THREE–DIMENSIONAL STATISTICAL GRAPHICS BASED ON INTERACTIVELY ANIMATED ANAGLYPHS

Jürgen Symanzik
Iowa State University, Department Of Statistics, Ames, Iowa 50011
e-mail: symanzik@iastate.edu

Abstract: First mentioned in 1853 by Rollmann, anaglyphs today are considered as an important means to represent three–dimensional pictures on flat surfaces in a variety of application areas such as geometry, chemistry, architecture, and mining. However, most statisticians ignore the power of printed anaglyphs and especially of interactively animated anaglyphs on CRT display systems. Generally, three–dimensional data displays such as three–dimensional scatterplots and three–dimensional graphs of bivariate probability density functions yield better impressions of the underlying data than two–dimensional displays. Relatively little additional mathematical calculus, i. e., two central projections with different COPs calculated independently, must be done and only cheap additional "hardware", i. e., red and green filter glasses, is required to produce impressing stereoscopic graphics.

Keywords: Stereoscopic Graphics, 3D Computer Graphics, Three–Dimensional Scatterplots.

1. Introduction

Every human being has a plastic impression of surrounding objects. Moreover, during our lifetime, we learn to anticipate three dimensions even if we only see two dimensions such as on drawings, pictures and photographs. However, this anticipation does not replace real three–dimensional objects – especially if we don't know what we are expected to see. Therefore, techniques such as reliefs, drawings with central perspectives, anaglyphs, and holograms were developed to provide three dimensions without using a solid model. A technique for this purpose can be considered as optimal if it only requires two dimensions, e. g. paper or a computer screen, and thus is easy and cheap to be reproduced and transported.

Already black and white stereoscopic graphics such as Figure 1 provide an acceptable impression of the underlying three–dimensional object, provided the figure is properly enlarged. Colored stereoscopic images can be found in Hodges (1992).

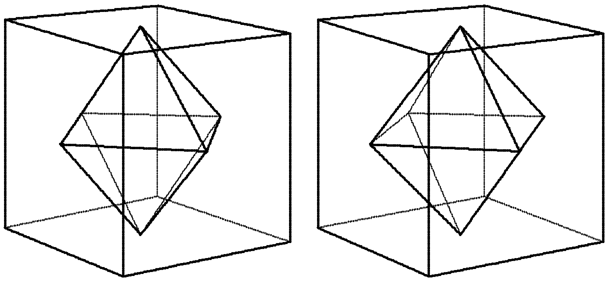

Figure 1: Cube with Inscribed Octaeder (Graf (1968), Abb. 295.1)

This article focuses on a technique called anaglyphs where each image is drawn in complementary colors. Additionally, colored filter glasses are required.

Section 2 introduces anaglyphs while the third section gives a historical overview on examples and applications of anaglyphs. In Section 4 the connection between anaglyphs and statistics is shown. The mathematical calculus required to draw anaglyphs is provided in the fifth section. A discussion of anaglyphs in comparision to other three–dimensional graphical representation techniques terminates this paper.

2. Viewing Anaglyphs

The word anaglyphs originates from the Greek. Its meaning is that an object is shaped relief–like, i. e., it has a non–flat surface. In science, the term anaglyphs describes two projections of a three-dimensional figure, usually drawn in complementary colors red and green, that are looked at through filter glasses of the same colors. The onlooker is expected to obtain the impression of a three-dimensional object.

Viewing anaglyphs can best be described when it is compared to natural viewing. Mucke (1970) is a good reference and can be summarized as follows:

All points of a common plane perpendicular to the viewing direction h, in Figure 2 points A, B, and C, produce a congruent mapping within each of our eyes (left: $\overline{a}\overline{b}\overline{c}$, right: $\tilde{a}\tilde{b}\tilde{c}$). At our imaginary eye E both mappings can be shifted to cover each other. However, a point P, here assumed to be in front of the plane, causes mappings that appear on different positions in the two eyes, i. e. $\overline{p}$ between $\overline{b}$ and $\overline{c}$, but $\tilde{p}$ between $\tilde{a}$ and $\tilde{b}$. This difference in location causes our brain to see that P is not located in the ABC plane but in front of it. The same holds for a point behind the plane.

