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On the Existence of Multiple, Single-Peaked Solutions for a Semilinear Neumann Problem

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§ 0. Introduction

In this paper, we are concerned with positive solutions of the following semilinear elliptic equation subject to homogeneous Neumann boundary conditions

$$(\mathrm{I})_d \qquad \begin{aligned} -d\Delta u + u &= |u|^{p-2} u \qquad &&\text{in } \Omega, \\ \partial u/\partial \nu &= 0 &&\text{on } \partial\Omega \end{aligned}$$

where d is a positive constant, Ω is a bounded domain in $\mathbf{R}^N (N \geqq 2)$ with a smooth boundary, and ν is the unit outer normal to $\partial\Omega$. p satisfies $2 < p < 2N/(N-2)$ if $N \geqq 3$, and $2 < p < +\infty$ if $N = 2$.

There have been a number of papers concerning this problem. See [LN, LNT, NT, T]. As was mentioned in [NT], this problem may be viewed as a prototype of pattern formation in biology and is related to the steady-state problem for a chemotactic aggregation model by KELLER & SEGEL [KS]. Moreover, $(\mathrm{I})_d$ plays an important role in the study of activator-inhibitor systems modeling biological pattern formation which were proposed by GIERER & MEINHARDT [GM].

In [LNT] and [NT], it is shown that $(\mathrm{I})_d$ has a nonconstant positive solution u_d (the so-called least-energy solution) if d is sufficiently small. It is also shown that u_d has only one local maximum over $\bar{\Omega}$ and hence it is the global maximum, which is achieved at one point on the boundary $\partial\Omega$. Note that $u \equiv 1$ is a trivial solution of $(\mathrm{I})_d$ and is the only positive solution if d is sufficiently large [LNT].

The goal of this paper is to establish a multiplicity result on the existence of nonconstant positive solutions of $(\mathrm{I})_d$ and to show how the number of positive solutions is affected by the topology of Ω or, more precisely, of $\partial\Omega$. Moreover, we can prove that all solutions we obtain have the property that each solution has at most one local maximum over $\bar{\Omega}$, which is achieved at a point on the boundary of Ω. Our main results are the following theorems.

Theorem 0.1. *If d is sufficiently small, $(\mathrm{I})_d$ has at least $\operatorname{cat}(\partial\Omega)$ distinct nonconstant positive solutions.*

Here, $\operatorname{cat}(\partial\Omega)$ denotes the Ljusternik-Schnirelman category of $\partial\Omega$ in itself. By definition (e.g., [CH]), the category of a subset $A \subset \partial\Omega$ is denoted by $\operatorname{cat}(A, \partial\Omega)$ and equals m, if A can be covered by m closed, contractible subsets in $\partial\Omega$, but not by $m-1$ such sets. By definition, $\operatorname{cat}(\partial\Omega) = \operatorname{cat}(\partial\Omega, \partial\Omega)$.

Theorem 0.2. *If d sufficiently small, all solutions obtained in Theorem 0.1 have the property that each solution has at most one local maximum on $\bar{\Omega}$, and this maximum is attained at exactly one point on the boundary $\partial\Omega$.*

Remark 0.1. In § 4, we have a more precise characterization of the shape of the solutions (see Theorem 4.1 and Corollary 4.1).

Remark 0.2. The conclusion of Theorem 0.2 was first obtained in [NT] for least-energy solutions and for d sufficiently small. In fact, a very precise 'one-point condensation' phenomenon was established there. See Theorems 2.1 and 2.3 in [NT]. Also in the Notes Added in Proof in [NT], NI & TAKAGI claim to have proved that as $d \to 0$, the maximum point of least-energy solutions tends to a point on $\partial\Omega$ where the boundary mean curvature assumes its maximum. Here I shall show that all solutions obtained in Theorem 0.1 share the 'one-point condensation' phenomenon (Theorem 0.2 and Theorem 4.1 in § 4), but I have not obtained any further information on the locations of those maximum points for solutions given by Theorem 0.1. I tend to believe that they are close to critical points of the mean curvature of the boundary.

Remark 0.3. Results relating the topology of the domain to the existence and the multiplicity of solutions have been obtained in recent years for some nonlinear elliptic equations subject to homogeneous Dirichlet boundary conditions [BaC, BeC, Da, Di]. Theorem 0.1 shows that, for Neumann boundary problems, the topology of the boundary $\partial\Omega$, rather than Ω itself, plays a role in the existence of multiple solutions. Nevertheless, Theorem 0.1 is motivated by these papers, especially [BeC], and can be regarded as an analogue in our setting of a multiplicity result established in [BeC] for a nonlinear Dirichlet problem.

§ 1. Preliminaries

Throughout this paper, let $\Omega \subset R^N$ be a bounded domain with a smooth boundary. We are seeking positive nonconstant solutions of

$$(\mathrm{I})_d \qquad \begin{aligned} -d\Delta u + u &= |u|^{p-2}u \quad &&\text{in } \Omega, \\ \partial u/\partial \nu &= 0 &&\text{on } \partial\Omega \end{aligned}$$

where p satisfies $2 < p < +\infty$ if $N = 2$, and $2 < p < 2N/(N-2)$ if $N \geqq 3$.

It is well known that solutions of $(\mathrm{I})_d$ correspond to critical points of the following functional defined in $W^{1,2}(\Omega)$:

$$(1.1) \qquad J_d(v) = \tfrac{1}{2}\int_\Omega (d|\nabla v|^2 + v^2)\,dx - \tfrac{1}{p}\int_\Omega |v|^p\,dx.$$

By using the Mountain Pass Theorem ([AR]) the authors of [LNT] and [NT] prove the existence of a positive nonconstant solution u_d of $(\mathrm{I})_d$. (In fact, a more general nonlinearity was treated in [LNT] and [NT].) $J_d(v)$ is defined as the 'energy' in [NT] and the solution obtained in [NT] is called the least-energy solution, which corresponds to the least positive critical value of $J_d(v)$.

In this paper, in order to establish multiplicity results we turn the problem around a bit and define a new functional

$$(1.2) \qquad E_d(u) = \int_\Omega (d|\nabla u|^2 + u^2)\,dx, \qquad u \in V_1(\Omega)$$

where

$$(1.3) \qquad V_1(\Omega) = \left\{ u \in W^{1,2}(\Omega) \,\middle|\, \int_\Omega |u|^p\,dx = 1 \right\}.$$

It is easy to check that if u is a critical point of E_d on $V_1(\Omega)$, then $v = [E_d(u)]^{1/(p-2)}u$ is a solution of $(\mathrm{I})_d$. Here we call $E_d(u)$ the 'energy' of u. It is related to $J_d(v)$ (the energy used in [NT]) by

$$J_d(v) = \frac{p-2}{2p}[E_d(u)]^{p/(p-2)}.$$

Note that this formula holds only for solutions of $(\mathrm{I})_d$. It is obvious that the absolute minimum of $E_d(u)$ corresponds to the least energy of $J_d(v)$. Let us define

$$(1.4) \qquad c_d = \min_{u \in V_1(\Omega)} E_d(u).$$

It is not difficult to verify that c_d is achieved on V_1, and that a critical point of $E_d(u)$ with absolute minimum critical value corresponds to a critical point of $J_d(v)$ and thus a solution of $(\mathrm{I})_d$ (the least-energy solution).

We shall prove the existence of multiple critical points of E_d (therefore multiple solutions of $(\mathrm{I})_d$) with critical values close to c_d.

The strategy here is to estimate the topology of a certain level set of E_d, say

$$(1.5) \qquad E_d^{c_d+\varepsilon} = \{u \in V_1(\Omega) \,|\, E_d(u) \leqq c_d + \varepsilon\},$$

for some appropriate $\varepsilon > 0$ depending on d. We shall prove that if d is small enough, then

$$(1.6) \qquad \operatorname{cat}(E_d^{c_d+\varepsilon}) \geqq 2\operatorname{cat}(\partial\Omega),$$

where $\operatorname{cat}(\cdot)$ denotes the Ljusternik-Schnirelman category. The standard critical-point theory yields the existence of at least $2\operatorname{cat}(\partial\Omega)$ critical points of E_d having critical values in $[c_d, c_d + \varepsilon]$. An energy estimate shows that none of these critical points changes sign in Ω so there exist at least $\operatorname{cat}(\partial\Omega)$

positive solutions (note that equation $(\mathrm{I})_d$ is odd in u). By the Maximum Principle these solutions are strictly positive on $\bar{\Omega}$.

More precisely, in § 2, we shall give an asymptotic estimate of c_d in terms of d. The proof of (1.6) will be based on constructing two continuous maps

$$\varphi_d : \partial\Omega \to E_d^{c_d+\varepsilon},$$
$$\beta : E_d^{c_d+\varepsilon} \to N_\rho(\partial\Omega),$$

where $\rho > 0$ is chosen and fixed throughout this paper such that the ρ-neighborhood of $\partial\Omega$, given by

$$N_\rho(\partial\Omega) = \{x \in \mathbf{R}^N \mid \operatorname{dist}(x, \partial\Omega) < \rho\}, \tag{1.7}$$

is homotopically equivalent to $\partial\Omega$. φ_d is made positive, i.e., for all $y \in \partial\Omega$, $\varphi_d(y)(x) \geqq 0$ for all $x \in \Omega$. Then a topological argument asserts that

$$\operatorname{cat}(\{u \in E_d^{c_d+\varepsilon} \mid u \geqq 0\}) \geqq \operatorname{cat}(\partial\Omega),$$

and consequently (1.6) holds. In § 2, we shall construct φ_d and β by means of some asymptotic estimates. The proof of the existence result will be carried out in § 3, and § 4 will be devoted to studying the shape of the solutions obtained in Theorem 0.1. Besides Theorem 0.2, we shall prove some further results characterizing the shape of the solutions.

