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Multiple Bound States of Nonlinear Schrödinger Systems

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Abstract: This paper is concerned with existence of bound states for Schrödinger systems which have appeared as several models from mathematical physics. We establish multiplicity results of bound states for both small and large interactions. This is done by different approaches depending upon the sizes of the interaction parameters in the systems. For small interactions we give a new approach to deal with multiple bound states. The novelty of our approach lies in establishing a certain type of invariant sets of the associated gradient flows. For large interactions we use a minimax procedure to distinguish solutions by analyzing their Morse indices.

1. Introduction

In this paper we study standing wave solutions $(\Phi_1, \dots, \Phi_N) : \mathbb{R}^n \to \mathbb{C}^N$ of the time-dependent system of N coupled nonlinear Schrödinger equations given by

$$\begin{cases} -i\dfrac{\partial}{\partial t}\Phi_j = \Delta\Phi_j + \displaystyle\sum_{i=1}^N \beta_{ij}|\Phi_i|^2\Phi_j & \text{for } x \in \mathbb{R}^n,\ t > 0, \\ \Phi_j(x,t) \to 0 & \text{as } |x| \to +\infty,\ t > 0, \end{cases} \tag{1}$$

$j = 1, \dots, N$, where β_{ij} are constants satisfying $\beta_{ij} = \beta_{ji}$, $n = 2, 3$, $N \geq 2$.

A solitary wave of this system is a solution with $\Phi_j(x,t) = e^{i\lambda_j t}u_j(x)$, $j = 1, \dots, N$. This ansatz leads to the elliptic system

$$-\Delta u_j + \lambda_j u_j = \sum_{i=1}^N \beta_{ij}u_i^2 u_j \text{ in } \mathbb{R}^n,\quad u_j(x) \to 0 \text{ as } |x| \to \infty,\quad j = 1, \dots, N. \tag{2}$$

Throughout the paper, λ_j and β_{jj} are positive constants for all j. The system (1) appears in many physical problems, especially in nonlinear optics. Physically, the solution Φ_j

denotes the j^{th} component of the beam in Kerr-like photorefractive media ([1]). The positive constant β_{jj} is for self-focusing in the j^{th} component of the beam. The coupling constant β_{ij} $(i \neq j)$ is the interaction between the i^{th} and the j^{th} components of the beam. When $N = 2$, problem (1) also arises in the Hartree-Fock theory for a double condensate, i. e., a binary mixture of Bose-Einstein condensates in two different hyperfine states $|1\rangle$ and $|2\rangle$ ([13]). Physically, Φ_j are the corresponding condensate amplitudes, β_{jj} and β_{12} are the intraspecies and interspecies scattering lengths. The sign of the scattering length β_{12} determines whether the interactions of states $|1\rangle$ and $|2\rangle$ are repulsive or attractive. For more references we refer the reader to [1,10,13–17,23,27].

The goal of this paper is to establish multiplicity results of the above nonlinear Schrödinger system (2) for small interaction constants (both positive and negative) and for large positive interaction constants. The existence theory of solutions has received great interest recently, see [2,3,7,8,18,21,26] for the existence of a ground state or bound state solution, [19,20,22,24] for semiclassical states or singularly perturbed settings, and [11,28,29] for existence of multiple solutions of two coupled equations with negative interactions. Our paper is devoted to establishing multiplicity of nontrivial bound state solutions (see Theorems 2.1 and 3.1). This is done for the parameter regime of both small interaction constants (both positive and negative) and large positive interaction constants (i.e., $\beta_{ij}, i \neq j$), a topic which has not been studied in the above works.

The difficulty in obtaining multiplicity results of nontrivial bound state solutions lies in distinguishing nontrivial solutions (we will call a solution nontrivial if all components of the solution are non-zero) from semi-nontrivial solutions (i.e., solutions where at least one component is zero). We give existence results of nontrivial solutions both for small interaction parameters $\beta_{ij}, i \neq j$ for a general N system with $N \geq 2$ and for large interactions for the case $N = 2$. In the case of small interactions we make use of invariant sets methods for gradient flows to distinguish nontrivial and semi-trivial solutions. For the case of large interactions, we use estimates of Morse indices of solutions to separate solutions. The reason for the interactions to be small or large may be justifiable in light of the nonexistence results of nontrivial positive solutions in [8] in which it is proved that there exist two positive numbers $\beta' < \beta''$ such that the two-system (system (2) with $N = 2$) has nontrivial positive solutions if and only if β_{12} is less than β' or greater than β''. Thus it seems to be natural to require β_{ij} to be small or large to obtain existence of nontrivial solutions. However, we do not know whether for some parameters the system (1) has no nontrivial solutions at all.

The paper is organized as follows. Section 2 is devoted to the case of small interactions and Sect. 3 to the case of large interactions. We finish each section with a few concluding remarks.

2. Small Interaction Constants

In this section we consider small interactions (both positive and negative) between the components. We assume $N \geq 2$.

