ takes a negative value and thus $B(\phi)(-1) < 0$. Consider the auxiliary function $v(x) = B(\phi)(-1)[\exp(-\eta(x+1)/\epsilon) + \exp(\eta(x-1)/\epsilon)]$ for $\theta > \eta > 0$ independent of $\epsilon$ where $\theta$ is the constant given in lemma 2. Note that $v < 0$, $v(\pm 1) \leq B(\phi)(\pm 1)$ and

$$\epsilon^2 v'' - (\epsilon^2(\phi')^2 + A(\phi))v = v[\eta^2 - \epsilon^2(\phi')^2 - A(\phi)] > 0.$$

By ODE comparison, $v(x) \leq B(\phi)(x)$ for all $x \in (-1, 1)$. Also,

$$\frac{2\epsilon}{\eta} B(\phi)(-1)(1 - e^{-2\eta/\epsilon}) = \int_{-1}^{1} v\,dx \leq \int_{-1}^{1} B(\phi)\,dx = \alpha - \beta.$$

It follows that $\int_{-1}^{1} e^{-\phi}\,dx \leq C\epsilon$ for some $C = C(\alpha, \beta, \eta)$ and thus $\phi(x_0) \geq \log(\epsilon^{-1}) + \log(C)$ for some $x_0 \in (-1, 1)$. □

Theorems 5 and 6 both characterize the case $\alpha \neq \beta$. Theorem 6 deals with general boundary values while theorem 5 deals with the specific zero voltage case, $\phi_0(-1) = \phi_0(1)$. Typical limiting solutions are plotted in figures 2 and 3

**Theorem 5.** *Let* $\phi \in C^\infty(\Omega) \cap C^0(\bar{\Omega})$ *satisfy* (13) *and* (14) *for* $\alpha < \beta$ *and* $\phi_0(-1) = \phi_0(1) = 0$. *Then* $\phi$ *is even, concave,*

$$\phi'(-1) = -\phi'(1) = \frac{(\beta - \alpha)}{2\epsilon^2} \tag{21}$$

*and*

$$\phi(0) \geq \log(\epsilon^{-2}) + C \tag{22}$$

*for some* $C$ *independent of* $\epsilon$.

*Proof.* Again, one may check that $\psi(x) = \phi(-x)$ also satisfies (13) and (14) and thus $\psi = \phi$ is an even function.

Differentiating $\epsilon^2\phi'' = B(\phi)$ with respect to $x$ and multiplying by $\phi'$ we find

$$\epsilon^2\phi'''\phi' = B'(\phi)(\phi')^2 = A(\phi)(\phi')^2.$$

Due to evenness, $\phi'(-x) = -\phi'(x)$ and $\phi''(x) = \phi''(-x)$ for all $x \in (-1, 1)$. Integrating the above expression over $(-a, a)$ for $0 < a < 1$ gives

$$2\epsilon^2\phi''(a)\phi'(a) = \epsilon^2\phi''\phi'\Big|_{-a}^{a} = \int_{-a}^{a} (\phi')^2 + A(\phi)(\phi')^2\,dx.$$

The right hand side is nonnegative so that $\phi''(a)\phi'(a) \geq 0$ for all $0 < a < 1$. In particular, this implies that $\phi''$ does not change sign on $(0, 1)$. Since

$$\alpha - \beta = \epsilon^2 \int_{-1}^{1} \phi''\,dx = 2\epsilon^2 \int_{-1}^{0} \phi''\,dx,$$

$\phi'' < 0$ if $\alpha < \beta$. Thus $\phi$ is convex and positive with maximum $\phi(0)$.

Integrating $\epsilon^2\phi'' = B(\phi)$ over $(-1, 1)$ with $\phi'(-1) = -\phi(1)$ will give (21). Mutliplying $\epsilon^2\phi'' = B(\phi)$ by $\phi'$ we find

$$\frac{\epsilon^2}{2}((\phi')^2)' = (A(\phi))'.$$

Integrating this expression over $(-1,0)$, $\phi'(0)=0$ and (21) imply that

$$\frac{(\alpha-\beta)^2}{8\epsilon^2} = \alpha\frac{1-e^{\phi(0)}}{\int_{-1}^1 e^{\phi}\,dx} + \beta\frac{1-e^{-\phi(0)}}{\int_{-1}^1 e^{-\phi}\,dx}.$$

Certainly, $\int_{-1}^1 e^{\phi}\,dx \le 2e^{\phi(0)}$ and $\int_{-1}^1 e^{-\phi}\,dx \ge 2e^{-\phi(0)}$, giving the upper bound

$$\frac{(\alpha-\beta)^2}{4\epsilon^2} \le \alpha e^{-\phi(0)} + \beta e^{\phi(0)} - (\alpha+\beta).$$