Now we give some preliminaries results. First, we summarize known facts about positive solutions to the equation

$$-\Delta\omega + \omega = \omega^{p-1} \quad \text{in } \mathbf{R}^N. \tag{1.8}$$

Proposition 1.1. *Equation* (1.8) *has a solution* ω *satisfying*

(i) $\omega \in C^2(\mathbf{R}^N) \cap W^{1,2}(\mathbf{R}^N)$ *and* $\omega > 0$ *in* $\mathbf{R}^N$.

(ii) ω *is spherically symmetric*: $\omega(z) = \omega(r)$ *with* $r = |z|$ *and* $d\omega/dr < 0$ *for* $r > 0$.

(iii) ω *and its first derivatives decay exponentially at infinity, i.e., there exist positive constants* C *and* μ *such that*

$$|D^\alpha \omega(z)| \leqq Ce^{-\mu|z|} \quad \textit{for } z \in \mathbf{R}^N$$

with $|\alpha| \leqq 1$.

(iv) *Let* $\tilde{\omega} = \omega/\|\omega\|_{L^p(\mathbf{R}^N)}$; *then*

$$\int_{\mathbf{R}^N} (|\nabla\tilde{\omega}|^2 + \tilde{\omega}^2)\,dx = \min\left\{\int_{\mathbf{R}^N} (|\nabla u|^2 + u^2)\,dx \,\middle|\, u \in W^{1,2}(\mathbf{R}^N),\ \int_{\mathbf{R}^N} |u|^p\,dx = 1\right\}.$$

For the proof see, eg., [CGM] or [BL] in the case of $N \geqq 3$, and [BGK] for $N = 2$.

Later we shall frequently rescale the problem $(\mathrm{I})_d$. In fact, there is a one-to-one correspondence between the solutions of $(\mathrm{I})_d$ and the solutions of

$$(\mathrm{II})_d \qquad \begin{aligned} -\Delta u + u &= |u|^{p-2}u \quad \text{in } \Omega_{1/\sqrt{d}}, \\ \partial u/\partial\nu &= 0 \qquad\qquad \text{on } \partial\Omega_{1/\sqrt{d}} \end{aligned}$$

where

$$\Omega_{1/\sqrt{d}} = \{x \in \mathbf{R}^N \mid \sqrt{d}x \in \Omega\}, \tag{1.9}$$

because $(\mathrm{II})_d$ is also associated with a functional defined by

$$\tilde{E}_{1/\sqrt{d}}(u) = \int_{\Omega_{1/\sqrt{d}}} (|\nabla u|^2 + u^2)\,dx, \ \forall\, u \in V_1(\Omega_{1/\sqrt{d}}) \tag{1.10}$$

where

$$V_1(\Omega_{1/\sqrt{d}}) = \left\{u \in W^{1,2}(\Omega_{1/\sqrt{d}}) \,\middle|\, \int_{\Omega_{1/\sqrt{d}}} |u|^p\,dx = 1\right\}. \tag{1.11}$$

For $u \in V_1(\Omega)$, let us define

$$\sigma(u)(x) = d^{N/2p}u(\sqrt{d}x). \tag{1.12}$$

Then it is not difficult to see $\sigma(u) \in V_1(\Omega_{1/\sqrt{d}})$. Moreover, we have

Lemma 1.1. *For any* $u \in V_1(\Omega)$

$$\tilde{E}_{1/\sqrt{d}}(\sigma(u)) = d^{-N(p-2)/2p}E_d(u), \tag{1.13}$$

and therefore

$$\min_{V_1(\Omega_{1/\sqrt{d}})} \tilde{E}_{1/\sqrt{d}} = d^{-N(p-2)/2p} \min_{V_1(\Omega)} E_d. \tag{1.14}$$

Proof. Compute directly

$$\begin{aligned} \tilde{E}_{1/\sqrt{d}}(\sigma(u)) &= d^{N/p} \int_{\Omega_{1/\sqrt{d}}} (d|\nabla u(\sqrt{d}x)|^2 + |u(\sqrt{d}x)|^2)\,dx \\ &= d^{N/p} \int_{\Omega} (d|\nabla u(y)|^2 + |u(y)|^2)\, d^{-N/2}\,dy \\ &= d^{-N(p-2)/2p}E_d(u). \end{aligned}$$

Since σ is a one-to-one correspondence, (1.14) follows from (1.13). □

Now let us define some notation for later use. For each $r \geqq 1$, $\alpha > 0$, set

$$m(r, \alpha) = \min_{u \in V_\alpha(\Omega_r)} \tilde{E}_r, \quad V_\alpha(\Omega_r) = \left\{u \in W^{1,2}(\Omega_r) \,\middle|\, \int_{\Omega_r} |u|^p\,dx = \alpha\right\},$$

$$m(+, \alpha) = \min\left\{\int_{\mathbf{R}^N_+} (|\nabla u|^2 + u^2)\,dx \,\middle|\, u \in W^{1,2}(\mathbf{R}^N_+),\ \int_{\mathbf{R}^N_+} |u|^p\,dx = \alpha\right\},$$

$$m(\infty, \alpha) = \min\left\{\int_{\mathbf{R}^N} (|\nabla u|^2 + u^2)\,dx \,\middle|\, u \in W^{1,2}(\mathbf{R}^N),\ \int_{\mathbf{R}^N} |u|^p\,dx = \alpha\right\}.$$

We have

Lemma 1.2. *For* $r \geqq 1,\ \alpha > 0$,

(i) $m(\infty, 1) = \int_{\mathbf{R}^N} (|\nabla \tilde{\omega}|^2 + |\tilde{\omega}|^2)\, dx$, *where* $\tilde{\omega}$ *is given in Proposition 1.1.*

(ii) $m(r, \alpha) = \alpha^{2/p} m(r, 1)$ *and the identity also holds when* r *is replaced by* $+$ *or* ∞.

(iii) $m(\infty, 2) = 2m(+, 1)$.

Proof. Statement (i) follows from the definition of $m(\infty, 1)$. Statement (ii) follows from the homogeneity of the functionals and the constraints involved. Statement (iii) can be proved by checking that, with $\tilde{\omega}$ given in Proposition 1.1, $u = 2^{1/p}\tilde{\omega}$ attains $m(\infty, 2)$ and

$$\tilde{u} = u|_{\mathbf{R}^N_+}$$

belongs to $W^{1,2}(\mathbf{R}^N_+)$ with $\int_{\mathbf{R}^N_+} |\tilde{u}|^p\, dx = 1$. Therefore $\tilde{u}$ attains $m(+, 1)$. □

§ 2. Some asymptotic estimates

In this section we give some asymptotic estimates as $d \to 0$. We state the results first, then follow with proofs.

Proposition 2.1. *As* $d \to 0$,

$$c_d = d^{N(p-2)/2p}(m(+, 1) + o(1)), \tag{2.1}$$

where c_d *is defined in* (1.4).

To get an upper bound for c_d, we need to define a family of functions, which also will be used later in the proof of the main theorem.

Let η be a smooth nonincreasing function defined on $[0, +\infty)$ such that $\eta(t) = 1$ if $0 \leqq t \leqq 1$, $\eta(t) = 0$ if $t \geqq 2$, and $|\eta'| \leqq 2$. Also let $\eta_r(\cdot) = \eta\left(\frac{\cdot}{r}\right)$.

Let $\rho > 0$ be given in (1.7). Define, for each $y \in \partial\Omega$, $\psi_d(y) \in W^{1,2}(\Omega)$ by

$$\psi_d(y)(x) = \eta_\rho(|x - y|)\, \omega\left(\frac{x - y}{\sqrt{d}}\right) \quad \forall x \in \Omega, \tag{2.2}$$

and $\varphi_d(y) \in V_1(\Omega)$ by

$$\varphi_d(y) = \frac{\psi_d(y)}{\|\psi_d(y)\|_{L^p(\Omega)}}, \tag{2.3}$$

where ω is given in Proposition 1.1. We have

Proposition 2.2. $\varphi_d \in C(\partial\Omega, V_1(\Omega))$ *and*

$$E_d(\varphi_d(y)) = d^{N(p-2)/2p}(m(+, 1) + o(1)) \quad \textit{as } d \to 0 \tag{2.4}$$

uniformly for $y \in \partial\Omega$.

Next, let $\beta(u)$ be the center of mass of $u \in V_1(\Omega)$ in terms of the L^p norm:

$$\beta(u) = \int_\Omega |u|^p x\, dx \quad \forall u \in V_1(\Omega). \tag{2.5}$$

Note that β is continuous in u and

$$\beta(u) \in \overline{\text{conv}\ \Omega} \quad \forall u \in V_1(\Omega),$$

where conv Ω is the convex closure of Ω.