Theorem 2.1. *Let $n = 2, 3$ and let λ_j and β_{jj} be fixed positive constants. Then for any $k \in \mathbb{N}$, there exists $\beta_k > 0$ such that for $|\beta_{ij}| \leq \beta_k$, $i \neq j$, the system*

$$\begin{cases} -\Delta u_j + \lambda_j u_j = \sum_{i=1}^{N} \beta_{ij} u_i^2 u_j, & \text{in } \mathbb{R}^n, \\ u_j(x) \to 0, \ \text{as } |x| \to \infty, & j = 1, 2, \cdots, N \end{cases} \tag{3}$$

has as least k pairs of nontrivial spherically symmetric solutions.

Proof. Let $E = H_r^1(\mathbb{R}^n)$ consist of spherically symmetric functions in $H^1(\mathbb{R}^n)$ in which we use the equivalent norms

$$\|u\|_j = \left(\int_{\mathbb{R}^n} |\nabla u|^2 + \lambda_j u^2\right)^{1/2}, \quad j = 1, 2, \cdots, N.$$

The product space $E^N = \overbrace{E \times E \times \cdots \times E}^{N}$ is endowed with the norm

$$\|\vec{u}\| = (\sum_{j=1}^{N} \|u_j\|_j^2)^{1/2}, \quad \vec{u} = (u_1, u_2, \cdots, u_N) \in E^N.$$

Spherically symmetric solutions of (3) correspond to critical points of the functional

$$J(\vec{u}) = \frac{1}{2}\|\vec{u}\|^2 - \frac{1}{4}\sum_{i,j=1}^{N} \beta_{ij} \int_{\mathbb{R}^n} u_i^2 u_j^2, \quad \vec{u} = (u_1, u_2, \cdots, u_N) \in E^N.$$

Note that $J \in C^2(E^N)$ and

$$\nabla J(\vec{u}) = \vec{u} - A(\vec{u}), \quad \vec{u} = (u_1, u_2, \cdots, u_N) \in E^N,$$

where $A(\vec{u}) = ((A(\vec{u}))_1, (A(\vec{u}))_2, \cdots, (A(\vec{u}))_N)$ and

$$(A(\vec{u}))_j = (-\Delta + \lambda_j I)^{-1}(\sum_{i=1}^{N} \beta_{ij} u_i^2 u_j).$$

Let $\varphi^t(\vec{u})$ with the maximal interval of existence $[0, \eta(\vec{u}))$ be the solution of the initial value problem

$$\begin{cases} \frac{d}{dt}\varphi^t = -\nabla J(\varphi^t), & \text{for } t \geq 0, \\ \varphi^0 = \vec{u}. \end{cases}$$

We say $\varphi : \{(t, \vec{u}) |\ \vec{u} \in E^N, t \in [0, \eta(\vec{u}))\} \to E^N$ is the gradient flow of J. A subset $\mathcal{F}$ of E^N is said to be an invariant set for the flow if $\varphi(t, \vec{u}) \in \mathcal{F}$ for all $\vec{u} \in \mathcal{F}$ and $t \in [0, \eta(\vec{u}))$. For two invariant sets $\mathcal{F} \subset \mathcal{G}$, we say $\mathcal{F}$ is strictly invariant with respect to $\mathcal{G}$ if $\varphi(t, \vec{u}) \in \mathrm{int}_{\mathcal{G}}\mathcal{F}$ for all $\vec{u} \in \mathcal{F}$ and $t \in (0, \eta(\vec{u}))$, where $\mathrm{int}_{\mathcal{G}}\mathcal{F}$ is the interior of $\mathcal{F}$ in $\mathcal{G}$. Define

$$\mathcal{A}_0 = \{\vec{u} \in E^N \mid \lim_{t \to \eta(\vec{u})-0} \varphi^t(\vec{u}) = 0\}.$$

Since 0 is a strict local minimizer of J, $\mathcal{A}_0$ is an open subset of E^N. We claim that $\partial\mathcal{A}_0$ is an invariant set. Otherwise, there would be $\vec{u}_0 \in \partial\mathcal{A}_0$ and $t_0 > 0$ such that $\varphi^{t_0}(\vec{u}_0) \notin \partial\mathcal{A}_0$. Then either $\varphi^{t_0}(\vec{u}_0) \in \mathcal{A}_0$ or $\varphi^{t_0}(\vec{u}_0) \in E^N \setminus \overline{\mathcal{A}_0}$. If $\varphi^{t_0}(\vec{u}_0) \in \mathcal{A}_0$ then $\lim_{t \to \eta(\vec{u}_0)-0} \varphi^t(\vec{u}_0) = 0$ and $\vec{u}_0 \in \mathcal{A}_0$, a contradiction. If $\varphi^{t_0}(\vec{u}_0) \in E^N \setminus \overline{\mathcal{A}_0}$, then the continuous dependence of φ on initial data would imply that $\varphi^{t_0}(\vec{u}_1) \in E^N \setminus \overline{\mathcal{A}_0}$ for any $\vec{u}_1 \in \mathcal{A}_0$ sufficiently close to $\vec{u}_0$, which is also a contradiction.

