9.4 SYSTEMS OF LINEAR INEQUALITIES; LINEAR PROGRAMMING

My earliest recollection of feeling that mathematics might some day be something special was perhaps in the fourth grade when I showed the arithmetic teachers that the squares always end in—well, whatever it is that they end in.
Irving Kaplansky

Consider the problem of assigning 70 men to 70 jobs. Unfortunately there are 70 factorial permutations, or ways to make the assignments. The problem is to compare 70 factorial ways and to select the one which is optimal, or "best" by some criterion. Even if the Earth were filled with nano-speed computers, all programmed in parallel from the time of the Big Bang until the sun grows cold, it would [be] impossible to examine all the possible solutions. The remarkable thing is that the simplex method with the aid of a modern computer can solve this problem in a split second.
George P. Dantzig

In earlier chapters we solved inequalities that involved single variables. We noted that the solution sets could be shown on a number line. In this section we are interested in solving inequalities in which two variables are involved. We shall see that the solution set may be shown as a region of the plane.

Linear Inequalities

In Section 9.1 we studied linear equations that can be written in the form $ax + by = c$. If we replace the equal sign by one of the inequality symbols, $\leq$, $<$, $\geq$, or $>$, we have a **linear inequality.** The example that follows illustrates a technique for representing the solution set for a linear inequality.

▶EXAMPLE 1 ***A line and Inequalities*** Show all points in the plane that satisfy **(a)** $-x + 2y = 4$, **(b)** $-x + 2y < 4$, and **(c)** $-x + 2y > 4$.

Solution

(a) The points (x, y) that satisfy the equation are on line L whose equation may also be written $y = \frac{1}{2}x + 2$. This appears in Figure 9, which also shows some typical points M, P, and Q, where M is on the line, P is below M, and Q is above M. Since $y_2 < y_1$ and $y_1 = \frac{1}{2}x_1 + 2$, then $y_2 < \frac{1}{2}x_1 + 2$. Similarly, $y_3 > \frac{1}{2}x_1 + 2$.

(b) The inequality can be written $y < \frac{1}{2}x + 2$. The diagram in Figure 10 shows that the coordinates of any point below the line L, such as $P(x_1, y_2)$, will satisfy the given inequality. Any point on or above the line will not. Therefore, the set of points (x, y) that satisfy $-x + 2y < 4$ consists of all points below L. This

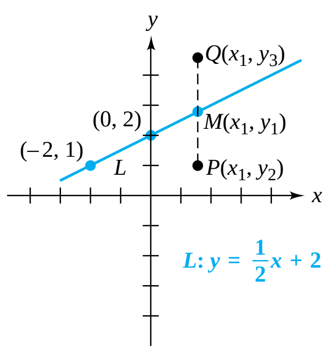


FIGURE 9

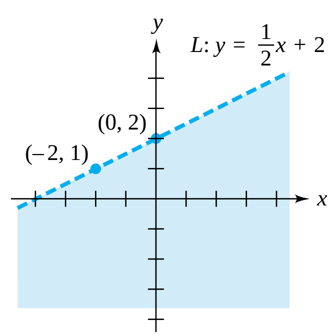


FIGURE 10

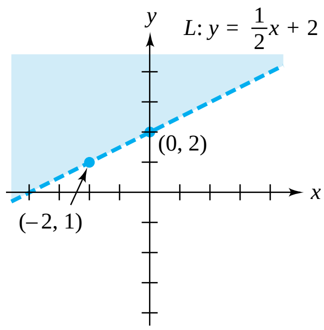


FIGURE 11

is the shaded region (or half-plane) in Figure 10, where L is shown as a broken line to indicate that the points on L are not included in the solution set. Your graphing calculator may be able to graph the kinds of shaded regions in Figures 10 and 11. Look for a DRAW menu.

(c) In a similar manner, the given inequality is equivalent to $y > \frac{1}{2}x + 2$, and the solution set consists of all points in the half-plane above L. See Figure 11. ◀

Parts **(b)** and **(c)** of Example 1 suggest the following definition.