Proposition 2.3. *For* $\rho > 0$ *given in* (1.7), *there exist* $\varepsilon_1 > 0$ *and* $d_1 > 0$ *such that for any* $0 < d \leqq d_1$,

$$\beta(u) \in N_\rho(\partial\Omega) \quad \forall u \in E_d^{c_d + \varepsilon d^{N(p-2)/2p}}.$$

We begin with the

Proof of Proposition 2.1. First, let us note that by using the map φ_d given in Proposition 2.2 we have

$$c_d \leqq d^{N(p-2)/2p}(m(+, 1) + o(1)) \quad \text{as } d \to 0.$$

We prove the reverse inequality by arguing indirectly. Suppose that there exists a constant A such that

$$\liminf_{d \to 0} d^{-N(p-2)/2p} c_d = A < m(+, 1).$$

Then there exist $d_n \to 0$ and $u_n \in V_1(\Omega)$, solutions of $(\mathrm{I})_{d_n}$, such that

$$c_{d_n} = E_{d_n}(u_n),$$

$$\lim_{n\to\infty} d_n^{-N(p-2)/2p} E_{d_n}(u_n) = A.$$

By Lemma 1.1, $v_n = \sigma(u_n) \in V_1(\Omega_{1/\sqrt{d_n}})$ satisfies

$$\lim_{n\to\infty} \int_{\Omega_{1/\sqrt{d_n}}} (|\nabla v_n|^2 + v_n^2)\, dx = A. \tag{2.6}$$

Now we need the Concentration-Compactness Lemma [L]:

Concentration-Compactness Lemma. *Suppose that* μ_n *is a sequence of measures on* $\mathbf{R}^N$: $\mu_n \geqq 0$, $\int_{\mathbf{R}^N} \mu_n\, dx = 1$. *There is a subsequence* (μ_n) *such that one of the following three mutually exclusive conditions holds:*

(1°) (*Compactness*) *There exists a sequence* $x_n \in \mathbf{R}^N$ *such that for any* $\varepsilon > 0$ *there is a radius* $R > 0$ *with the property that*

$$\lim_{n\to\infty} \int_{B_R(x_n)} \mu_n\, dx \geqq 1 - \varepsilon.$$

(2°) (*Vanishing*) *For all* $R > 0$,

$$\lim_{n\to\infty}\left(\sup_{x\in\mathbf{R}^N}\int_{B_R(x)}\mu_n\,dx\right) = 0.$$

(3°) (*Dichotomy*) *There exists a number* λ, $0 < \lambda < 1$, *such that for any* $\varepsilon > 0$ *there is a number* $R > 0$ *and a sequence* (x_n) *with the following property: Given* $R' > R$ *there are non-negative measures* μ_n^1, μ_n^2 *such that*

$$0 \leqq \mu_n^1 + \mu_n^2 \leqq \mu_n,$$

$$\operatorname{supp}(\mu_n^1) \subset B_R(x_n), \quad \operatorname{supp}(\mu_n^2) \subset \mathbf{R}^N \setminus B_{R'}(x_n),$$

$$\limsup_{n\to\infty}\left(\left|\lambda - \int_{\mathbf{R}^N}\mu_n^1\,dx\right| + \left|(1-\lambda) - \int_{\mathbf{R}^N}\mu_n^2\,dx\right|\right) \leqq \varepsilon.$$

Returning to the proof of Proposition 2.1, we define $\mu_n = \chi_n|v_n|^p$, where χ_n is the characteristic function of Ω_n, i.e., $\chi_n(x) = 1$ if $x \in \Omega_n$, $\chi_n(x) = 0$ if $x \notin \Omega_n$. For simplicity, we write $\Omega_{1/\sqrt{d_n}}$ as Ω_n. By the Concentrated-Compactness Lemma there are three possibilities: compactness, vanishing, and dichotomy. We are going to rule out the last two possibilities so that compactness holds.

To show that vanishing does not happen we need the following lemma, which is a slight extension of a result in [L].

Lemma 2.1. *Let* v_n *be as above. If for any* $R > 0$,

$$\lim_{n\to\infty}\left(\sup_{y\in\mathbf{R}^N}\int_{B_R(y)}\chi_n|v_n|^p\,dx\right) = 0,$$

then

$$\lim_{n\to\infty}\int_{\mathbf{R}^N}\chi_n|v_n|^p\,dx = 0.$$

Now, since $\int_{\mathbf{R}^N}\chi_n|v_n|^p\,dx \equiv 1$, vanishing cannot happen. The proof of Lemma 2.1 will be given at the end of this paper.

Next we have

Assertion 1. *Dichotomy does not happen.*

If dichotomy were to happen, there would exist $\lambda \in (0, 1)$ such that the concentration functions

$$Q_n(t) = \sup_{y\in\mathbf{R}^N}\int_{y+B_t}\chi_n|v_n|^p\,dx,$$

$$Q(t) = \lim_{n\to\infty} Q_n(t) \quad \text{(a nondecreasing function)},$$

would satisfy

$$\lim_{t\to\infty} Q(t) = \lambda.$$

Therefore for any $\varepsilon > 0$, there exists $R_0 > 0$ such that $Q(R_0) \geqq \lambda - \varepsilon/4$, and then there exist $y_n \in \mathbf{R}^N$, $n_0 \geqq 0$ such that

$$Q_n(R_0) = \int_{y_n + B_{R_0}}\chi_n|v_n|^p\,dx \geqq \lambda - \frac{\varepsilon}{2} \qquad \forall\, n \geqq n_0.$$

Also, by definition, there exist $R_n \to \infty$, such that $Q_n(2R_n) \leqq \lambda + \varepsilon/2$. Let η be the function defined at the beginning of this section and define

$$\xi(t) = 1 - \eta(t),$$

a nondecreasing function on $[0, \infty)$. Set

$$v_n^1(x) = \chi_n(x)\,\eta\left(\frac{|x-y_n|}{R_0}\right)v_n(x),$$

$$v_n^2(x) = \chi_n(x)\,\xi\left(\frac{|x-y_n|}{R_n}\right)v_n(x).$$

Then it is not difficult to check that

$$\left|1 - \int_{\mathbf{R}^N}(|v_n^1|^p + |v_n^2|^p)\,dx\right| \leqq \varepsilon \qquad \forall\, n \geqq n_0.$$

Therefore we have

$$\left|\int_{\mathbf{R}^N}|v_n^1|^p\,dx - \lambda\right| \leqq \tfrac{1}{2}\varepsilon, \quad \left|\int_{\mathbf{R}^N}|v_n^2|^p\,dx - (1-\lambda)\right| \leqq \tfrac{3}{2}\varepsilon.$$

Thus

$$\int_{\mathbf{R}^N}|v_n^1|^p\,dx = \lambda + \varepsilon_n^1, \quad |\varepsilon_n^1| \leqq 2\varepsilon, \tag{2.7}$$

$$\int_{\mathbf{R}^N}|v_n^2|^p\,dx = (1-\lambda) + \varepsilon_n^2, \quad |\varepsilon_n^2| \leqq 2\varepsilon. \tag{2.8}$$

On the other hand, we may choose R_0 large enough (for fixed $\varepsilon > 0$) so that

$$\int_{\Omega_n}(|\nabla v_n|^2 + |v_n|^2)\,dx - \int_{\Omega_n}(|\nabla v_n^1|^2 + |v_n^1|^2)\,dx - \int_{\Omega_n}(|\nabla v_n^2|^2 + |v_n^2|^2)\,dx \geqq -2\varepsilon. \tag{2.9}$$

Now by (2.7), (2.8), (2.9) and Lemma 1.2,

$$\begin{aligned} A &= \lim_{n\to\infty} m\left(\frac{1}{\sqrt{d_n}}, 1\right) \\ &= \lim_{n\to\infty}\int_{\Omega_n}(|\nabla v_n|^2 + v_n^2)\,dx \\ &\geqq \lim_{n\to\infty}\left(\int_{\Omega_n}(|\nabla v_n^1|^2 + |v_n^1|^2)\,dx + \int_{\Omega_n}(|\nabla v_n^2|^2 + |v_n^2|^2)\,dx - 2\varepsilon\right) \geqq \end{aligned}$$

$$\geqq \lim_{n\to\infty}\left(m\left(\frac{1}{\sqrt{d_n}}, \lambda - 2\varepsilon\right) + m\left(\frac{1}{\sqrt{d_n}}, (1-\lambda) - 2\varepsilon\right) - 2\varepsilon\right)$$

$$= \lim_{n\to\infty}\left((\lambda - 2\varepsilon)^{2/p} m\left(\frac{1}{\sqrt{d_n}}, 1\right) + (1-\lambda-2\varepsilon)^{2/p} m\left(\frac{1}{\sqrt{d_n}}, 1\right) - 2\varepsilon\right)$$

$$= (\lambda - 2\varepsilon)^{2/p} A + (1-\lambda-2\varepsilon)^{2/p} A - 2\varepsilon .$$

By the embedding $W^{1,2}(\Omega_n) \hookrightarrow L^p(\Omega_n)$ (cf. [A]) we have $m(1/\sqrt{d_n}, 1) \geqq c > 0$, where c is independent of n since Ω_n satisfies the uniform cone condition for all n. As a consequence, $A > 0$. Letting $\varepsilon \to 0$, we get a contradiction

$$1 \geqq \lambda^{2/p} + (1-\lambda)^{2/p} > 1 .$$

Thus Assertion 1 is proved.