Of course, $\mathcal{A}_0$ depends on β_{ij}, $i \neq j$. Assume $|\beta_{ij}| \leq \beta^*$ $(i \neq j)$ and β^* is small enough such that $b_1 I \leq (\beta_{ij}) \leq b_2 I$, where $b_1 = \frac{1}{2}\min\{\beta_{11}, \cdots, \beta_{NN}\}$, $b_2 = 2\max\{\beta_{11}, \cdots, \beta_{NN}\}$, and I is the $N \times N$ identity matrix. Then a careful inspection shows that there exist $r > 0$ and $\alpha > 0$ independent of β_{ij} $(i \neq j)$ such that $B_r(0) \subset \mathcal{A}_0$ and $\inf_{\partial \mathcal{A}_0} J \geq \alpha$, and for any finitely dimensional subspace F of E^N, there exists $R > 0$ independent of β_{ij} $(i \neq j)$ such that $\mathcal{A}_0 \cap F \subset \{\vec{u} \mid \|\vec{u}\| \leq R\}$.

Now fix $k \in \mathbb{N}$ and let E_1 be a k-dimensional subspace of E. Define

$$c_k = \sup_{E_1^N} \tilde{J},$$

where $\tilde{J}$ is the functional J in which (β_{ij}) is replaced with $b_1 I$. Clearly, $c_k < \infty$, c_k is independent of β_{ij} $(i \neq j)$, and for $|\beta_{ij}| \leq \beta^*$ $(i \neq j)$,

$$\sup_{E_1^N} J \leq c_k.$$

For any $\vec{u} \in E^N$, if $J(\vec{u}) \leq c_k$ and $\|\nabla J(\vec{u})\| \leq 1$ then

$$4c_k + \|\vec{u}\| \geq 4J(\vec{u}) - (\nabla J(\vec{u}), \vec{u}) \geq \|\vec{u}\|^2,$$

which implies $\|\vec{u}\| < 2 + c_k$. Therefore,

$$\|\nabla J(\vec{u})\| > 1, \qquad \text{if } J(\vec{u}) \leq c_k \text{ and } \|\vec{u}\| \geq 2 + c_k. \tag{4}$$

We show that

$$\|\varphi^t(\vec{u})\| \geq 2 + c_k, \qquad \text{if } 0 < J(\vec{u}) \leq c_k \text{ and } \|\vec{u}\| \geq 2 + 2c_k \text{ and } J(\varphi^t(\vec{u})) \geq 0. \tag{5}$$

If not then there would be $\vec{u}$ and $t_0 > 0$ such that $0 < J(\vec{u}) \leq c_k$, $\|\vec{u}\| \geq 2 + 2c_k$, and $J(\varphi^{t_0}(\vec{u})) \geq 0$, but $\|\varphi^{t_0}(\vec{u})\| < 2 + c_k$. Let $t_1 \in (0, t_0)$ be such that $\|\varphi^{t_1}(\vec{u})\| = 2 + c_k$ and $\|\varphi^t(\vec{u})\| > 2 + c_k$ for $t \in (0, t_1)$. Then a contradiction can be obtained as

$$\begin{aligned} 2 + c_k = \|\varphi^{t_1}(\vec{u})\| &\geq \|\vec{u}\| - \|\vec{u} - \varphi^{t_1}(\vec{u})\| \\ &\geq 2 + 2c_k - \int_0^{t_1} \|\nabla J(\varphi^t(\vec{u}))\| dt \\ &\geq 2 + 2c_k - \int_0^{t_1} \|\nabla J(\varphi^t(\vec{u}))\|^2 dt \\ &= 2 + 2c_k - (J(\vec{u}) - J(\varphi^{t_1}(\vec{u}))) \\ &> 2 + 2c_k - (J(\vec{u}) - J(\varphi^{t_0}(\vec{u}))) \geq 2 + c_k. \end{aligned}$$

Using (4) and (5), a standard argument shows that if $0 < J(\vec{u}) \leq c_k$ and $\|\vec{u}\| \geq 2 + 2c_k$ then there exists $t > 0$ such that $J(\varphi^t(\vec{u})) \leq 0$. Therefore, since $\partial \mathcal{A}_0$ is an invariant set and $\inf_{\partial \mathcal{A}_0} J \geq \alpha > 0$, if $\vec{u} \in \partial \mathcal{A}_0$ and $J(\vec{u}) \leq c_k$ then $\|\vec{u}\| < 2 + 2c_k$. Or, in other word,

$$\partial \mathcal{A}_0 \cap J^{c_k} \subset \operatorname{int}(B_{2+2c_k}(0)), \tag{6}$$