Definition: half-plane

The solution set for a linear inequality, such as $ax + by < c$, consists of all points on one side of the defining line, $ax + by = c$. The graph of the linear inequality is a **half-plane.**

▶EXAMPLE 2 ***A linear inequality*** Graph the inequality $3x - 2y \le 6$.

Strategy: Begin with the boundary line L, $3x - 2y = 6$, and choose a test point that is not on the line. If the coordinates satisfy the desired inequality, the solution is the half-plane that contains the test point; if not, choose the other half-plane.

Solution

We want all points (x, y) that satisfy $3x - 2y < 6$ and all those that satisfy $3x - 2y = 6$. The graph will consist of all points in a half-plane together with the points *on* the boundary line.

Follow the strategy, referring to Figure 12. We must decide which half-plane (above or below L) satisfies the inequality. To do this, take a test point not on L, say $(0, 0)$, and see if it satisfies the inequality.

$$3 \cdot 0 - 2 \cdot 0 \le 6 \quad \text{or} \quad 0 \le 6$$

Since $0 \le 6$ is a true statement, the half-plane that contains $(0, 0)$ is the one we want, the portion of the plane above and to the left of L. The shaded region in Figure 12 including the line L (drawn solid) is the graph of the inequality. ◀

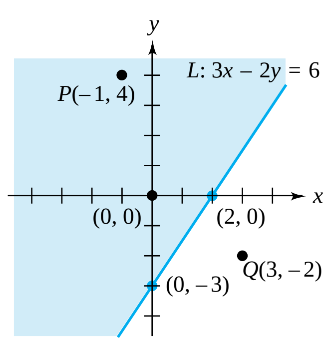


FIGURE 12

The technique for determining the solution set by drawing a graph of a linear inequality, as illustrated in the above example, can be expressed in algorithmic form.

Algorithm for solving a linear inequality

1. Replace the inequality symbol by an equal sign and graph the corresponding line L (broken, for a strict inequality, solid otherwise).
2. Take a test point P not on line L and see if it satisfies the inequality. If it does, then the desired solution set includes all points in the half-plane that contains P; if not, then the solution set consists of the half-plane on the other side of L.

Systems of Inequalities

A **system of linear inequalities** consists of two or more linear inequalities that must be satisfied simultaneously. The following two examples illustrate techniques for determining the solution set or the graph of such a system.

▶EXAMPLE 3 ***System of linear inequalities*** Solve the system of inequalities and show the solution set as a graph in the plane.

$$\begin{aligned} x + 2y &\le 3 \\ -3x + y &< 5 \\ -3x + 8y &\ge -23 \end{aligned}$$

Strategy: Each inequality defines a half-plane, so the solution set for the system is the intersection of three half-planes. Draw each boundary line, find the coordinates of the intersections, and identify the correct half-planes by taking test points.

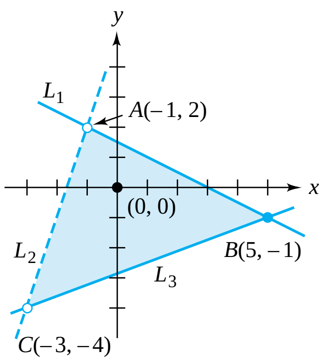


FIGURE 13

Solution

Follow the strategy. First draw graphs of the three lines L_1, L_2, and L_3:

$$L_1\colon x + 2y = 3 \qquad L_2\colon -3x + y = 5 \qquad L_3\colon -3x + 8y = -23.$$

The points of intersection of these three lines, called **corner points,** are obtained by solving the equations in pairs.

$$A\begin{cases} x + 2y = 3 \\ -3x + y = 5 \end{cases} \qquad B\begin{cases} x + 2y = 3 \\ -3x + 8y = -23 \end{cases} \qquad C\begin{cases} -3x + y = 5 \\ -3x + 8y = -23 \end{cases}$$

The three corner points are $A(-1, 2)$, $B(5, -1)$, and $C(-3, -4)$. In Figure 13 L_2 is shown as a broken line, and points A and C are indicated by open circles, since the points on L_2 are not in the solution set.