With vanishing and dichotomy ruled out, we obtain the compactness of sequence $\mu_n = \chi_n |v_n|^p$, i.e., there exist $y_n \in \mathbf{R}^N$ and for each $\varepsilon > 0$, there exists $R > 0$ such that

$$\int_{y_n + B_R} \chi_n |v_n|^p \, dx \geqq 1 - \varepsilon . \tag{2.10}$$

Assertion 2. *There exists a constant $\bar{C} > 0$ such that*

$$\operatorname{dist}(y_n, \partial\Omega_n) \leqq \bar{C}. \tag{2.11}$$

If Assertion 2 were not true, then $\operatorname{dist}(y_n, \partial\Omega_n) \to +\infty$. By (2.10), $y_n \in \Omega_n$. Now for a fixed $\varepsilon > 0$, there exists $R > 0$ such that (2.10) holds. For this $R > 0$, if n is large enough, we have

$$y_n + B_{2R} \subset \Omega_n .$$

Set

$$w_n(x) = \eta\left(\frac{|x - y_n|}{R}\right) v_n(x);$$

then $w_n(x) \in W^{1,2}(\mathbf{R}^N)$. Also, as noted above, we may choose R so large (for fixed $\varepsilon > 0$) that

$$\int_{\Omega_n} (|\nabla v_n|^2 + v_n^2)\, dx - \int_{\mathbf{R}^N} (|\nabla w_n|^2 + w_n^2)\, dx \geqq -2\varepsilon .$$

Therefore, by setting $\lambda_n = \int_{\mathbf{R}^N} |w_n|^p \, dx$, we have

$$\int_{\Omega_n} (|\nabla v_n|^2 + v_n^2)\, dx \geqq \int_{\mathbf{R}^N} (|\nabla w_n|^2 + w_n^2)\, dx - 2\varepsilon$$

$$\geqq m(\infty, \lambda_n) - 2\varepsilon = \lambda_n^{2/p} m(\infty, 1) - 2\varepsilon \geqq (1-\varepsilon)^{2/p}\, 2^{(p-2)/p} m(+, 1) - 2\varepsilon .$$

Letting $n \to \infty$ and $\varepsilon \to 0$ we get a contradiction for $A < m(+, 1)$. Assertion 2 is proved.

Now, by (2.11) there exists $t_n \in \partial\Omega$ such that

$$\operatorname{dist}(y_n, \bar{t}_n) \leqq \bar{C} \tag{2.12}$$

where $\bar{t}_n = t_n/\sqrt{d_n} \in \partial\Omega_n$. For each n, let us choose a unitary matrix T_n such that $\tilde{\Omega}_n = T_n(\Omega_n - \bar{t}_n)$ has y^N as inner normal direction of $\partial\tilde{\Omega}_n$ at θ. We have

Assertion 3. *For any fixed $R_1 > 0$, as $n \to \infty$, $T_n(\Omega_n - \bar{t}_n) \cap B_{R_1}(\theta)$ converges to $B_{R_1}^+(\theta) = \{x \in B_{R_1}(\theta) \mid x = (x^1, \ldots, x^N)$ with $x^N \geqq 0\}$ in the following sense: There exists $K_1 > 0$, and for each $\delta > 0$ there exists $n_\delta > 0$ such that*

$$\{x \in B_{R_1}^+(\theta) \mid x^N \geqq \delta\} \subset T_n(\Omega_n - \bar{t}_n) \quad \text{for } n \geqq n_\delta , \tag{2.13}$$

$$L^N\{x \in T_n(\Omega_n - \bar{t}_n) \cap B_{R_1}(\theta) \mid x^N \leqq \delta\} \leqq K_1 \delta \tag{2.14}$$

where $L^N\{\cdot\}$ denotes the Lebesgue measure.

In fact, without loss of generality we may assume that $t_n = \theta$ is the origin and that y^N is the inner normal direction of $\partial\Omega$ at θ. Then there is a function $x^N = \psi(x')$, which is defined in $B_a^{N-1} = \{x \in \mathbf{R}^N \mid x^N = 0,\ |x| \leqq a\}$ for some $a > 0$, such that the boundary $\partial\Omega$ near θ is given by

$$x^N = \psi(x'), \quad x' \in B_a^{N-1}(\theta), \tag{2.15}$$

with the property that $\psi(\theta) = \nabla\psi(\theta) = 0$. Moreover, for fixed $R_1 > 0$, $\partial\Omega_n \cap B_{R_1}(\theta)$ is given by

$$x^N = \frac{\psi(x')}{\sqrt{d_n}} \quad \text{for } x' \in B^{N-1}_{\sqrt{d_n}R_1}(\theta).$$

Thus for n large, $|x'| \leqq \sqrt{d_n}\, R_1$ and

$$|x^N| = \frac{|\psi(x')|}{\sqrt{d_n}} \leqq C\sqrt{d_n}\, R_1^2 \to 0 \quad \text{as } n \to \infty \tag{2.16}$$

where C is a constant depending only on $D^2\psi$. Since $\partial\Omega$ is smooth and compact, we may apply the argument above to any points on $\partial\Omega$ with uniform constants being involved, say, a in (2.15) and C in (2.16). And (2.16) is sufficient to give (2.13) and (2.14). Assertion 3 is proved.

Assertion 4. *$\|v_n\|_{L^\infty(\Omega_n)}$ is uniformly bounded (independent of n).*

To prove Assertion 4, let us note that $[E_{d_n}(u_n)]^{1/(p-2)} u_n(x)$ satisfies $(\mathrm{I})_{d_n}$. By Theorem 3 in [LNT], $[E_{d_n}(u_n)]^{1/(p-2)} u_n(x)$ is uniformly bounded in $L^\infty(\Omega)$. By Lemma 1.1,

$$E_{d_n}(u_n) = d_n^{N(p-2)/2p} \tilde{E}_{1/\sqrt{d_n}}(v_n),$$

and therefore

$$[E_{d_n}(u_n)]^{1/(p-2)} = d_n^{N/2p}(A + o(1))^{1/(p-2)} \quad \text{as } d_n \to 0 .$$

Since by definition $v_n(x) = d_n^{N/2p} u_n(\sqrt{d_n}\, x)$, v_n is also uniformly bounded in $L^\infty(\Omega_n)$. Assertion 4 is proved.

Now we are ready to finish the proof of Proposition 2.1. By the compactness of v_n for all $\varepsilon > 0$, there exists $R > 0$ such that

$$\int_{B_R(y_n)} \psi_n |v_n|^p \, dx \geqq 1 - \varepsilon \quad \text{for } n \text{ large}.$$

By Assertion 2 and by (2.12), when setting $R_1 = R + \bar{C}$, we have

$$\int_{B_{R_1}(\bar{t}_n)} \chi_n |v_n|^p \, dx \geqq 1 - \varepsilon .$$

From (2.13), (2.14) and Assertion 4, for the given $\varepsilon > 0$, there exists $\delta_1 > 0$ and n_{δ_1} such that

$$(2.17) \qquad \int_{\{x \in B^+_{R_1}(\theta) | x^N \geqq \delta_1\}} |v_n(T_n^{-1}x + \bar{t}_n)|^p \, dx \geqq 1 - 2\varepsilon \quad \text{for } n \geqq n_{\delta_1}.$$

Define $x_{\delta_1} = (0, 0, \dots, 0, \delta_1)$ and

$$\tilde{v}_n(x) = \eta_{R_1}(|x|)\, v_n(T_n(x + x_{\delta_1}) + \bar{t}_n), \quad x \in \mathbf{R}^N_+ .$$

By (2.13) $\tilde{v}_n(x)$ is well defined for n large. As noted before, we may require R large enough such that

$$\int_{\Omega_n} (|\nabla v_n|^2 + v_n^2) \, dx - \int_{\mathbf{R}^N_+} (|\nabla \tilde{v}_n|^2 + \tilde{v}_n^2) \, dx \geqq -2\varepsilon .$$

Then, for n large, by (2.17),

$$\begin{aligned} A &= \lim_{n\to\infty} \int_{\Omega_n} (|\nabla v_n|^2 + v_n^2) \, dx \\ &\geqq \lim_{n\to\infty} \int_{\mathbf{R}^N_+} (|\nabla \tilde{v}_n|^2 + \tilde{v}_n^2) \, dx - 2\varepsilon \\ &\geqq (1 - 2\varepsilon)^{2/p} m(+, 1) - 2\varepsilon . \end{aligned}$$

Letting $\varepsilon \to 0$, we get a contradiction since we assumed $A < m(+, 1)$. So $A = m(+, 1)$, and the proof of Proposition 2.1 is complete. □

Following the proof above, we can obtain another result which will be used repeatedly for sequences that may not correspond to minimizers of $E_d(u)$:

Lemma 2.2. *Let $d_n \to 0$ as $n \to \infty$. Assume that $v_n \in V_1(\Omega_{1/\sqrt{d_n}})$ satisfies*

$$\int_{\Omega_{1/\sqrt{d_n}}} (|\nabla v_n|^2 + v_n^2) \, dx \to m(+, 1) \quad \text{as } n \to \infty .$$

Then there exist a subsequence of v_n (still denoted by v_n), $y_n \in \mathbf{R}^N$ and a constant $\bar{C}$ independent of n such that for each $\varepsilon > 0$ there exists $R > 0$ such that

$$\lim_{n\to\infty} \int_{B_R(y_n) \cap \Omega_{1/\sqrt{d_n}}} |v_n|^p \, dx \geqq 1 - \varepsilon ,$$

$$\operatorname{dist}(y_n, \partial\Omega_{1/\sqrt{d_n}}) \leqq \bar{C} .$$

Proof of Proposition 2.2. First, let us note that

$$E_d(\varphi_d(y)) = \frac{E_d(\psi_d(y))}{\|\psi_d(y)\|^2_{L^p(\Omega)}} .$$

By definition, $\psi_d(y)(x) = \eta_\rho(|x - y|)\, w((x - y)/\sqrt{d})$; then we have

$$|\nabla \psi_d(y)|^2 = |\nabla \eta_\rho|^2 \omega^2 + \frac{1}{d} \eta_\rho^2 |\nabla \omega|^2 + 2\eta_\rho \omega \nabla \eta_\rho \nabla \omega .$$