where $J^{c_k} = \{\vec{u} \mid \vec{u} \in E^N,\ J(\vec{u}) \le c_k\}$. For $\|\vec{u}\| \le 4 + 4c_k$, we have

$$\begin{aligned}
\|(A(\vec{u}))_j\|_j &= C_1 \|(-\Delta + \lambda_j I)^{-1}(\sum_{i=1}^{N} \beta_{ij} u_i^2 u_j)\|_{W^{2,2}(\mathbb{R}^n)} \\
&\le C_2 \|\sum_{i=1}^{N} \beta_{ij} u_i^2 u_j\|_{L^2(\mathbb{R}^n)} \\
&\le C_3 \sum_{i=1}^{N} |\beta_{ij}| \|u_i\|_i^2 \|u_j\|_j \\
&\le C_3 \beta_{jj} \|u_j\|_j^3 + 16 C_3 \max\{|\beta_{ij}| :\ i \ne j\}(1 + c_k)^2 \|u_j\|_j,
\end{aligned}$$

where C_i are constants independent of $\vec{u}$, k, and β_{ij}. Let $\beta_k = \min\{\beta^*, \frac{1}{64C_3(1+c_k)^2}\}$ and fix β_{ij} with $|\beta_{ij}| \le \beta_k$ for $i \ne j$. There exists $\varepsilon_k \in (0, r/\sqrt{N})$ such that if $\|\vec{u}\| \le 4+4c_k$ and $0 < \|u_j\|_j \le \varepsilon_k$ then

$$\|(A(\vec{u}))_j\|_j < \|u_j\|_j.$$

Define, for $\varepsilon > 0$ and $j = 1, 2, \cdots, N$,

$$\mathcal{D}_j^{\varepsilon} = \{\vec{u} \mid \vec{u} = (u_1, u_2, \cdots, u_N) \in E^N,\ \|u_j\|_j \le \varepsilon\}.$$

Then for $0 < \varepsilon \le \varepsilon_k$,

$$A(\mathcal{D}_j^{\varepsilon} \cap B_{4+4c_k}(0)) \subset \mathrm{int}(\mathcal{D}_j^{\varepsilon}), \quad j = 1, 2, \cdots, N.$$

According to [12, §4], for any $\vec{u} \in \mathcal{D}_j^{\varepsilon} \cap B_{3+3c_k}(0)$ and any $0 < \varepsilon \le \varepsilon_k$ there exists $t_0 > 0$ such that $\varphi^t(\vec{u}) \in \mathrm{int}(\mathcal{D}_j^{\varepsilon}) \cap B_{4+4c_k}(0)$ for $t \in (0, t_0)$. This observation together with (6) implies that $\varphi^t(\vec{u}) \in \partial\mathcal{A}_0 \cap J^{c_k} \cap \mathrm{int}(\mathcal{D}_j^{\varepsilon})$ if $\vec{u} \in \partial\mathcal{A}_0 \cap J^{c_k} \cap \mathcal{D}_j^{\varepsilon}$ and $t \in (0, \eta(\vec{u}))$ for any $j = 1, 2, \cdots, N$, $0 < \varepsilon \le \varepsilon_k$. Therefore, $\partial\mathcal{A}_0 \cap J^{c_k} \cap \mathcal{D}_j^{\varepsilon}$, $j = 1, 2, \cdots, N$, $0 < \varepsilon \le \varepsilon_k$, are strictly invariant sets with respect to $\partial\mathcal{A}_0 \cap J^{c_k}$. Define

$$\mathcal{A}_1 = \{\vec{u} \in \partial\mathcal{A}_0 \cap J^{c_k} \mid \exists\, t > 0 \text{ such that } \varphi^t(\vec{u}) \in \cup_{j=1}^{N} \mathrm{int}(\mathcal{D}_j^{\varepsilon_k})\}.$$

Then $\mathcal{A}_1$ is an open subset of $\partial\mathcal{A}_0 \cap J^{c_k}$, and $\partial\mathcal{A}_0 \cap J^{c_k} \setminus \mathcal{A}_1$ is closed and invariant for the flow. We want to prove $\mathrm{gen}(\partial\mathcal{A}_0 \cap J^{c_k} \setminus \mathcal{A}_1) \ge k$, where $\mathrm{gen}(\cdot)$ is the genus of a closed symmetric subset of E^N. Note that

$$E_1^N \cap \partial\mathcal{A}_0 \setminus \mathcal{A}_1 \subset J^{c_k} \cap \partial\mathcal{A}_0 \setminus \mathcal{A}_1.$$