Returning to the inequalities, identify the points that belong to all three half-planes. Using $(0, 0)$ as a test point, the desired half-planes are below L_1, below L_2, and above L_3. The intersection of the three half-planes, the solution set, is shown as the shaded region in the figure. Any other test point not on any of the three lines would serve as well to identify the three half-planes and their intersection. ◀

▶EXAMPLE 4 ***Mixture problem*** A dietitian wishes to combine two foods, A and B, to make a mixture that contains at least 50 g of protein, at least 130 mg of calcium, and not more than 550 calories. The nutrient values of foods A and B are given in the table.

Food	*Protein (g/cup)*	*Calcium (mg/cup)*	*Calories (cup)*
A	20	20	100
B	10	50	150

How many cups of each of the foods should the dietitian use?

Strategy: We want numbers of cups of A and B, so assign letters (variables). Write inequalities for grams of protein (≥ 50), milligrams of calcium (≥ 130), and number of calories (≤ 550), then draw a graph to show the solution set.

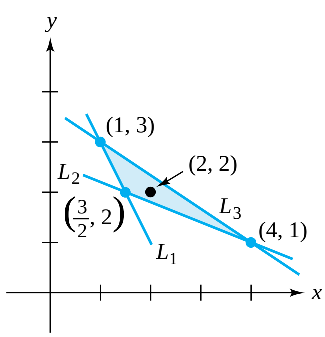


FIGURE 14

Solution

Follow the strategy. Let x be the number of cups of food A and y be the number of cups of food B. The three conditions to be met can be written as inequalities:

$$\begin{aligned} &\text{Protein:} && 20x + 10y \ge 50 \\ &\text{Calcium:} && 20x + 50y \ge 130 \\ &\text{Calories:} && 100x + 150y \le 550. \end{aligned}$$

Simplify the inequalities by dividing each of the first two by 10 and the third by 50, and then graph the three lines L_1, L_2, and L_3,

$$L_1\colon 2x + y = 5 \qquad L_2\colon 2x + 5y = 13 \qquad L_3\colon 2x + 3y = 11.$$

Find the points of intersection of L_1, L_2, and L_3 and draw the lines, as shown in Figure 14. The solution set for the system of inequalities is the region shown. Therefore, any point in the region will give a combination of foods A and B that will satisfy the given constraints. For instance, point $(2, 2)$ is in the region. Taking two cups of each type of food will provide 60 g of protein, 140 mg of calcium, and 500 calories. ◀

HISTORICAL NOTE

SIMPLEX AND KARMARKAR ALGORITHMS FOR LINEAR PROGRAMMING

Example 5 illustrates a kind of problem that modern industry and government face all the time—that of maximizing or minimizing some function subject to constraints or restrictions. An oil refinery, for example, may produce a dozen products (grades of engine oil, gasoline, diesel, and so on), each of which requires different crude oil purchases, refining processes, and storage. Transportation costs and customer demand vary. Refinery and storage capacity and raw material availability also affect what can be produced and the profitability of the whole operation.

The constraints can usually be described by a set of linear inequalities such as those in Example 5. The set of points satisfying the system of inequalities forms some kind of polyhedral region in a high dimensional space like the regions pictured in Figure 15. It turns out that the desired maximum or minimum always occurs at a corner point of the graph. Many industrial or economic applications may present dozens or even hundreds of variables, and locating and testing corner points becomes a staggering problem.

Algorithms for linear programming are used to solve complex problems that face oil companies and firms in other industries

In 1947 an American mathematician, George B. Dantzig, developed a new method for dealing with such problems called the *simplex algorithm for linear programming.* The algorithm uses computers to manipulate matrices in a way that essentially moves from one corner to the next, improving the result at each step. The simplex algorithm has saved untold billions of dollars for industries and consumers worldwide.