Therefore

$$\begin{aligned} E_d(\psi_d(y)) &= \int_\Omega (d|\nabla \psi_d(y)|^2 + \psi_d^2) \, dx \\ &= \int_\Omega \eta_\rho^2 (|\nabla \omega|^2 + \omega^2) \, dx + \int_\Omega (|\nabla \eta_\rho|^2 \omega^2 + 2\eta_\rho \omega \nabla \eta_\rho \nabla \omega) \, dx \\ &= I_1 + I_2 . \end{aligned}$$

We want to show that

$$(2.18) \qquad I_1 = d^{N/2} \left(\int_{\mathbf{R}^N_+} (|\nabla \omega|^2 + \omega^2) \, dx + o(1) \right) \quad \text{as } d \to 0 ,$$

$$(2.19) \qquad I_2 = d^{N/2}(o(1)) \quad \text{as } d \to 0 ,$$

both uniformly for $y \in \partial\Omega$. In fact, computing directly, we obtain

$$\begin{aligned} (2.20) \quad I_1 &= \int_{(\Omega - y)} \eta_\rho^2(|z|) \left(|\nabla \omega|^2 \left(\frac{z}{\sqrt{d}} \right) + \omega^2 \left(\frac{z}{\sqrt{d}} \right) \right) dz \\ &= d^{N/2} \int_{(\Omega - y)_{1/\sqrt{d}}} \eta^2 \left(\frac{|h\sqrt{d}|}{\rho} \right) (|\nabla \omega|^2(h) + \omega^2(h)) \, dh . \end{aligned}$$

By Proposition 1.1, for every $\varepsilon > 0$, there exists $R > 0$ such that

$$(2.21) \qquad \int_{(\Omega - y)_{1/\sqrt{d}} \cap \{h | |h| \geqq R\}} \eta^2 \left(\frac{|h\sqrt{d}|}{\rho} \right) (|\nabla \omega|^2(h) + \omega^2(h)) \, dh < \frac{\varepsilon}{2} \quad \forall y \in \partial\Omega .$$

For this $R > 0$,

$$(\Omega - y)_{1/\sqrt{d}} \cap \{h | |h| \leqq R\} \to B^+_R \quad \text{in measure as } d \to 0 ,$$

uniformly for $y \in \partial\Omega$. Thus there exists a $d_1 > 0$ such that for all $d \leqq d_1$ (with d_1 satisfying $\sqrt{d_1} \leqq \rho/R$),

$$(2.22) \qquad \left| \int_{(\Omega - y)_{1/\sqrt{d}} \cap \{h | |h| \leqq R\}} (|\nabla \omega|^2 + \omega^2) \, dh - \int_{B^+_R} (|\nabla \omega|^2 + \omega^2) \, dh \right| \leqq \frac{\varepsilon}{2} ,$$

uniformly for $y \in \partial\Omega$. Therefore (2.18) follows from (2.20), (2.21), and (2.22).

To prove (2.19), we calculate as above, obtaining

$$I_2 \leqq d^{N/2} \int_{\{h | \rho/\sqrt{d} \leqq |h| \leqq 2\rho/\sqrt{d}\}} \left(\frac{4}{\rho^2} \omega^2 + \omega^2 + |\nabla \omega|^2 \right) dh .$$

By using Proposition 1.1 again, we obtain

$$I_2 \leqq d^{N/2}(o(1)) \quad \text{as } d \to 0,$$

uniformly for $y \in \partial\Omega$. Similarly, we have

$$\| \psi_d(y) \|^2_{L^p(\Omega)} = d^{N/p} \left(\left\{ \int_{\mathbf{R}^N_+} \omega^p \, dx \right\}^{2/p} + o(1) \right) \quad \text{as } d \to 0$$

uniformly for $y \in \partial\Omega$. And finally,

$$E_d(\varphi_d(y)) = \frac{d^{N/2} \left[\int_{\mathbf{R}^N_+} (|\nabla \omega|^2 + \omega^2) \, dx + o(1) \right]}{d^{N/p} \left[\| \omega \|^2_{L^p(\mathbf{R}^N_+)} + o(1) \right]}$$
$$= d^{N(p-2)/2p} [m(+,1) + o(1)] \quad \text{as } d \to 0$$

uniformly for $y \in \partial\Omega$. □

Proof of Proposition 2.3. We argue indirectly. If Proposition 2.3 were not true, there would exist $d_n \to 0$, $u_n \in V_1(\Omega)$ such that

(2.23) $$c_n = \beta(u_n) \notin N_\rho(\partial\Omega),$$

(2.24) $$\lim_{n\to\infty} d^{-N(p-2)/2p} E_{d_n}(u_n) = m(+,1);$$

here we have used Proposition 2.1. By Lemma 1.1,

$$v_n(x) = d^{N/2p} u_n(\sqrt{d_n} x) \in V_1(\Omega_{1/\sqrt{d_n}}),$$
$$\lim_{n\to\infty} \int_{\Omega_{1/\sqrt{d_n}}} (|\nabla v_n|^2 + |v_n|^2) \, dx = m(+,1).$$

For simplicity we write $\Omega_{1/\sqrt{d_n}}$ as Ω_n. By Lemma 2.2, there exist a subsequence of v_n (still denoted by v_n), $y_n \in \mathbf{R}^N$, and a constant $\bar{C} > 0$ independent of n, and for each $\varepsilon > 0$ there exists an $R > 0$ such that

(2.25) $$\lim_{n\to\infty} \int_{B_R(y_n)\cap\Omega_n} |v_n|^p \, dx \geqq 1 - \varepsilon,$$

(2.26) $$\operatorname{dist}(y_n, \partial\Omega_n) \leqq \bar{C}.$$

Therefore, there exists $t_n \in \partial\Omega$ such that

(2.27) $$\operatorname{dist}\left(y_n, \frac{t_n}{\sqrt{d_n}} \right) \leqq \bar{C}.$$

By passing to a subsequence, we may assume that $t_n \to t \in \partial\Omega$.

Without loss of generality, we assume that $c_n = \beta(u_n)$ satisfies $c_n \to \theta$ in $\mathbf{R}^N$. By direct computation, we have

$$\int_{\Omega_n} |v_n|^p x \, dx = \frac{c_n}{\sqrt{d_n}}.$$

By the assumption that $c_n = \beta(u_n) \notin N_\rho(\partial\Omega)$, we have $t \neq \theta$. For simplicity we assume that $t = (t^1, t^2, \ldots, t^N)$ with $t^1 > 0$. From (2.25) and (2.27) it follows that for all $\varepsilon > 0$, there exists $R_1 > 0$

$$\lim_{n\to\infty} \int_{B_{R_1}(t_n/\sqrt{d_n})\cap\Omega_n} |v_n|^p \, dx \geqq 1 - \varepsilon.$$

Let

$$s = \min\{y^1 \mid (y^1, y^2, \ldots, y^N) \in \partial\Omega.$$

Then for n large we have

$$\frac{c_n^1}{\sqrt{d_n}} = \int_{\Omega_n} |v_n|^p x^1 \, dx$$
$$\geqq \int_{B_{R_1}(t_n/\sqrt{d_n})\cap\Omega_n} |v_n|^p x^1 \, dx + \int_{\Omega_n \setminus B_{R_1}(t_n/\sqrt{d_n})} |v_n|^p x^1 \, dx$$
$$\geqq \left(\frac{t_n^1}{\sqrt{d_n}} - R_1 \right)(1-\varepsilon) - \frac{|s|}{\sqrt{d_n}} \varepsilon.$$

Hence, we get

$$c_n^1 \geqq (t_n^1 - R_1\sqrt{d_n})(1-\varepsilon) - |s|\varepsilon.$$

Letting $n \to \infty$ and $\varepsilon \to 0$, we get $0 \geqq t^1 > 0$, a contradiction. This completes the proof of Proposition 2.3. □

§ 3. The Proof of Theorem 0.1

The proof is carried out in three steps. The first step is to obtain the estimate

(3.1) $$\operatorname{cat}(E_d^{c_d+\varepsilon_d}) \geqq 2 \operatorname{cat}(\partial\Omega)$$

for some $\varepsilon_d > 0$ depending on d. Once we have (3.1) we may use standard variational techniques in the level set $E_d^{c_d+\varepsilon_d}$ and obtain the existence of at least $2 \operatorname{cat}(\partial\Omega)$ critical points of E_d in $E_d^{c_d+\varepsilon_d}$. Finally, an energy estimate shows that none of these solutions changes sign, and consequently we find at least $\operatorname{cat}(\partial\Omega)$ positive solutions of $(\mathrm{I})_d$.