For $\vec{u} \in \mathcal{A}_1$, since $\partial\mathcal{A}_0 \cap J^{c_k} \cap \mathcal{D}_j^{\varepsilon}$, $j = 1, 2, \cdots, N$, $0 < \varepsilon \le \varepsilon_k$, are strictly invariant sets with respect to $\partial\mathcal{A}_0 \cap J^{c_k}$, there exists $t > 0$ such that $\varphi^t(\vec{u}) \in \cup_{j=1}^{N} \mathcal{D}_j^{\varepsilon_k/2}$ and the function $\tau : \mathcal{A}_1 \to \mathbb{R}^+$ defined by

$$\tau(\vec{u}) = \inf\{t \ge 0 :\ \varphi^t(\vec{u}) \in \cup_{j=1}^{N} \mathcal{D}_j^{\varepsilon_k/2}\}$$

is even and continuous. Define a map $h : E_1^N \cap \mathcal{A}_1 \to E_1^N$ as

$$h(\vec{u}) = (\gamma_1(\vec{u})u_1, \gamma_2(\vec{u})u_2, \cdots, \gamma_N(\vec{u})u_N),$$

where

$$\gamma_i(\vec{u}) = \begin{cases} 1, & \text{if } |\varphi_i^{\tau(\vec{u})}(\vec{u})\|_i \geq \varepsilon_k, \\ \frac{2}{\varepsilon_k}\|\varphi_i^{\tau(\vec{u})}(\vec{u})\|_i - 1, & \text{if } \varepsilon_k/2 \leq \|\varphi_i^{\tau(\vec{u})}(\vec{u})\|_i < \varepsilon_k, \\ 0, & \text{if } \|\varphi_i^{\tau(\vec{u})}(\vec{u})\|_i < \varepsilon_k/2, \end{cases}$$

and φ_i^t is the i^{th} component of φ^t. Then $h : E_1^N \cap \mathcal{A}_1 \to E_1^N$ is odd and continuous. For any $\vec{u} \in E_1^N \cap \mathcal{A}_1$, the definition of h implies that at least one component of $h(\vec{u})$ is zero. On the other hand, since $B_r(0) \subset \mathcal{A}_0$ and $\varepsilon_k \in (0, r/\sqrt{N})$ and $\varphi^{\tau(\vec{u})}(\vec{u}) \in \mathcal{A}_1 \subset \partial\mathcal{A}_0$, at least one component of $h(\vec{u})$ is nonzero. Define

$$W = \{\vec{u} \in E_1^N : u_1 = u_2 = \cdots = u_N\}$$

and let V be the orthogonal complement of W in E_1^N. Thus any $\vec{u} \in E_1^N$ can be uniquely decomposed as $\vec{u} = \vec{w} + \vec{v},\ \vec{w} \in W,\ \vec{v} \in V$. Define $g : E_1^N \to V$ as $g(\vec{u}) = \vec{v}$. Since $W \cap h(E_1^N \cap \mathcal{A}_1) = \emptyset$, $g \circ h : E_1^N \cap \mathcal{A}_1 \to V \setminus \{0\}$ is odd and continuous. Therefore

$$\text{gen}(E_1^N \cap \mathcal{A}_1) \leq \dim V = (N-1)k,$$

which implies

$$\begin{aligned} \text{gen}(J^{c_k} \cap \partial\mathcal{A}_0 \setminus \mathcal{A}_1) &\geq \text{gen}(E_1^N \cap \partial\mathcal{A}_0) - \text{gen}(E_1^N \cap \mathcal{A}_1) \\ &\geq Nk - (N-1)k = k. \end{aligned}$$

Define for $i = 1, 2, \cdots, k$,

$$d_i = \inf_{A \in \Sigma_i} \sup_{\vec{u} \in A} J(\vec{u}),$$

where

$$\Sigma_i = \{A \mid A \subset J^{c_k} \cap \partial\mathcal{A}_0 \setminus \mathcal{A}_1,\ \text{gen}(A) \geq i\}.$$

Now standard arguments (see, for example, [25]) can be used to obtain the conclusion. □

Remark 2.2. a) The result holds for general potential $V_j(x)$ in place of λ_j for $j = 1, \ldots, N$. We only need $\inf_{\mathbb{R}^n} V_j > 0$ and some sort of compactness condition such that the embedding from $E_j := \{u \in H^1(\mathbb{R}^n) | \int V_j u^2 < \infty\}$ into $L^q(\mathbb{R}^n)$ is compact for all $q \in [2, 2^*)$. This is assured for example by assuming $V_j(x) \to \infty$ as $|x| \to \infty$. A more general one (e.g., [6]) is that there is $r > 0$ such that for each $b > 0$, $\lim_{|y|\to\infty} \mu(\{x \in \mathbb{R}^n | V_j(x) \leq b\} \cap B_r(y)) = 0$, where $\mu(\cdot)$ is the Lebesgue measure in $\mathbb{R}^n$.