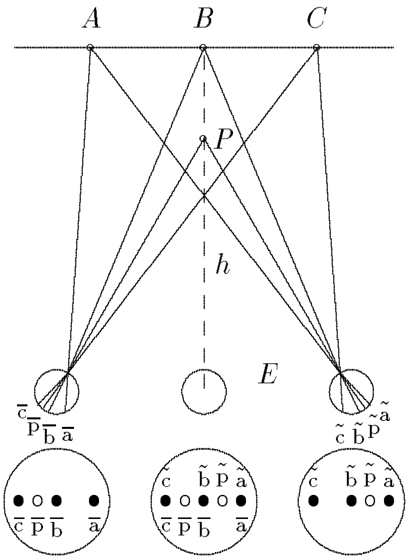

Figure 2: Natural Viewing

Now assume we are drawing two partial mappings in complementary colors on the same surface. Also assume we are wearing filter glasses where, e. g., the left glass is red and the right glass is green.

Consider the situation in Figure 3. If all colors have been selected adequately, then point $\tilde{p}$ of the red mapping is invisible for the left eye as well as point $\overline{p}$ is invisible for the right eye. However, when looking through the red glass the green point $\overline{p}$ appears black for the left eye. Also the red point $\tilde{p}$ and the green glass merge to black for the right eye.

Thus both eyes only see what they would see if a real object would appear at position P. Hence anaglyphs provide real three–dimensional images by only using additional filter glasses.

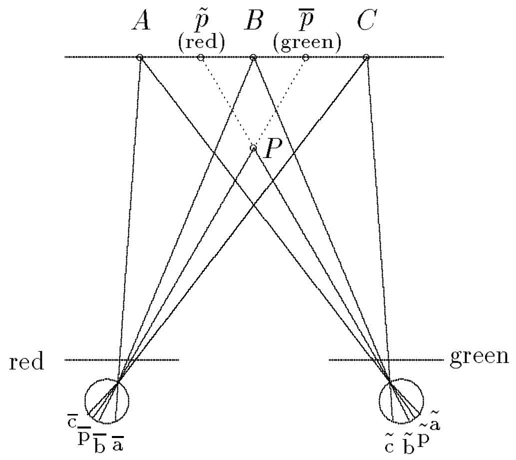

Figure 3: Viewing Anaglyphs

However, human viewing is also effected by some other factors that are different from natural viewing when looking at anaglyphs. Often, the inexperienced onlooker quickly feels tired. But the human brain is capable of learning how to best see anaglyphs for a longer time period. Also, if colors of filter glasses and printed colors don't match exactly, eyes retain a noise picture that makes it harder to see the expected three–dimensional object.

3. History and Examples

This section reviews books and articles that have made major contributions to the distribution of anaglyphs in several sciences. This overview doesn't claim to be complete, but the applications and references cited here provide good examples for further use of anaglyphs, especially in statistics.

Anaglyphs first appeared in 1853. A German teacher, Wilhelm Rollmann, described the effect of stereoscopic graphics drawn in red and green colors that are looked at with the naked eye (Rollmann (1853a)). Rollmann's (1853b) article describing the effect of looking at such colored pictures using filter glasses of corresponding complementary colors has been judged by Vuibert (1912) and Rösch (1954) as the birth of anaglyphs. Independently from Rollmann, the French teacher Joseph Charles d'Almeida proposed in 1858 to use differently colored light to produce anaglyphs. However, the name anaglyphs was introduced by the French Ducos du Hauron not earlier than in 1891. Vuibert

(1912) is one of the first books that entirely deals with anaglyphs.

More than two decades later, Köhler, Graf & Calov (1938) succeeded in printing anaglyphs of high quality. Their so–called *Plastoreoskop–Verfahren* almost perfectly prevents the remainder of an additional noise picture. The main application of anaglyphs at that epoque was geometry. Further references for this topic are Graf (1938, 1941).