Lemma 3.1. *Let $\varepsilon_1 > 0$ be given as in Proposition 2.3. For any $\varepsilon \in (0, \varepsilon_1)$, there exists $d_\varepsilon > 0$ such that*

(3.2) $$\operatorname{cat}(E_d^{c_d+\varepsilon_d}) \geqq 2 \operatorname{cat}(\partial\Omega)$$

for

(3.3) $$\varepsilon_d = d^{N(p-2)/2p} \varepsilon, \quad 0 < d \leqq d_\varepsilon.$$

Proof. By Propositions 2.1, 2.2, and 2.3, for $0 < \varepsilon \leqq \varepsilon_1$ there exists $d_\varepsilon > 0$ such that for all d in $(0, d_\varepsilon)$,

$$\varphi_d : \partial\Omega \to E_d^{c_d+\varepsilon_d} \cap \{u \in V_1(\Omega) \mid u \geqq 0, \text{ a.e. in } \Omega\},$$
$$\beta : E_d^{c_d+\varepsilon_d} \to N_p(\partial\Omega)$$

are both well-defined continuous maps, where ε_d is given in (3.3). By the construction of φ_d, for any $y \in \partial\Omega$,

$$\beta \circ \varphi_d(y) \in N_{2p}(y). \tag{3.4}$$

Let $P : N_{2p}(\partial\Omega) \to \partial\Omega$ be the homotopical equivalence map with $P|_{\partial\Omega} = \mathrm{id}_{\partial\Omega}$. Set

$$A_+ = E_d^{c_d+\varepsilon_d} \cap \{u \in V_1(\Omega) \mid u \geqq 0 \text{ a.e. in } \Omega\}$$

and assume $\operatorname{cat}(A_+) = k$. Then there exist k closed and contractible subsets of $A_+, A_1, \ldots, A_k$, such that

$$A_+ = \bigcup_{i=1}^k A_i.$$

Let

$$Y_i = \varphi_d^{-1}(A_i) \subset \partial\Omega, \quad i = 1, 2, \ldots, k;$$

then

$$\bigcup_{i=1}^k Y_i = \partial\Omega.$$

Therefore

$$\operatorname{cat}(\partial\Omega) \leqq \sum_{i=1}^k \operatorname{cat}_{\partial\Omega}(Y_i). \tag{3.5}$$

We shall show that, if $Y_i \neq \phi$, then Y_i is contractible in $\partial\Omega$ and $\operatorname{cat}_{\partial\Omega}(Y_i) = 1$. Since A_i is contractible in A_+, there exists $H_i \in C([0, 1] \times A_i, A_+)$ such that

$$H_i(0, a) = a \quad \forall\, a \in A_i,$$
$$H_i(1, a) = a_i \in A_+ \quad \forall\, a \in A_i.$$

Define a map $M : [0, 2] \times Y_i \to \partial\Omega$ by

$$M(t, y) = \begin{cases} P[y - t(y - \beta \circ H_i(0, \cdot) \circ \varphi_d(y))] & \text{for } 0 \leqq t \leqq 1,\ y \in Y_i, \\ P \circ \beta \circ H_i(t-1, \cdot) \circ \varphi_d(y) & \text{for } 1 \leqq t \leqq 2,\ y \in Y_i. \end{cases}$$

Then we verify that

$$M(0, y) = y \qquad \forall\, y \in Y_i,$$
$$M(2, y) = P \circ \beta(a_i) \in \partial\Omega \quad \forall\, y \in Y_i.$$

By (3.4), M is well-defined and consequently Y_i is contractible in $\partial\Omega$. So by (3.5), $\operatorname{cat}(\partial\Omega) \leqq k = \operatorname{cat}(A_+)$. Using $-\varphi_d$ and the same argument one can show that

$$\operatorname{cat}(A_-) \geqq \operatorname{cat}(\partial\Omega),$$

where $A_- = E_d^{c_d+\varepsilon_d} \cap \{u \in V_1(\Omega) \mid u \leqq 0 \text{ a.e. in } \Omega\}$. Since A_- and A_+ are disjoint in $E_d^{c_d+\varepsilon_d}$, we get

$$\operatorname{cat}(E_d^{c_d+\varepsilon_d}) \geqq \operatorname{cat}(A_+ \cup A_-) = \operatorname{cat}(A_+) + \operatorname{cat}(A_-) \geqq 2\operatorname{cat}(\partial\Omega). \quad \square$$

Lemma 3.2. *Let u be a critical point of E_d with*

$$E_d(u) < 2^{(p-2)/p} c_d. \tag{3.6}$$

Then u does not change sign.

Proof. If the conclusion were not true, we would have $u = u_+ + u_-$, with $u_+ \not\equiv 0$ and $u_- \not\equiv 0$. By the definition of c_d,

$$\|u_\pm\|^2_{L^p(\Omega)} \cdot c_d \leqq \int_\Omega (d|\nabla u_\pm|^2 + u_\pm^2)\, dx.$$

By equation $(\mathrm{I})_d$,

$$\int_\Omega (d|\nabla u_\pm|^2 + u_\pm^2)\, dx = E_d(u) \int_\Omega |u_\pm|^p\, dx.$$

From this and the assumption (3.6), we obtain

$$\|u_\pm\|^p_{L^p(\Omega)} \geqq \left[\frac{c_d}{E_d(u)}\right]^{p/(p-2)} > \frac{1}{2}.$$

Therefore we get a contradiction by

$$1 = \|u\|^p_{L^p(\Omega)} = \|u_+\|^p_{L^p(\Omega)} + \|u_-\|^p_{L^p(\Omega)} > 1. \quad \square$$

Proof of Theorem 0.1. By Proposition 2.1, $c_d = d^{N(p-2)/2p}(m(+, 1) + o(1))$ as $d \to 0$. For $\varepsilon_1 > 0$ given in Proposition 2.3, we choose $0 < \varepsilon_0 \leqq \varepsilon_1$ with the property that

$$\varepsilon_0 < (2^{(p-2)/p} - 1)\, m(+, 1).$$

Then there exists $d_0 > 0$ such that for all $d \in (0, d_0)$,

$$c_d + d^{N(p-2)/2p} \varepsilon_0 < 2^{(p-2)/p} c_d.$$

For this $\varepsilon_0 > 0$, by Lemma 3.1 there exists a $d_0' > 0$ such that

$$\operatorname{cat}(E_d^{c_d+\varepsilon_d}) \geqq 2\operatorname{cat}(\partial\Omega) \qquad \forall\, d \in (0, d_0')$$

with $\varepsilon_d = d^{N(p-2)/2p} \varepsilon_0$.

Applying the minimax method here one may get that there exist at least $2\operatorname{cat}(\partial\Omega)$ critical points of E_d in $E_d^{c_d+\varepsilon_d}$ for $d \leqq \min\{d_0, d_0'\}$. By Lemma 3.2 and (3.7) none of these critical points changes sign, and therefore there exist at least $\operatorname{cat}(\partial\Omega)$ positive critical points and thus $\operatorname{cat}(\partial\Omega)$ solutions of $(\mathrm{I})_d$. $\square$

Remark 3.1. From this proof we can see that if u_d is a solution of $(\mathrm{I})_d$ given by Theorem 0.1, then

$$E_d\left(\frac{u_d}{\|u_d\|_{L^p(\Omega)}}\right) = d^{N(p-2)/2p}(m(+,1)+\varepsilon(u_d)) \quad \text{as } d \to 0$$

where $\varepsilon(u_d) \to 0$ as $d \to 0$ uniformly for solutions given by Theorem 0.1.

§ 4. The Proof of Theorem 0.2 and More on the Shape of Solutions

In this section we study the shape of the solutions we obtain in Theorem 0.1. The property of the least-energy solution, which corresponds to c_d here, was studied in [NT] where the shape of the least-energy solution was characterized quite clearly. We prove that all solutions obtained above in Theorem 0.1 have the same properties as the least-energy solution does in the following sense: Each solution has at most one local maximum over Ω, which is achieved at a point on the boundary. At the end of this section we state two more results in this direction, which are analogues of Theorem 2.3 and Corollary 2.4 in [NT]. It is asserted in [NT] that as $d \to 0$, the maximum points of the least-energy solutions tend to a point on $\partial\Omega$ where the boundary mean curvature assumes its maximum. Here, for solutions obtained in Theorem 0.1 I tend to believe that the maxima points are close to critical points of the mean curvature of the boundary. The proof of Theorem 0.2 is carried out in three steps. Step 1 is a key estimate on the location of local maxima of solutions. Once having established Step 1 we can adopt the arguments in [NT] to prove that the local maximum must be on the boundary of Ω (Step 2). Finally, the uniqueness of local maxima is proved in Step 3. In Steps 1 and 3 the concentration-compactness argument is employed again.

Proposition 4.1. *There exists a constant $C > 0$, which is independent of d and solutions of $(\mathrm{I})_d$ obtained in Theorem 0.1, such that if u_d is a solution of $(\mathrm{I})_d$ given by Theorem 0.1 that achieves a local maximum at $P_d \in \bar{\Omega}$, then*

$$\operatorname{dist}(P_d, \partial\Omega) \leqq C\sqrt{d}, \quad \text{as } d \to 0. \tag{4.1}$$

Proof. If not, there exists $d_n \to 0$ and solutions u_n of $(\mathrm{I})_{d_n}$ that achieve local maxima at $P_n \in \bar{\Omega}$, such that

$$\frac{\operatorname{dist}(P_n, \partial\Omega)}{\sqrt{d_n}} \to +\infty \quad \text{as } d_n \to 0. \tag{4.2}$$

By Remark 3.1, we have

$$E_{d_n}\left(\frac{u_n}{\|u_n\|_{L^p(\Omega)}}\right) = d_n^{N(p-2)/2p}(m(+,1)+o(1)) \quad \text{as } n \to \infty.$$

By Lemma 1.1 and Proposition 2.1, $v_n = \sigma(u_n/\|u_n\|_{L^p(\Omega)}) \in V_1(\Omega_{1/\sqrt{d_n}})$ and satisfies

$$\int_{\Omega_{1/\sqrt{d_n}}} (|\nabla v_n|^2 + v_n^2)\, dx \to m(+,1) \quad \text{as } n \to \infty.$$

By Lemma 2.1, there exist a subsequence v_n (still denoted by v_n), $y_n \in \mathbf{R}^n$, and a constant $\bar{C} > 0$ independent of n, such that for each $\varepsilon > 0$ there exists $R > 0$ for which

$$\lim_{n\to\infty} \int_{\mathbf{R}^N} \chi_n |v_n|^p\, dx \geqq 1 - \varepsilon,$$

$$\operatorname{dist}(y_n, \partial\Omega_{1/\sqrt{d_n}}) \leqq \bar{C}.$$

Now we need Lemma 4.5 in [LNT] or Lemma 4.1 in [NT].