b) Our methods allow extensions of the results to more general nonlinearities. We consider, for example, the system

$$\begin{cases} -\Delta u_j + \lambda_j u_j = \frac{\partial F}{\partial u_j}(u_1, u_2, \cdots, u_N), & \text{in } \mathbb{R}^n, \\ u_j(x) \to 0, \text{ as } |x| \to \infty, & j = 1, 2, \cdots, N, \end{cases} \tag{7}$$

where $n = 2, 3$, $\lambda_j > 0$, and

$$F(u) = \sum_{i,j=1}^{N} \sum_{l=1}^{m_{ij}} \beta_{ijl} |u_i|^{r_{ijl}} |u_j|^{s_{ijl}}$$

with $m_{ij} \in \mathbb{N}$, $r_{ijl} \geq 2$, $s_{ijl} \geq 2$, and $r_{ijl} + s_{ijl} < 6$ in case $n = 3$. We define

$$\mathscr{S} = \{(i, j, l) \mid l = 1, 2, \cdots, m_{ij} \text{ and } i, j = 1, 2, \cdots, N\}$$

and

$$\mathscr{T} = \{(i, j, l) \in \mathscr{S} \mid i = j\} \cup \{(i, j, l) \in \mathscr{S} \mid r_{ijl} > 2,\ s_{ijl} > 2\}.$$

The proof of Theorem 2.1 can be adapted to prove the following result. Assume that $(r_{ijl}+s_{ijl})$ is a constant independent of (i, j, l). Let $\beta_{ijl} > 0$ be fixed for all $(i, j, l) \in \mathscr{T}$. Then for any $k \in \mathbb{N}$, there exists $\beta_k > 0$ such that if $|\beta_{ijl}| \leq \beta_k$ for all $(i, j, l) \in \mathscr{S} \setminus \mathscr{T}$, (7) has as least k pairs of nontrivial spherically symmetric solutions.

Similar results hold for a system in which the potential takes the form in a) and the nonlinearity takes the form above.

c) We do not know how small β_k's are, but it seems β_k is non-increasing.

3. Large Interaction Constants

In this section we consider large interaction parameters. We will deal with only the case of two coupled equations, i.e., $N = 2$. Rewriting $\mu_i = \beta_{ii}$ and $\beta = \beta_{12}$ we have a system

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 u^3 + \beta v^2 u, & \text{in } \mathbb{R}^n, \\ -\Delta v + \lambda_2 v = \mu_2 v^3 + \beta u^2 v, & \text{in } \mathbb{R}^n, \\ u(x) \to 0, \quad v(x) \to 0, & \text{as } |x| \to \infty. \end{cases} \tag{8}$$

Theorem 3.1. *Let $n = 2, 3$ and for $i = 1, 2$ let λ_i and μ_i be fixed positive constants. Then for any $k \in \mathbb{N}$, there exists $\beta^k > 0$ such that for $\beta > \beta^k$ the system (8) has at least k pairs of nontrivial spherically symmetric solutions.*

For the proof of this theorem we need some preliminaries and preparation. First let us recall that solutions of (8) correspond to critical points of the C^2 functional on E^2 with $E = H_r^1(\mathbb{R}^n)$

$$\begin{aligned} J((u, v)) = &\frac{1}{2}\int_{\mathbb{R}^n} (|\nabla u|^2 + |\nabla v|^2) + \frac{1}{2}\int_{\mathbb{R}^n} (\lambda_1 u^2 + \lambda_2 v^2) \\ &- \frac{1}{4}\int_{\mathbb{R}^n} \mu_1 u^4 - \frac{1}{4}\int_{\mathbb{R}^n} \mu_2 v^4 - \frac{\beta}{2}\int_{\mathbb{R}^n} u^2 v^2. \end{aligned} \tag{9}$$

Proposition 3.2. *For any $\beta > 0$, (8) has a sequence of solutions (u_k, v_k) such that $J((u_k, v_k)) \to \infty$ as $k \to \infty$ and the Morse index of (u_k, v_k) is no more than k.*

Proof. The existence of a sequence of solutions follows from standard methods. By the symmetric mountain pass theorem or by a linking argument ([4]) we get the existence of a sequence of critical points of J having critical values tending to infinity. This sequence of solutions can also be constructed as critical points of a constraint problem, namely, critical points of

$$I((u, v)) = \int_{\mathbb{R}^n} |\nabla u|^2 + \lambda_1 u^2 + |\nabla v|^2 + \lambda_2 v^2$$

subject to

$$M = \{(u, v) \in E^2 \mid (u, v) \neq (0, 0), (\nabla J((u, v)), (u, v)) = 0\}.$$

Let (c_k) be the sequence of critical values of I, which tends to infinity, characterized as

$$c_k = \inf_{A \in \Gamma_k} \sup_{(u,v) \in A} I((u, v)),$$

where $\Gamma_k = \{A \subset M \mid A = -A,\ \text{closed}, \text{gen}(A) \geq k\}$. By the arguments in Chang's book [9], Chap. 2, Sect. 2.2, if $K_{c_k} = \{(u, v) \in M \mid I(u, v) = c_k, \nabla I(u, v) = 0\}$ is isolated then there exists a critical point (u_k, v_k) such that its Morse index is no more than $k - 1$.