Taking $e^{-\phi(0)} < 1$,

$$\phi(0) \ge \ln\left(\frac{(\alpha-\beta)^2}{4\beta\epsilon^2} + 1\right).$$

□

**Theorem 6.** *Let $\phi \in C^\infty(\Omega)\cap C^0(\bar{\Omega})$ satisfy (13) and (14) for $\alpha<\beta$ and $\phi_0(-1) < \phi_0(1)$. Then there exist $\epsilon_* > 0$ such that for $\epsilon < \epsilon_*$, $\phi$ is convex and*

$$\max_{x\in(-1,1)} \phi(x) \ge \log(\epsilon^{-2}) + C_1 \tag{23}$$

*for some $C_1$ independent of $\epsilon$. Further,*

$$|\phi(y)-\phi(x)| \le \frac{C_2}{\epsilon}e^{-\eta/\epsilon}(\sinh(\eta y/\epsilon) - \sinh(\eta x/\epsilon)), \quad 0<x,y<1 \tag{24}$$

*for some $C_2, \eta > 0$ independent of $\epsilon$.*

**Remark 3.** The estimate (24) shows that for any compact subset $K$ of $(-1,1)$, the difference between any two values in this set converges exponentialy to zero, since $-1+\delta < x$ and $y \le 1-\delta$ for some $\delta$ depending only on $K$. Consequently, $\phi$ converges uniformly to a constant value, e.g. $\phi(0)$, on $K$.

*Proof.* By lemma 3, there exists $C$ independent of $\epsilon$ and $\epsilon_* > 0$, such that for all $\epsilon < \epsilon_*$, there exists a $y_0 \in (-1,1)$ for which $\phi(y_0) > \phi_0(1)$. In particular, $\phi$ has an interior maximum for some $x_0 \in (-1,1)$. Then

$$0 \ge \epsilon^2\phi''(x_0) = B(\phi)(x_0).$$

However, by lemma 1 (3), $B(\phi)$ is monotone with respect to $\phi$, so that $B(\phi)(y) \le B(\phi)(x_0) \le 0$ for all $y \in (-1,1)$. Consequently, $\phi$ is convex and $-\phi'(-1)$ and $\phi'(1)$ share the same sign. Thus $\max\{\phi'(-1), -\phi'(1)\} \ge \epsilon^{-2}(\beta-\alpha)/2$. Without loss of generality, assume that $\phi'(-1) = \max\{\phi'(-1), -\phi'(1)\}$ and $\phi_0(-1) = 0$. Following theorem 5, multiplying $\epsilon^2\phi'' = B(\phi)$ by $\phi'$ and integrating over $(-1,x_0)$, gives

$$\frac{(\alpha-\beta)^2}{8\epsilon^2} \le \frac{\epsilon^2}{2}(\phi'(-1))^2 = \alpha\frac{1-e^{\phi(x_0)}}{\int_\Omega e^{\phi}\,dx} + \beta\frac{1-e^{-\phi(x_0)}}{\int_\Omega e^{-\phi}\,dx}.$$

Then the bound $\phi(x_0) \ge \log(\epsilon^{-2}) + C$ follows exactly as in theorem 5.

Consider, $\epsilon^2((\phi')^2)''/2 = \epsilon^2(\phi'')^2 + A(\phi)(\phi')^2$. By ODE comparison, $(\phi')^2/2 \le v$ where $v = (\phi'(-1))^2/2[\exp(-\eta(x+1)/\epsilon) + \exp(\eta(x-1)/\epsilon)]$ for $\theta \ge \eta > 0$. Then

$$\begin{aligned} |\phi(y)-\phi(x)| &= \left|\int_x^y \phi'(s)\,ds\right| \le \left(\int_x^y (\phi'(s))^2\,ds\right)^{\frac{1}{2}} \\ &\le |\phi'(-1)|\int_x^y \exp(-\eta(s+1)/\epsilon) + \exp(\eta(s-1)/\epsilon)\,ds \\ &= |\phi'(-1)|\frac{2\epsilon}{\eta}e^{-\eta/\epsilon}(\sinh(\eta y/\epsilon) - \sinh(\eta x/\epsilon)). \end{aligned}$$

The theorem follows by noting that $|\phi'(-1)| \leq (\beta - \alpha)/\epsilon^2$ for $\epsilon \leq \epsilon_*$. □

4. **Conclusion.** In summary, we have demonstrated that the limiting behaviour of solutions to (13), (14) depends only the ratio of total charges. When $\alpha = \beta$, then solutions remain bounded and converge exponentially to the average value of the boudnary data. Further, the boundary layer is exponential of thickness $\epsilon$, so that $\phi$ may be expanded as a matched inner and outer asymptotic solution. If $\alpha \neq \beta$, then solutions diverge uniformly in the interior to $\pm\infty$ and form a boundary layer of thickness $\epsilon^2$. From theorem 6, solutions converge exponentially on compact sets to a constant value which grows at the order $\log(\epsilon^{-2})$. We are currently not able to show that this is also an upper bound for growth, which is strongly suggested by our numerical simulations. In figure 3 we see that the interior values of three representative cases are asymptotic to $\log(\epsilon^{-2})$.