Lemma 4.1. *If u_d is a solution of $(\mathrm{I})_d$ which attains a local maximum at $x_0 \in \bar{\Omega}$, then there is $\eta_0 > 0$ independent of x_0 and d such that $u_d(x) \geqq \eta_0$ for $x \in B_{\sqrt{d}}(x_0) \cap \Omega$ provided that d is sufficiently small.*

Applying Lemma 4.1 to our sequence u_n, we get that there exists $\lambda_0 > 0$ such that

$$u_n(x) \geqq \lambda_0 \quad \forall x \in B_{\sqrt{d_n}}(P_n) \cap \Omega.$$

Since u_n satisfies $(\mathrm{I})_{d_n}$, we have

$$\|u_n\|_{L^p(\Omega)} = \left[E_{d_n}\left(\frac{u_n}{\|u_n\|_{L^p(\Omega)}}\right)\right]^{1/(p-2)}.$$

Combining this and Remark 3.1, we get

$$u_n(\sqrt{d_n}x) = (m(+,1)+o(1))\, v_n(x).$$

Therefore there is $\lambda_1 > 0$ such that for n large,

$$v_n(x) \geqq \lambda_1 \quad \forall x \in B_1(\bar{P}_n) \cap \Omega_{1/\sqrt{d_n}}$$

where $\bar{P}_n = P_n/\sqrt{d_n}$. By (4.2),

$$\operatorname{dist}(B_1(\bar{P}_n), \partial\Omega_{1/\sqrt{d_n}}) \to \infty \quad \text{as } n \to \infty.$$

Therefore there exists $\lambda_2 > 0$ such that for n large

$$\int_{B_1(\bar{P}_n)} |v_n|^p\, dx \geqq \lambda_2. \tag{4.3}$$

Take $\varepsilon > 0$ satisfying $\varepsilon < \lambda_2$; then there is $R > 0$ such that

$$\int_{B_R(y_n)\cap\Omega_{1/\sqrt{d}}} |v_n|^p\, dx \geqq 1 - \varepsilon \quad \text{for } n \text{ large}. \tag{4.4}$$

However, from (4.3) and (4.4),

$$1 = \int_{\Omega_{1/\sqrt{d_n}}} |v_n|^p \, dx \geqq \int_{B_R(y_n)\cap\Omega_{1/\sqrt{d}}} |v_n|^p \, dx + \int_{B_1(\bar{P}_n)} |v_n|^p \, dx \geqq 1 - \varepsilon + \lambda_2,$$

which yields a contradiction. The proof of Proposition 4.1 is complete. □

Proposition 4.2. *There exists a $d_1 > 0$ such that for all $d \leqq d_1$, the solutions of $(\mathrm{I})_d$ obtained in Theorem 0.1 achieve their local maxima only on the boundary $\partial\Omega$.*

Proof. If not, there would exist a decreasing sequence $d_n \to 0$ and a sequence u_n of solutions of $(\mathrm{I})_{d_n}$ such that u_n achieves its local maximum at $P_n \in \bar{\Omega}$, but $P_n \notin \partial\Omega$. By Proposition 4.1, we may assume that $P_n \to P_0 \in \partial\Omega$ as $n \to \infty$ by passing to a subsequence if necessary. Now a contradiction can be deduced by applying the argument in Step 2 of the proof of Theorem 2.1 in [NT], which we do not include here. □

Proposition 4.3. *There exists a $d_2 > 0$ such that for all $d \leqq d_2$, any solution of $(\mathrm{I})_d$ obtained in Theorem 0.1 has at most one local maximum.*

Proof. If not, there would exist a decreasing sequence $d_n \to 0$ and a sequence of solutions u_n of $(\mathrm{I})_{d_n}$ such that u_n has at least two local maxima at P_n and P_n'. By Propositions 4.1 and 4.2, we know that P_n' and P_n must be on the boundary. Moreover, the argument in the proof of Proposition 4.2 (also see [NT]) can show that

$$\frac{|P_n - P_n'|}{\sqrt{d_n}} \to +\infty \quad \text{as } n \to \infty. \tag{4.5}$$

By rescaling, we obtain $v_n \in V_1(\Omega_{1/\sqrt{d_n}})$ from u_n such that

$$\int_{\Omega_{1/\sqrt{d_n}}} (|\nabla v_n|^2 + v_n^2) \, dx \to m(+,1) \quad \text{as } n \to \infty.$$

By Lemma 2.1, we obtain the compactness of v_n; i.e., there exist a subsequence of v_n (still denoted by v_n), $y_n \in \mathbf{R}^N$, and a constant $\bar{C} > 0$ such that for each $\varepsilon > 0$ there exists $R > 0$ such that

$$\lim_{n\to\infty} \int_{B_R(y_n)\cap\Omega_{1/\sqrt{d_n}}} |v_n|^p \, dx \geqq 1 - \varepsilon,$$

$$\operatorname{dist}(y_n, \partial\Omega_{1/\sqrt{d_n}}) \leqq \bar{C}.$$

Let $\bar{P}_n = P_n/\sqrt{d_n}$ and $\bar{P}_n' = P_n'/\sqrt{d_n}$. By using (4.5) we may also assume that at least one of the following holds:

$$\operatorname{dist}(y_n, \bar{P}_n) \to +\infty \quad \text{as } n \to \infty, \tag{4.6}$$

$$\operatorname{dist}(y_n, \bar{P}_n') \to +\infty \quad \text{as } n \to \infty. \tag{4.7}$$

Let us assume that (4.6) holds.

As noted in the proof of Proposition 4.1, we can use Lemma 4.1 to show there exists a $\lambda_1 > 0$ such that

$$v_n(x) \geqq \lambda_1 \quad \forall\, x \in B_1(\bar{P}_n) \cap \Omega_{1/\sqrt{d_n}}.$$

By the fact used in § 2 that

$$L^n\{B_1(\bar{P}_n) \cap \Omega_{1/\sqrt{d_n}}\} \to L^n\{B_1^+\} \quad \text{as } n \to \infty$$

(see (2.13) and (2.14)), we may conclude that there exists $\lambda_2 > 0$ such that

$$\int_{B_1(\bar{P}_n)\cap\Omega_{1/\sqrt{d_n}}} |v_n|^p \, dx \geqq \lambda_2 \quad \text{for } n \text{ large}. \tag{4.8}$$

Take $\varepsilon > 0$ such that $\varepsilon < \lambda_2$; then there exists $R > 0$ such that

$$\int_{B_R(y_n)\cap\Omega_{1/\sqrt{d_n}}} |v_n|^p \, dx \geqq 1 - \varepsilon. \tag{4.9}$$

By (4.6), (4.8), and (4.9) we get a contradiction for n large:

$$1 = \int_{\Omega_{1/\sqrt{d_n}}} |v_n|^p dx \geqq \int_{B_R(y_n)\cap\Omega_{1/\sqrt{d_n}}} |v_n|^p dx + \int_{B_1(\bar{P}_n)\cap\Omega_{1/\sqrt{d_n}}} |v_n|^p dx \geqq 1 - \varepsilon + \lambda_2 > 1.$$

The proof of Proposition 4.3 is complete. □

The proof of Theorem 0.2 follows immediately from Propositions 4.1, 4.2, and 4.3.

In order to give another result we introduce a diffeomorphism that straightens a boundary portion near $P \in \partial\Omega$ (this was first used in [NT]). We may assume that P is the origin, and that the inner normal to $\partial\Omega$ at P points in the direction of the positive x^N-axis. Then there exists a smooth function $\psi(x')$, $x' = (x_1, \ldots, x^{N-1})$, defined for $|x'| \leqq a$ such that (i) $\psi(0) = 0$ and $\nabla\psi(0) = 0$, and (ii) $\partial\Omega \cap \mathscr{N} = \{(x', x^N) \mid x^N = \psi(x')\}$ and $\Omega \cap \mathscr{N} = \{(x', x^N) \mid x^N > \psi(x')\}$, where $\mathscr{N}$ is a neighborhood of P. For $y \in \mathbf{R}^N$ with $|y|$ sufficiently small, we define a mapping $x = \Phi(y)$ with $\Phi(y) = (\Phi_1(y), \ldots, \Phi_N(y))$ by

$$\Phi_j(y) = \begin{cases} y^j - y^N \dfrac{\partial \Psi}{\partial x^j}(y') & \text{for } j = 1, \ldots, N-1, \\ y^N + \psi(y') & \text{for } j = N. \end{cases}$$

Since $D\Phi(0) = I$, the identity map, Φ has the inverse mapping $y = \Phi^{-1}(x)$ for $|x| < a'$. We write it as $\Psi(x) = (\Psi_1(x), \ldots, \Psi_N(x))$ instead of $\Phi^{-1}(x)$.