Rellensmann/Jung (1939), Rellensmann (1940), and Linhard (1940) deal with early applications of anaglyphs for mining and related applications. The Hungarian Pál is credited with the distribution of anaglyphs all over Europe in a variety of sciences. The books Pál (1961, 1974) contain anaglyphs from fields such as geometry, mechanical engineering, architecture, chemistry, and spatial mathematical problems. Also the anaglyphs in Fejes Tóth (1965) have been created by Pál.

The first textbook, according to its author, that only deals with anaglyphs is Mucke (1970). There, applications are given for ten subdisciplines from mathematics/natural sciences. Mucke/Simon (1966, 1967) also publish books only presenting anaglyphs from geometry. Schmidt (1977) is a late book of high quality anaglyphs dealing with the same topic.

Architecture is an obvious field for the use of anaglyphs. Probably one of the technically best anaglyph pictures printed so far is Schwenkel's (1972) drawing of the Munich Olympics buildings.

During the 60's anaglyphs were introduced to chemistry. Examples are the textbooks of Hollemann/Wiberg (1963) and Klages (1965). Anaglyphs have many potential uses in cartography, biology, medicine, and for CAD applications.

Burkhardt (1963, 1972, 1974) should be mentioned here, too. His work covers technical aspects and problems of printed anaglyphs such as optimal colors, best filter glasses, etc.

Surprisingly, the literature reveals only little use of anaglyphs within statistics, although statistical data sets and problems often are well suited to be drawn as anaglyphs. The articles of Banchoff (1986), Carr, Littlefield & Hall (1986), and Gabriel/Odoroff (1986), all published in Wegman/DePriest (1986), give some impression on the value of anaglyphs within statistics.

4. Anaglyphs and Statistics

Huber (1987), p. 450, also noticed the lack of anaglyphs within statistics [1]. Independently, in 1988, research on anaglyphs within statictics started at the Statistical Department, University of Dortmund, Germany. The goal of this research is to develop computer software to interactively animate anaglyph images for multivariate statistical applications.

A report on anaglyphs in statstics and the description of the PC–program *Analyse*[2] is given in Hering/von der Weydt (1989). Hering/Symanzik (1992) and Symanzik (1993) describe software developed for Sun/Sparc workstations using the XWindow/XView environment. Meanwhile the program *Anaglyphen 3D 2.0* succesfully has been tested on other hardware environments such as DECstations and on IBM compatibles equipped with the LINUX operating system. Symanzik (1992) provides an extended overview on anaglyphs and related software and also presents the *Anaglyphen 3D 2.0* user's manual. The program can be obtained from the author at no cost.

The use of computers to interactively produce anaglyphs provides a way to find good projections before plotting on paper. With adequate hardware real–time three–dimensional animation is possible.

Of course, every application that has been created for interactive systems for graphical data analysis since Tukey's PRIM-9 in 1972 can also be done with anaglyph programs. The big advantage, however, is that anaglyphs are real three–dimensional images.

Interactive analysis of data sets, using anaglyphs, allows the use of rotation to identify useful viewpoints. The use of three–dimensional scatterplots or biplots instead of bivariate scatterplots enables the statistician to obtain a better idea of the distribution of the provided data.

[1] "Statisticians still lag behind other scientists in their use of stereo pairs. [...] The statisticians' efforts (e. g., the plates in Wegman and DePriest 1986) come late and pale in comparison."

[2] A free copy of this program can be obtained from Franz Hering, e–mail: *hering@amadeus.statistik.uni-dortmund.de*

Four–dimensional data sets can be represented as four three–dimensional scatterplots. This obviously is more informative than the six two–dimensional scatterplots that are usually drawn. If all operations such as highlighting, deleting and marking of points, that might be done on two–dimensional scatterplots, are available for three–dimensional ones, this will be a technique worthy of consideration. However, note that the number of different three–dimensional scatterplots increases with magnitude $\binom{p}{3}$ whereas only $\binom{p}{2}$ different two–dimensional scatterplots exist if p is the dimension of the data set.

Aside from three–dimensional scatterplots, mixed mappings can also be best represented as anaglyphs. Cartographic mappings, spatial statistics, time series, three–dimensional histograms, and growth curves are only a few possible examples. Even more sophisticated applications such as development of population pyramids over time, which are impossible when using only two drawing dimensions, now become realistic.