Now a new algorithm under investigation promises to deal with even larger problems in less time. This new algorithm, named for its developer, Narendra Karmarkar of Bell Laboratories, intuitively takes shortcuts through the polyhedron, instead of moving along the edges. Scientists, engineers, and economists are working and experimenting to see if computer utilization of the Karmarkar algorithm can significantly improve on the simplex algorithm.

Linear Programming

The Historical Note in this section describes some applications of linear programming. For most such problems we want to maximize or minimize a function, called the **objective function,** subject to conditions (linear inequalities) called **constraints.** The constraints define a set (the set satisfying the system of inequalities) referred to as the **feasible set.** The remarkable fact that makes it possible to solve such optimization problems effectively is the following theorem.

Linear programming theorem

If the objective function of a linear programming problem has a maximum or minimum value on the feasible set, then the extreme value must occur at a corner point of the feasible set.

Some of the problems that linear programming helps solve can include dozens of variables and even more constraints. Such complex problems require sophisticated computer techniques, but we can illustrate all of the key ideas with much simpler problems. We begin by outlining the basic ideas for solving a linear programming problem.

Solving a linear programming problem

1. Name the variables; express the constraints and the objective function in terms of the variables.
2. Sketch the boundaries of the feasible set (one boundary for each constraint).
3. Find the corners of the feasible set.
4. Evaluate the objective function at each corner point to identify maximum and minimum values.

▶EXAMPLE 5 ***Linear programming*** A farmer planning spring planting has decided to plant up to a total of 120 acres in corn and soybeans. An estimate of the investment required and the expected return per acre for each appears in the table.

Crop	*Investment*	*Return*
Corn	\$20	\$50
Soybeans	\$35	\$80

Because corn is needed for feed purposes on the farm, the farmer needs at least 38 acres of corn, and the budget can cover at most \$3000 for both corn and soybeans. How many acres of corn and how many acres of soybeans should be planted to maximize the return from these two crops?

Solution

Let x be the number of acres to be planted in soybeans and y the number of acres of corn. Then we must have $x \geq 0$ and the need for corn as feed implies $y \geq 38$. The total allowable acreage for the two crops is 120 acres, so $x + y \leq 120$. The investment required by x acres of soybeans and y acres of corn is $35x + 20y$, so $35x + 20y \leq 3000$. Finally, the objective function is the expected return, which is $R(x, y) = 80x + 50y$.

We want to maximize $R(x, y)$ on the feasible set, which is defined by the inequalities

$$x \geq 0 \qquad y \geq 38 \qquad x + y \leq 120 \qquad 35x + 20y \leq 3000.$$

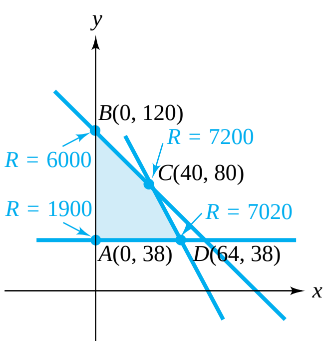


FIGURE 15

Draw a diagram and shade the feasible set. See Figure 15. To find coordinates of the corner points, find the intersections of the boundary lines. The corner points are: $A(0, 38)$, $B(0, 120)$, $C(40, 80)$, and $D(64, 38)$. Finally, determine the estimated return for each choice, that is, evaluate $R(x, y) = 80x + 50y$ at each corner point:

$$R(0, 38) = 1900 \qquad R(0, 120) = 6000$$
$$R(40, 80) = 7200 \qquad R(64, 38) = 7020.$$

The farmer will get the greatest return, subject to the given constraints, by planting 40 acres of soybeans and 80 acres of corn, for a return of \$7200. ◀

EXERCISES 9.4

Check Your Understanding

Exercises 1–6 True or False. Give reasons.