Let U be an open set in $\mathbf{R}^N$. For $v \in C^2(\bar{U})$ and $d > 0$, put

$$\|v\|_{C_d^2(\bar{U})} = \sum_{|\alpha|\leqq 2} d^{|\alpha|/2} \|D^\alpha v\|_{C^0(\bar{U})}.$$

Now we can state an approximation theorem for all solutions given by Theorem 0.1. This is an analogue of Theorem 2.3 in [NT], where least-energy solutions were studied. The proof in [NT] applies here without any difficulty so we omit the proof.

Theorem 4.1 *Assume that u_d is a solution of $(\mathrm{I})_d$ given by Theorem 0.1 and that u_d achieves its maximum at $P_d \in \partial\Omega$. Then for any $\varepsilon > 0$ there exist $d_0 > 0$ and a subdomain $\Omega_d^{(i)} \subset \Omega$ such that for $d \in (0, d_0)$ the following hold:*
(i) $P_d \in \partial\Omega_d^{(i)}$,
(ii) $\| u_d(\cdot) - \omega(\Psi(\cdot)/\sqrt{d})\|_{C_d^2(\Omega_d^{(i)})} \leqq \varepsilon$,
(iii) $|u_d(x)| \leqq C_1 \varepsilon e^{-\mu_1\delta(x)/\sqrt{d}}$ for $x \in \Omega_d^{(o)} = \Omega \backslash \Omega_d^{(i)}$,
where $\delta(x) = \min\{\operatorname{dist}(x, \partial\Omega_d^{(i)}), \eta_0\}$ and C_1, μ_1, η_0 are positive constants depending only on Ω.

In particular, if we define

$$\Omega_{\eta,d} = \{x \in \Omega \mid u_d(x) > \eta\}$$

for $\eta > 0$, then $\Omega_{\eta,d}$ has at most one connected component, by Theorem 0.2. We also have

Corollary 4.1. *Under the same condition as in Theorem 0.2,*

$$\frac{1}{\sqrt{d}}\,\Psi(\Omega_{\eta,d}) \to B^+_{\omega^{-1}(\eta)} \quad \text{as } d \to 0$$

for every $\eta \in (0, \max \omega)$, where ω is the solution in Proposition 1.1 and $B^+_{\omega^{-1}(\eta)} = \{z \in \mathbf{R}^N_+ \mid \omega(z) > \eta\}$.

§ 5. The Proof of Lemma 2.1

To prove Lemma 2.1, we need the following technical lemma.

Lemma 5.1. *Let $R > 0$ be given. Then there exist an integer $k \geqq 1$ and a cone $\mathscr{C}$ such that for n large one can find a covering Λ_n of Ω_n consisting of balls of radius either R or $2R$ with the following properties:*
(a) *Any $x \in \Omega_n$ is contained in at most k balls in Λ_n.*
(b) *For any ball B in Λ_n, $B \cap \Omega_n$ satisfies the cone $\mathscr{C}$ condition.*

Proof. First we choose a covering Λ of $\mathbf{R}^N$ consisting of all balls with radius R centered at points of form $(l_1 R, l_2 R, \ldots, l_N R)$ where $l_j = 0, \pm 1, \pm 2, \ldots$, $j = 1, 2, \ldots, N$. Then it is easy to check that any $x \in \mathbf{R}^N$ is contained in at most 2^N balls in Λ. By (2.13) and (2.14) we can choose a $n_1 > 0$ such that for all $n \geqq n_1$, $B_{2R}(y) \cap \Omega_n$ has only one connected component for all $y \in \partial\Omega_n$. Moreover, for this n_1 we may also find a cone $\mathscr{C}$ such that for all $y \in \partial\Omega_{n_1}$, $B_{2R}(y) \cap \Omega_{n_1}$ has the cone $\mathscr{C}$ condition. This can be seen from the smoothness and compactness of $\partial\Omega_{n_1}$. By (2.13) and (2.14) again we find that for $n \geqq n_1$, $B_{2R}(y) \cap \Omega_n$ still has the cone $\mathscr{C}$ condition for $y \in \partial\Omega_n$. Now for $n \geqq n_1$, we define Λ_n to be a covering of Ω_n as follows: For a ball in Λ, say $B_R(x)$, if $B_R(x) \cap \partial\Omega_n = \phi$, then $B_R(x) \in \Lambda_n$; if $B_R(x) \cap \partial\Omega_n \neq \phi$, we replace $B_R(x)$ by $B_{2R}(y)$ and add $B_{2R}(y)$ in Λ_n, where $y \in \partial\Omega_n$ satisfies $\operatorname{dist}(x, y) = \operatorname{dist}(x, \partial\Omega_n) \leqq R$ (so $B_R(x) \subset B_{2R}(y)$). Therefore Λ_n is a covering of Ω_n (in fact a covering of $\mathbf{R}^N$). It is not difficult to verify that, for any $x \in \mathbf{R}^N$, x is contained to at most 6^N balls in Λ_n. Moreover, (b) is also satisfied with the cone $\mathscr{C}$ we chose above for n_1. $\square$

Proof of Lemma 2.1. Take $\bar{p} = (p+2)/2$, $2 < \bar{p} < p$; then by the Hölder inequality,

$$\sup_{y\in\mathbf{R}^N} \int_{\Omega_n \cap B_{iR}(y)} |v_n|^{\bar{p}}\, dx \to 0 \quad \text{as } n \to \infty,\ i = 1, 2. \tag{5.1}$$

Also by the Hölder inequality,

$$\sup_{y\in\mathbf{R}^N} \int_{\Omega_n \cap B_{iR}(y)} |v_n|^{\bar{p}-1} |\nabla v_n|\, dx \tag{5.2}$$

$$\leqq \sup_{y\in\mathbf{R}^N} \left\{\int_{\Omega_n \cap B_{iR}(y)} |v_n|^{p}\, dx\right\}^{1/2} \left\{\int_{\Omega_n \cap B_{iR}(y)} |\nabla v_n|^2\, dx\right\}^{1/2}$$

$$\to 0 \text{ as } n \to \infty,\ i = 1, 2.$$

Limits (5.1) and (5.2) show that $|v_n|^{\bar{p}} \in W^{1,1}(\Omega_n)$; by the Sobolev Embedding Theorem (see [A]),

$$\int_{\Omega_n \cap B_{iR}(y)} |v_n|^{\bar{p}\gamma}\, dx \leqq c(\gamma) \left\{\int_{\Omega_n \cap B_{iR}(y)} (|v_n|^{\bar{p}} + \bar{p}|v_n|^{\bar{p}-1}|\nabla v_n|)\, dx\right\}^{\gamma}$$

$$\leqq c(\gamma)\, \varepsilon_n^{\gamma-1} \int_{\Omega_n \cap B_{iR}(y)} (|v_n|^{\bar{p}} + \bar{p}|v_n|^{\bar{p}-1}|\nabla v_n|)\, dx$$

for $\gamma \in \left(1, \dfrac{N}{N-1}\right)$, where $c(\gamma)$ is a constant depending on the cone condition of $\Omega_n \cap B_{iR}(y)$, and $\varepsilon_n \to 0$ as $n \to \infty$ uniformly for $y \in \mathbf{R}^N$.

Now, we use the covering Λ_n of Ω_n given in Lemma 5.1. By (b) there exists a $C_1 > 0$ depending only upon the cone $\mathscr{C}$ in (b), such that for any n,

$$\int_{B\cap\Omega_n} |v_n|^{\bar{p}\gamma}\, dx \leqq C_1 \| v_n^{\bar{p}} \|^{\gamma}_{W^{1,1}(B\cap\Omega_n)} \quad \forall B \in \Lambda_n. \tag{5.3}$$

Then by (5.3) and (a) in Lemma 5.1.

$$\int_{\Omega_n} |v_n|^{\bar{p}\gamma}\, dx \leqq \sum_{B\in\Lambda_n} \int_{\Omega_n \cap B} |v_n|^{\bar{p}\gamma}\, dx$$

$$\leqq \sum_{B\in\Lambda_n} C_1 \varepsilon_n^{\gamma-1} \int_{B\cap\Omega_n} (|v_n|^{\bar{p}} + \bar{p}|v_n|^{\bar{p}-1}|\nabla v_n|)\, dx$$

$$\leqq C_1 \varepsilon_n^{\gamma-1} k \int_{\Omega_n} (|v_n|^{\bar{p}} + \bar{p}|v_n|^{\bar{p}-1}|\nabla v_n|)\, dx$$

$$\leqq C_1 \varepsilon_n^{\gamma-1} k C_2 \;\alpha\; 0 \quad \text{as } n \to \infty,$$

where $\| v_n^{\bar{p}} \|_{W^{1,1}(\Omega_n)} \leqq C_2$ is independent of n. Now let us take $\gamma = p/\bar{p} = 2p/(p+2)$; then $1 < \gamma < N/(N-1)$ for $2 < p < 2N/(N-2)$. We get

$$\int_{\Omega_n} |v_n|^{p}\, dx \to 0 \quad \text{as } n \to \infty. \quad \square$$

Notes Added in Proof

1. While revising this paper, I learned that in a recent preprint, G. MANCINI & R. MUSINA have obtained the same existence result as Theorem 0.1.
2. I have generalized the existence result of Theorem 0.1 to exterior-domain problems [W1], and under an additional assumption on the geometry of the boundary, to critical exponent problems in bounded domains [W2].

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