Also bivariate probability/cumulative density functions or any bivariate function as well as contour plots are worthy of display as anaglyphs. Future versions of statistical software packages such as *Splus* hopefully will provide functions such as *persp3D* (Three–dimensional Perspective Plots) or *contour3D* (Contour Plots) that are based on anaglyphs.

5. Mathematical Calculus

Anaglyphs are constructed using two central projections onto the same projection plane with two different centers of projections (COPs). This section summarizes mathematical calculus that can be found for example in Salmon/Slater (1987). Special requirements for anaglyphs are presented in Mucke (1970), Graf (1987), and Hodges (1992).

While standard central projections only require one COP, anaglyphs require two. Assuming a 3D object is between the eyes of the observer, i. e. the COPs Z_l and Z_r, and the projection plane, each point P of the object has to be projected twice onto this plane (see Figure 4). Drawing these projections using red and green colors and looking through filter glasses of the complementary colors makes only one part of the image visible to each eye. The human brain assembles a three–dimensional graphic from these partial images as described in Section 2.

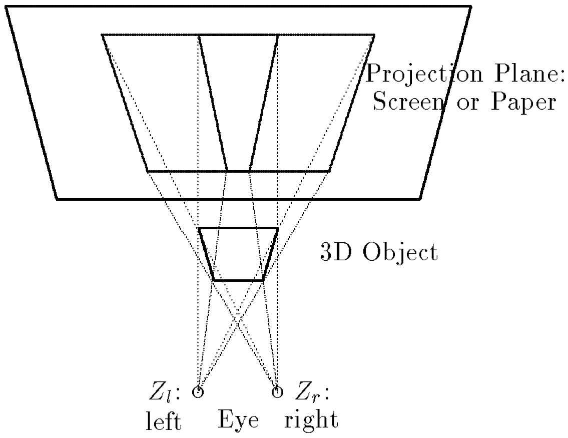

Figure 4: Model of Two Central Projections

For the required mathematical calculus, assume spatial coordinates according to a right hand Cartesian coordinate system.

First, only consider the case of one COP: A projection beam starting at $Z = (0, -a, 0)$ and perpendicular to the projection plane intersects this plane in point $O = (0, 0, 0)$. The distance $a > 0$ is the so called accommodation distance.

A point $E = (x_s, 0, z_s)$ in the projection plane has to be determined for every point $P = (x_0, y_0, z_0)$ as the intersection of the line running through the points Z and P and the projection plane. To calculate E's two–dimensional CRT coordinates (x_p, y_p), projections into the xy and zy plane can be considered independently. Figure 5 shows projections into the xy plane for points P, P_1, P_2, and P_3.
Since

$$\frac{\mid PB \mid}{\mid EO \mid} = \frac{\mid ZB \mid}{\mid ZO \mid} \Leftrightarrow \mid EO \mid = \mid PB \mid \cdot \frac{\mid ZO \mid}{\mid ZB \mid}$$

it follows that $x_p = x_s = x_0 \cdot \frac{a}{y_0+a}$. By analogy, $y_p = z_s = z_0 \cdot \frac{a}{y_0+a}$, can be calculated as the projection into the zy plane.

Some extensions are required for two central projections with COPs Z_l and Z_r (see Figure 6).

Assuming two of the three values, distance from eyes d, accommodation distance a, and angle of convergence α are known, the relations