In general we use a perturbation argument. Let $A := K_{c_k-1}^{c_k+1} = \{(u, v) \in M \mid c_k - 1 \leq I(u, v) \leq c_k + 1, \nabla I(u, v) = 0\}$ then A is compact. If the Morse indices of all critical points of A are greater than k, for each $(u, v) \in A$ there is $a > 0$ such that the linearized eigenvalue problem at (u, v) has at least $k+1$ eigenvalues less than or equal to $-a$. Due to compactness of A, there exist $\delta > 0$ and $a_0 > 0$ such that for all $(u, v) \in A_\delta$, the δ-neighborhood of A, the linearized eigenvalue problem at (u, v) has at least $k + 1$ eigenvalues less than or equal to $-a_0$. Choose $\varepsilon > 0$ such that $\varepsilon < \frac{1}{4} \max\{a_0, 1\}$. Fix these $\delta > 0$ and $\varepsilon > 0$. Then we perturb I to an even functional $\tilde{I}$ which agrees with I outside A_δ, $\|I - \tilde{I}\|_{C^2} \leq \varepsilon$ and $\tilde{I}$ has only non-degenerate critical points in A_δ. For this perturbed functional we apply the minimax procedure again to define $\tilde{c}_k$. Then $c_k - \varepsilon \leq \tilde{c}_k \leq c_k + \varepsilon$ by the property of $\tilde{I}$. Thus we may assume for $\tilde{I}$, $K_{\tilde{c}_k}$ is isolated. Then there exists a critical point of $\tilde{I}$ at this level whose Morse index is no more than k. This is a contradiction since this critical point of $\tilde{I}$ is in A_δ and its Morse index has to be at least $k + 1$. □

Proposition 3.3. *Consider*

$$-\Delta U = c_0 |U|^{p-1} U, \quad U \in C_b^2(\mathbb{R}^n), \tag{10}$$

where $1 < p < 2^* - 1$, $c_0 > 0$ and $C_b^2(\mathbb{R}^n)$ consists of bounded C^2 functions. Then any radial solution of (10) with finite Morse index (with respect to the class of radial functions) must be zero. I.e., if $i(U) < \infty$, then $U = 0$, where $i(U)$ is the number of negative eigenvalues of $-\Delta - c_0 p |U|^{p-1}$ in $H_r^2(\mathbb{R}^n)$.

Proof. This was proved in [5] for the equation considered without radial symmetry. Checking the proofs there one sees that all the arguments are still valid when confined in the class of radially symmetric functions. □

Proposition 3.4. *Consider*

$$-\Delta u + \lambda_1 u = \mu_1 u^3, \tag{11}$$

and define for each integer k,

$$\mathcal{S}_k^1 := \{u \in E \mid u \text{ is a solution of (11) with Morse index no more than } k\}.$$

Then for any k, $\mathcal{S}_k^1$ is bounded in $L^\infty(\mathbb{R}^n)$.

There is a similar statement by defining $\mathcal{S}_k^2$ for the second equation,

$$-\Delta u + \lambda_2 u = \mu_2 u^3.$$

Proof. First, for any $u \in \mathcal{S}_k^1$, $|u(0)|$ is a global maximum of $|u(x)|$. In fact, if $s \geq 0$ is a zero of u' then multiplying the equation $-u'' - (n-1)u'/r + \lambda_1 u = \mu_1 u^3$ with u' and taking the integral from s to ∞ yields $|u(s)| > \sqrt{2\lambda_1/\mu_1}$. Let $s_2 > s_1 \geq 0$ be two local maximizers of $|u|$. By multiplying the equation with u' and taking the integral from s_1 to s_2 we have $(u^2(s_2) - u^2(s_1))(2\lambda_1 - \mu_1(u^2(s_2) + u^2(s_1))) > 0$ which together with $|u(s_i)| > \sqrt{2\lambda_1/\mu_1}$ implies $|u(s_1)| > |u(s_2)|$.

We prove the proposition by contradiction. Let us assume there is a sequence $u_m \in \mathcal{S}_k^1$ such that $M_m := |u_m(0)| = \max_{\mathbb{R}^n} |u_m| \to \infty$ as $m \to \infty$. Then we define

$$w_m(x) = M_m^{-1} u_m(M_m^{-1} x),$$

which satisfies the following equation

$$-\Delta w_m + M_m^{-2} \lambda_1 w_m = \mu_1 w_m^3.$$

By elliptic estimates, for a subsequence we have $w_m \to U$ in $C^2_{loc}(\mathbb{R}^n)$ as $m \to \infty$ and U is a non-zero solution in $C_b^2(\mathbb{R}^n)$ of (10) in the case $p = 3$. Since $i(u_m) \leq k$ we have $i(U) \leq k$. Thus $U = 0$, which is a contradiction with $|U(0)| = 1$. □