1. The point $(-1, 2)$ is in the solution set for $2x + 3y < 4$.
2. The solution set for the system $x < 0$, $y > 0$, $x + y > 1$ contains only points in the second quadrant.
3. The solution set for the system $x < 0$, $y > 0$, $x + y > 1$ is the empty set.
4. The solution set for the system $x < 0$, $x - y > 1$ contains points in the third quadrant only.
5. The solution set for the system $x < 1$, $y < x$ contains no points in the fourth quadrant.
6. Point $(2, -3)$ is a corner point for the system of inequalities, $2x + 3y \leq -5$, $3x - y \geq 9$, $x - y \leq 1$.

Exercises 7–10 Fill in the blank with the quadrant(s) that make the resulting statement true.

7. The solution set for $y > x + 2$ contains no points in ______.
8. The solution set for $y \geq 2x + 1$ contains points in ______.
9. The solution set for the system $y \geq x$, $y \leq -x$ contains points in ______.
10. The expression $\sqrt{x - y - 2}$ is a real number for some points (x, y) in ______.

Develop Mastery

Exercises 1–4 ***Locating Points*** *Determine whether or not the given pair of numbers (x, y) belongs to the solution set of the system of inequalities.*

1. $x - 3y < 4$, $2x + y < 3$ **(a)** $(1, 1)$ **(b)** $(\sqrt{2}, -0.5)$
2. $-2x + y > -3$, $5x + 2y < 1$ **(a)** $(-1, 2)$ **(b)** $(1, -5)$
3. $x - 3y \geq 1$, $4x - y \leq \pi$ **(a)** $(1, -1)$ **(b)** $(\sqrt{2}, \pi)$
4. $y \leq 2x$, $3x + y > 0$ **(a)** $(0, 0)$ **(b)** $(-1, 3)$

Exercises 5–12 ***Graphing Inequalities*** ***(a)*** *Draw a graph showing all points (x, y) in the solution set of the given inequality.* ***(b)*** *Give coordinates of any two specific pairs (x, y) that satisfy the inequality.*

5. $x + 2y < 4$
6. $-x + y > 3$
7. $2x - 3y \geq 6$
8. $4x - 2y \leq 9$
9. $x + y + 4 < 0$
10. $2x > y - 4$
11. $y \geq 2x$
12. $2y < 3x - 4$

Exercises 13–24 ***Solving Inequalities with Graphs*** *Draw a graph showing the solution set for the system of inequalities. Determine the coordinates of any corner points and show them on your diagram. Indicate which boundary curves and corner points belong and which do not belong to the solution set.*

13. $x + y < 4$, $2x - y < -1$
14. $3x - 2y > 5$, $-x - y < -5$
15. $x - 2y \geq 4$, $|x| > 2$
16. $3x - 4y < 6$, $|x| < 2$, $|y| < 3$
17. $-x + 2y < 5$, $2x + y > 0$, $3x - y < 5$
18. $4x + 3y \leq 16$, $-x + y > -4$, $6x + y \geq 10$
19. $y > 0$, $x + y > 1$
20. $x < 0$, $x + y > 1$
21. $x < 0$, $y > 0$, $x + y > 1$
22. $-1 < x - y \leq 2$, $-2 < x + y \leq 2$
23. $x > 2$, $y > -1$, $x + y < 3$
24. $|x - y| \leq 2$, $|x + y| \leq 2$

Exercises 25–28 ***Which Quadrants?*** *For the system of inequalities, determine which quadrants contain points in the solution set.*

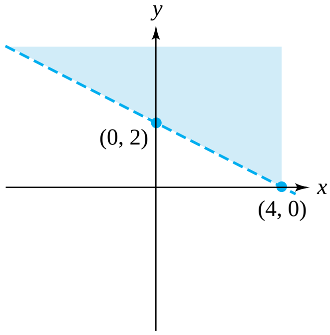
25. $y > 2x$, $y > 4 - x$
26. $x > 1$, $y > x$
27. $x + y \leq 1$, $x - y \leq -1$
28. $x - y \geq 2$, $2x + y \geq 4$