Many of the results in this paper may be generalized to higher dimensions. In particular, lemma 3 holds for all dimensions and domains. With some additional control on the geometry of $\partial\Omega$, a result similar to theorem 6 follows for the case when $\phi_0$ is a constant. In the cases when $\Omega$ and $\phi_0$ are radially symmetric, most of the above arguments follow with slight modifications. Nevertheless, to be coherent, we have illustrated the properties for solutions of the one dimensional problem only. In [16], we will present, along with a proofs of theorems 2 and 3, these higher dimensional generalizations.

**Acknowledgements.** This paper was completed while the authors were visiting the Institute for Mathematics and its Applications at the University of Minnesota for the thematic program *Mathematics of Materials and Macromolecules*. We would like to thank the institute for its help and hospitality. We would also like to thank Professors Fang-Hua Lin, H. C. Chang, and Martin Bazant for many useful discussions.

## REFERENCES

[1] Bazant, M., Chu, K. and Bayly, B., *Current-voltage relations for electrochemical thin films*, SIAM J. Appl. Math, **65** (2005) 1463–1484.

[2] Bazant, M. and Squires, T., *Induced-charge electro-kinetic phenomena: Theory and microfluidic applications*, Phys. Rev. Lett., **92** 2004, 066101.

[3] Bazant, M., Thornton, K. and Armand, A., *Diffuse-charge dynamics in electrochemical systems*, Phys. Rev. E, **70**, (2004), 021506.

[4] Ben, Y. and Chang, H.-C., *A Spectral Theory for Miscible Fingering*, Phys of Fluids, **14**, (2002).

[5] Ben, Y. and Chang, H.-C., *Nonlinear Smoluchowski slip velocity and micro-votex generation*, J. Fluid Mech., **461** (2002), 229–238.

[6] Bonnefont, A., Argoul, F. and Bazant, M., *Analysis of diffuse-layer effects on time-dependent interfacial kinetics*, J. Electroanal. Chem., **500** (2001), 52–61.

[7] Brenner, M., Constantin, P., Kadanoff, L., Schenkel, A. and Venkataramani, S., *Diffusion, attraction and collapse*, Nonlinearity, **12** (1999), 1071–1098.

[8] Bruin, G., *Recent developments in electrokinetically driven analysis on microfabricated devices*, Electrophoresis, **21** (2000), 3931–3951.

[9] Chu, K. and Bazant, M., *Electrochemical thin films at and above the classical limiting current*, SIAM J. Appl. Math **65** (2005), 1485–1505.

[10] Dolnik, V. and Hutterer, K., *Capillary electrophoresis of protiens 1999 - 2001*, Electrophoresis, **22** (2001), 4163–4178.

[11] Dukhin, S., *Electrokinetic phenomena of the second kind and there applications*, Adv. Colloid Interface Sci., **35** (2001) 173–196.

[12] Lin, F. H. and Liu, C., *Nonparabolic Dissipative Systems, Modeling The Flow Of Liquid Crystals*, Comm. Pure Appl. Math., **XLVIII** (1995), 501–537.

[13] Minerick, A., Takhistov, P., Zhou, R. and Chang, H.-C., *Manipulation and Characterization of Red Blood Cells with AC Fields in Micro-Devices*, Electrophoresis, **24**, (2003).
[14] Probstein, R. F., *Physicochemical Hydrodynamics, An Introduction*, John Wiley and Sons, INC., (1994).
[15] Rubinstein, I., *Electro-Diffusion of Ions*, SIAM, Philadelphia, (1990).
[16] Ryham, R., Liu, C. and Wang, Z-Q., *Mathematical Modeling for Electro-Kinetic Fluids: Analysis and Simulation*, preprint, 2005.
[17] Schasfoort, R., Schlautmann, S., Hendrikse, L. and Van Den Berg, A., *Field-effect flow control for micrfabricated fluidic networks*, Science, **286** (1999), 942–945.
[18] Squires, T. and Bazant, M., *Induced-charge electro-osmosis*, J. Fluid Mech., **509** (2004), 217–252.
[19] Stone, H. and Kim, S., *Microfluidics: basic issues, applications, and challenges*, AICHE J., **47** (2001), 1250–1254.
[20] Thamida, S. and Chang, H.-C., *Nonlinear Electrokinetic Ejection and Entrainment due to Polarization at Nearly Insulated Wedges*, Phys Fluids, **14** (2002), 4315–4328.
[21] Xiao, Y. and Bhattacharya, K., *Modeling electromechanical properties of ionic polymers*, Proc. SPIE: Electroactive Polymers, Actuators and Devices - Smart Structures and Materials, (2001) 292–300.
[22] Xiao, Y. and Bhattacharya, K., *Interaction of oxygen vacancies with domain walls and its impact on fatigue in ferroelectric thin films*, Smart Structures and Materials 2004: Active Materials: Behavior and Mechanics Proc. SPIE, (2004).
[23] Xu, J. and Zikatanov, L., *A monotone finite element scheme for convection-diffusion equations*, Mathematics of Computation, **68** (1999), 1429–1446.

Received March 2005; revised September 2005.

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