Let i be fixed. For $w \in \mathcal{S}_i^1$ and for $l = 1, 2, \ldots$, let $\nu_l(w)$ denote the l^{th} eigenvalue of the following weighted eigenvalue problem in E:

$$-\Delta\varphi + \lambda_2 \varphi = \nu_l(w) w^2 \varphi. \tag{12}$$

Proposition 3.5. *For each i and l, there exists $C_{i,l}^1 > 0$ such that $\nu_l(w) \leq C_{i,l}^1$ for all $w \in \mathcal{S}_i^1$.*

Again a similar statement can be proved for the eigenvalue problem $-\Delta\varphi + \lambda_1 \varphi = \nu_l(w) w^2 \varphi$, where $w \in \mathcal{S}_i^2$, and, accordingly, a bound $C_{i,l}^2$ can be obtained such that $\nu_l(w) \leq C_{i,l}^2$.

Proof. Note that $|w(0)| > \sqrt{2\lambda_1/\mu_1}$ for all $w \in \mathcal{S}_i^1$. From Proposition 3.4, the elliptic estimate shows that $\mathcal{S}_k^1$ is bounded in $C^1(\mathbb{R}^n)$. Therefore, there exist $r > 0$ and $a_0 > 0$ such that $|w(x)| \geq a_0$ for $x \in B_r(0)$ and all $w \in \mathcal{S}_i^1$. Now since $\nu_l(w)$ can be characterized by

$$\nu_l(w) = \inf_{F \subset E,\ \dim F = l} \ \sup_{\varphi \in F,\ \varphi \neq 0} \frac{\int_{\mathbb{R}^n} |\nabla\varphi|^2 + \lambda_2 \varphi^2}{\int_{\mathbb{R}^n} w^2 \varphi^2},$$

we see

$$\nu_l(w) \leq C_{i,\,l}^1 := \frac{1}{a_0^2} \inf_{F \subset H^1_{0,\,r}(B_r(0)),\ \dim F = l} \ \sup_{\varphi \in F,\ \varphi \neq 0} \frac{\int_{\mathbb{R}^n} |\nabla\varphi|^2 + \lambda_2 \varphi^2}{\int_{\mathbb{R}^n} \varphi^2}.$$

□

Finally we complete the proof of Theorem 3.1.

Proof of Theorem 3.1. Let k be fixed. We show that there is a $\beta^k > 0$ such that for $\beta > \beta^k$, (u_i, v_i) for $i = 1, \dots, k$ in Proposition 3.2 are all nontrivial, i.e., all components of (u_i, v_i) are nonzero for $i = 1, 2, \dots k$. We need to rule out the possibility that (u_i, v_i) is of the form $(w_1, 0)$ or $(0, w_2)$ with w_j $(j = 1, 2)$ being a nonzero solution of $-\Delta w_j + \lambda_j w_j = \mu_j w_j^3$. We will analyze the Morse indices of these semi-trivial solutions and compare them with the Morse index of (u_i, v_i) which is no more than k for $i = 1, 2, \dots, k$.

First, if w_1 is a nonzero solution to $-\Delta u + \lambda_1 u = \mu_1 u^3$ having Morse index m such that $m > k$, it is easy to see that $(w_1, 0)$ as a solution of the system (8) has Morse index at least m for functional J and this is independent of the size of β.

Next we claim that there is β^k such that for $\beta > \beta^k$, any critical point of J of the form $(w_1, 0)$ or $(0, w_2)$ with $w_1 \in \mathcal{S}_k^1$ and $w_2 \in \mathcal{S}_k^2$ have Morse index greater than k. This follows from Proposition 3.5. In fact, when we linearize the system (8) at a solution of the form $(w_1, 0)$ with $w_1 \in \mathcal{S}_k^1$ we have

$$\begin{cases} \Delta\varphi_1 - \lambda_1\varphi_1 + 3\mu_1 w_1^2\varphi_1 = 0, & \varphi_1 \in E, \\ \Delta\varphi_2 - \lambda_2\varphi_2 + \beta w_1^2\varphi_2 \quad = 0, & \varphi_2 \in E. \end{cases}$$

The first equation has an eigenvector corresponding to a negative eigenvalue.

According to Proposition 3.5 we see that for $\beta \geq C_{k,k}^1$ the Morse index of $(w_1, 0)$ as a solution of the system (8) is at least $k + 1$. Similarly, for $\beta \geq C_{k,\,k}^2$ the Morse index of $(0, w_2)$ as a solution of the system (8) is at least $k + 1$. If we set $\beta^k = \max\{C_{k,\,k}^1, C_{k,\,k}^2\}$, then for $\beta \geq \beta^k$, the critical point (u_i, v_i) for $i = 1, 2, \dots, k$ provided by Proposition 3.2 must be nontrivial since we know the Morse index of (u_i, v_i) is no more than k for $i = 1, 2, \dots, k$. $\square$

Remark 3.6. a) We do not know how large β^k's are.

b) The solutions given by our results likely are not one-sign solutions. But we do not have a more precise nodal property of the solutions.

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