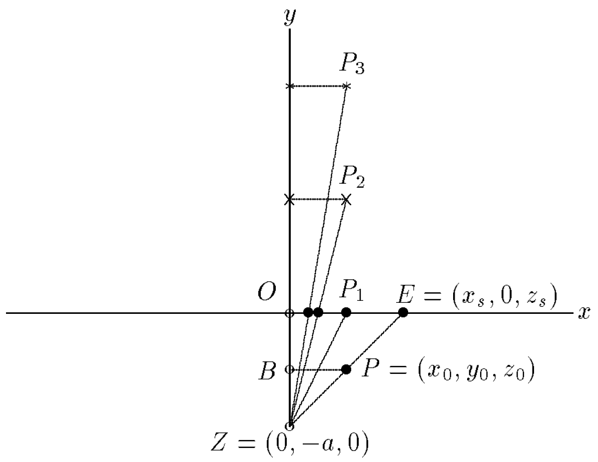

Figure 5: Projection for a Single COP

$\tan\frac{\alpha}{2} = \frac{d/2}{a}$, resp. $\alpha = 2\arctan\frac{d/2}{a}$ provide the third value. COPs for both eyes then are located at the points $Z_l = (-\frac{d}{2}, -a, 0)$ and $Z_r = (\frac{d}{2}, -a, 0)$.

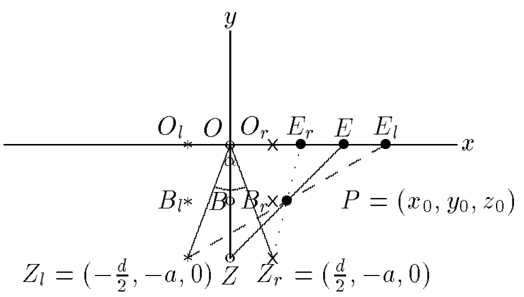

Figure 6: Projection for Two COPs

Here

$$| E_lO_l | = | PB_l | \cdot \frac{| Z_lO_l |}{| Z_lB_l |} = \left| \left(x_0 + \frac{d}{2} \right) \cdot \frac{a}{y_0 + a} \right|$$

and

$$| E_rO_r | = | PB_r | \cdot \frac{| Z_rO_r |}{| Z_rB_r |} = \left| \left(x_0 - \frac{d}{2} \right) \cdot \frac{a}{y_0 + a} \right| .$$

Therefore, it follows that

$$x_{l_p} = \left(x_0 + \frac{d}{2} \right) \cdot \frac{a}{y_0 + a} - \frac{d}{2}$$

and

$$x_{r_p} = \left(x_0 - \frac{d}{2} \right) \cdot \frac{a}{y_0 + a} + \frac{d}{2}.$$

The y coordinate remains unchanged for both COPs, i. e. $y_{l_p} = y_{r_p} = y_p = z_0 \cdot \frac{a}{y_0 + a}$. This holds since vectorial decomposition of line segments through points Z_l and P resp. Z_r and P yields $\Delta z = z_0$ and $\Delta y = y_0 + a$, such that in the distance a the height $\dfrac{\Delta z}{\Delta y} a$ is reached.

6. Discussion

Hodges' (1992) comparision of different stereoscopic CRT display systems is republished as Figure 7. Time multiplexed systems where two different images appear in alternating order require expensive additional hardware. Time parallel/Separate image systems, especially Head–mounted ones, probably will gain more importance in future. Today, they are still too expensive and the technique is not yet as good as it should be. All these techniques have the disadvantage that they are available only for CRTs, but not for printed 3D graphics.

Anaglyphs are available for both CRT and paper display, and they have the big advantage that the required additional "hardware", i. e. the filter glasses, only costs a few cents. Given the variety of important uses for the display and analysis of multivariate data, the development of anaglyphs as a prominent element of statistical graphics must be given serious consideration.

Acknowledgements

I wish to thank Dr. F. Hering for the support given during my research on anaglyphs, Dr. K.–H. Schwefel for his help while writing my computer science diploma thesis, and finally U. Hermes for all his hints on XView, XWindows and UNIX during the programming of *Anaglyphen 3D 2.0*. Thanks are also due to Dr. K. J. Koehler for his final review of the paper.

References

[1] Banchoff, T.F. (1986) Visualizing Two–Dimensional Phenomena in Four–Dimensional Space: A Computer Graphics Approach; in: Wegman, E.J., DePriest, D.J. (Editors) *Statistical Image Processing and Graphics*, Marcel Dekker, New York, pp. 187–202.

[2] Burkhardt, R. (1963) Untersuchungen zur Verbesserung von Anaglyphenbildern, *Bildmessung und Luftbildwesen*, **31**, **Heft 4**, pp. 190–195, (in German).