Exercises 29–36 ***Domains*** *Show on a graph all points (x, y) for which the expression will be a real number.*

29. $\sqrt{2x - y - 4}$
30. $\sqrt{x - y + 1}$
31. $\ln(2x + y - 2)$
32. $\log(x - 2y - 4)$
33. $\text{Arcsin}(y - x)$
34. $\text{Arcsin}(x + y + 1)$
35. $\ln x + \ln(y - x)$
36. $\log(x + y) - \log(2x - y)$

Exercises 37–39 ***Write an Inequality*** *Write a linear inequality whose solution set is the shaded region in the diagram.*

37.



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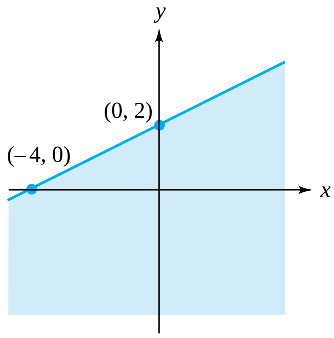

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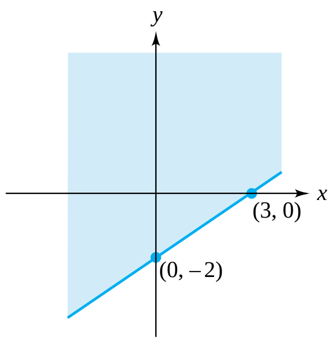

Exercises 40–42 ***Write a System*** *Find a system of inequalities whose solution set is the shaded region in the diagram and give the coordinates of the corner points.*

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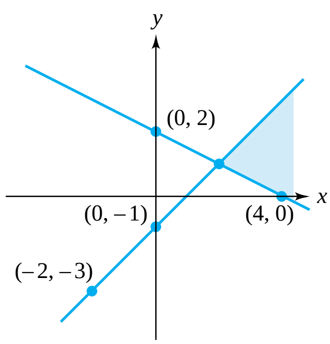

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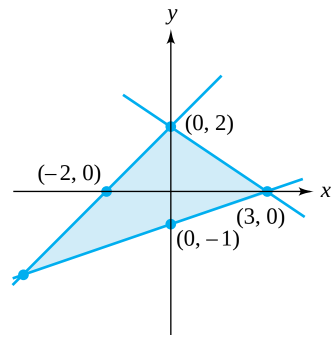

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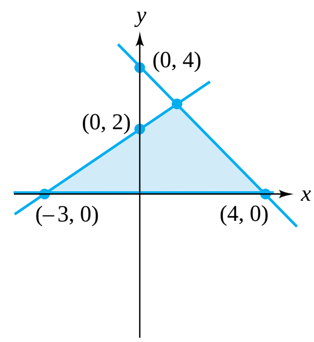

Exercises 43–46 ***System Defining Triangular Region*** ***(a)*** *Draw a diagram showing the set of all points inside the triangle whose vertices are the points A, B, and C.* ***(b)*** *Find a system of inequalities whose solution set consists of all points inside the triangle.*

43. $A(-2, 0)$ $B(0, 4)$ $C(4, -2)$

44. $A(-3, 2)$ $B(3, -2)$ $C(5, 2)$

45. $A(-3, 0)$ $B(0, 4)$ $C(2, 0)$

46. $A(0, 0)$ $B(-2, 2)$ $C(4, 2)$

Exercises 47–48 ***Verbal to System*** *Sketch a graph for the set described and find a system of inequalities for which the set described is the solution set.*

47. All points above the line $2x - y = 1$ and below the line $x + 2y = 4$.

48. All points above the line $y = 2x$ and below the line $x + 2y = 5$.

Exercises 49–54 ***Linear Programming*** *Find the minimum and maximum values of the objective function subject to the given constraints. (Hint: First draw a diagram showing the feasible set and use the linear programming theorem.)*