[3] Burkhardt, R. (1972) Die farbliche Bemessung von Anaglyphenbrillen, *Die Farbe*, **21**, **Nr. 1/6**, pp. 26–32, (in German).

[4] Burkhardt, R. (1974) Studie über Anaglyphen–Druckfarben, *Die Farbe*, **23**, **Nr. 1/4**, pp. 161–172, (in German).

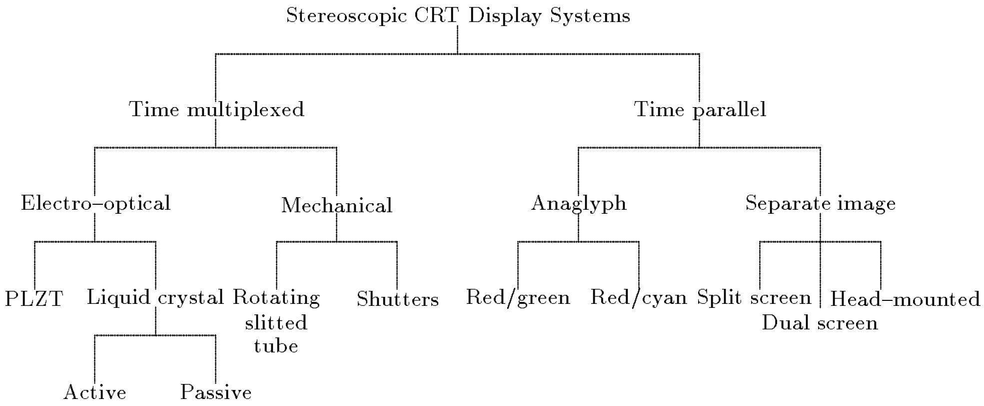

Figure 7: Types of Stereoscopic CRT Display Systems (Hodges (1992), Figure 4)

[5] Carr, D.B., Nicholson, W.L., Littlefield, R.J., Hall, D.L. (1986) Interactive Color Display Methods for Multivariate Data; in: Wegman, E.J., DePriest, D.J. (Editors) *Statistical Image Processing and Graphics*, Marcel Dekker, New York, pp. 215–250.

[6] Fejes Tóth, L. (1965) *Reguläre Figuren*, Akadémiai Kiadó, Budapest, (in German).

[7] Gabriel, K.R., Odoroff, C.L. (1986) Illustrations of Model Diagnosis by Means of Three–Dimensional Biplots; in: Wegman, E.J., DePriest, D.J. (Editors) *Statistical Image Processing and Graphics*, Marcel Dekker, New York, pp. 257–274.

[8] Graf, M. (1987) Drehen und Wenden, Ein Verfahren zur Manipulation räumlicher Objekte, *c't*, **Heft 7**, pp. 126–130, (in German).

[9] Graf, U. (1938) *Sphärische Geometrie, Trigonometrie der Ebene und Kartenentwürfe*, Quelle & Meyer, Leipzig, (in German).

[10] Graf, U. (1941) Konstruierte Anaglyphen, *Bildmessung und Luftbildwesen*, **16**, **Heft 2**, pp. 59–66; erschienen als Beilage in: *Allgemeine Vermessungs–Nachrichten*, **53**, (in German).

[11] Graf, U. (1968) *Darstellende Geometrie*, Neunte Auflage, Quelle & Meyer, Heidelberg, (in German).

[12] Hering, F., Symanzik, J. (1992) *Anaglyphen 3D — Ein Programm zur interaktiven Anaglyphendarstellung*, Forschungsbericht 92/1, Fachbereich Statistik, Universität Dortmund, (in German).

[13] Hering, F., von der Weydt, S. (1989) *Interaktive Anaglyphendarstellungen als Hilfsmittel zur Analyse mehrdimensionaler Daten*, Forschungsbericht 89/7, Fachbereich Statistik, Universität Dortmund, (in German).

[14] Hodges, L.F. (1992) Tutorial: Time–Multiplexed Stereoscopic Computer Graphics, March 1992, *IEEE Computer Graphics & Applications*, pp. 20–30.

[15] Holleman, A.F., Wiberg, E. (1963) *Lehrbuch der anorganischen Chemie*, 57.–70. Auflage, de Gruyter, Berlin, (in German).

[16] Huber, P.J. (1987) Experiences With Three–Dimensional Scatterplots, *Journal of the American Statistical Association*, **82**, **No. 398**, pp. 448–453.

[17] Klages, F. (1965) *Einführung in die organische Chemie*, 2. Auflage, de Gruyter, Berlin, (in German).

[18] Köhler, O., Graf, U., Calov, C. (1938) *Mathematische Raumbilder*, Dreyer, Berlin, (in German).

[19] Linhard, M. (1940) Die räumliche Darstellung von physikalisch–chemischen Prozessen, insbesondere von Schmelz– und Reduktionsvorgängen nach dem Anaglyphenverfahren, *Metall und Erz*, **37**, **Heft 12**, pp. 228–231, (in German).

[20] Mucke, H. (1970) *Anaglyphen Raumzeichnungen*, Teubner, Leipzig, (in German).

[21] Mucke, H., Simon, H. (1966) *Anaglyphen zur darstellenden Geometrie: Zweitafelprojektion*, Volk und Wissen Volkseigener Verlag, Berlin, (in German).

[22] Mucke, H., Simon, H. (1967) *Anaglyphen zur darstellenden Geometrie: Dreitafelprojektion, Axonometrie, Zentralperspektivische Kartenprojektion*, Volk und Wissen Volkseigener Verlag, Berlin, (in German).

[23] Pál, I. (1961) *Darstellende Geometrie in Raumbildern*, 1. Auflage der deutschen Übersetzung, Lindauer, München, (in German).

[24] Pál, I. (1974) *Raumgeometrie in der technischen Praxis*, Akadémiai Kiadó, Budapest, (in German).

[25] Rellensmann, O. (1940) Die Bedeutung des Anaglyphen–Raumbildes für das Grubenrißwesen im Erzbergbau, *Metall und Erz*, **37**, **Heft 12**, pp. 225–228, (in German).

[26] Rellensmann, O., Jung, H. (1939) Das Anaglyphen–Raumbild als Hilfsmittel für die Veranschaulichung von bergmännischen Aufgaben, *Glückauf*, **75**, **Nr. 24**, pp. 530–533, (in German).

[27] Rösch, S. (1954) 100 Jahre Anaglyphen, *Die Farbe*, **3**, **Heft 1/2**, pp. 1–6, (in German).

[28] Rollmann, W. (1853a) Notiz zur Stereoskopie, *Annalen der Physik und Chemie*, **Band 89**, pp. 350–351, (in German).

[29] Rollmann, W. (1853b) Zwei neue stereoskopische Methoden, *Annalen der Physik und Chemie*, **Band 90**, pp. 186–187, (in German).

[30] Salmon, R., Slater, M. (1987) *Computer Graphics: Systems & Concepts*, Addison–Wesley, Wokingham, England, Reading, Massachusetts.

[31] Schmidt, R. (1977) *Darstellende Geometrie mit Stereo–Bildern*, Bauverlag, Wiesbaden, Berlin, (in German).

[32] Schwenkel, D. (1972) Räumlich perspektive Darstellung von Bauwerken durch automatisch gezeichnete Anaglyphenbilder, *Bildmessung und Luftbildwesen*, **40**, **Heft 3**, pp. 144–147, (in German).

[33] Symanzik, J. (1992) *Computerdarstellungen von Anaglyphen – Ein Hilfsmittel der multivariaten Statistik*, Diplomarbeit, Fachbereich Informatik, Universität Dortmund, (in German).

[34] Symanzik, J. (1993) Anaglyphen 3D – A Program for the Interactive Representation of Three–Dimensional Perspective Plots of Statistical Data, (to appear in the 1992 GfKL-Proceedings).

[35] Vuibert, H. (1912) *Les Anaglyphes Géométriques*, Librairie Vuibert, Paris, (in French).

[36] Wegman, E.J., DePriest, D.J. (Editors) (1986) *Statistical Image Processing and Graphics*, Marcel Dekker, New York.