49. Objective function: $T = 48x + 56y + 120$
$x + y \geq 4,\ y \leq 2x + 1,\ 4x + y \leq 13$

50. Objective function: $T = 36x + 73y - 16$
$x \geq 1,\ y \leq x,\ y \geq 3x - 8$

51. Objective function: $T = 67x + 35y$
$y \leq 2$, $y \leq 2x$, $y \geq x - 4$

52. Objective function: $T = 65x + 124y - 200$
$x + y \geq 3$, $y \leq 2x$, $4x + y \leq 12$

53. Objective function: $T = 84x + 73y - 78$
$x \geq 0$, $y \geq 0$, $x - 3y + 14 \geq 0$, $5x + 2y \leq 32$, $4x + 5y \geq 12$

54. Objective function: $T = 47x + 56y - 24$
$x - 3y + 11 \geq 0$, $4x + y \leq 21$, $3x + 4y \geq 6$

Exercises 55–58 ***Applied Inequalities***

55. A concert is to be presented in an auditorium that has a seating capacity of 800. The price per ticket for 200 of the seats is \$6, and \$3 each for the remaining 600 seats. The total cost for putting on the concert will be \$2100. Draw a graph to show the various possible pairs of numbers of \$6 and \$3 tickets that must be sold for the concert to avoid financial loss.

56. A rancher wants to purchase some lambs and goats—at least five lambs and at least four goats—but cannot spend more than \$800. Each lamb costs \$80, and goats cost \$50 each. How many of each can the rancher buy? Draw a graph to list all possible pairs, keeping in mind that lambs and goats come in whole numbers.

57. A sheep rancher raises two different kinds of sheep for market, Rambis and Eustis, with only enough summer range to support 3000 animals for sale each year. To satisfy loyal customers, the rancher must have at least 750 of each breed available, and because of different range demands, at least a third of the herd should be Rambis. The average profit for the Rambis breed is \$8 per animal, while each Eustis should yield an average of \$10. How many of each breed should the rancher raise to maximize the profit? (*Hint:* If x is the number of Rambis sheep and y is the number of Eustis, the condition that at least a third should be Rambis can be expressed as $x \geq \frac{(x + y)}{3}$ or $y \leq 2x$.)

58. A fish cannery packs tuna in two ways, chunk style and solid pack. Limits on storage space and customer demand lead to these constraints:

The total number of cases produced per day must not exceed 3000.

The number of cases of chunk style must be at least twice the number of cases of solid pack.

At least 600 cases of solid pack must be produced each day.

How many cases of each type can be produced per day if all constraints are to be satisfied? Draw a graph of the solution set and show the coordinates of the corner points.

Exercises 59–61 ***Mixture Problems*** *Use the information from the following table, which gives nutrient values for four foods, A, B, C, and D. Each unit is 100 grams.*

Food	*Energy (calories/unit)*	*Vitamin C (mg/unit)*	*Iron (mg/unit)*
A	200	2	0.5
B	100	3	1.5
C	300	0	2.0
D	400	1	0.0

Food	*Calcium (mg/unit)*	*Protein (g/unit)*	*Carbohydrate (g/unit)*
A	10	2	15
B	4	3	30
C	20	9	10
D	5	3	10

59. In preparing a menu, determine how many units of *A* and of *B* can be included so that the combined nutrient values will satisfy the following constraints:

At least 8 milligrams of vitamin C

At least 18 milligrams of calcium

Not more than 800 calories

60. How many units of *A* and *C* can be included in a menu to contribute:

At least 3 milligrams of vitamin C

At least 40 milligrams of calcium

Not more than 60 grams of carbohydrates

61. How many units of *C* and *D* will give a combined total that satisfies these constraints:

At least 2 milligrams of vitamin C

At least 15 grams of protein

Not more than 6 milligrams of iron

Not more than 2100 calories

Exercises 62–66 ***Acreage and Fertilizer Choices***

62. Would the farmer's decision in Example 5 be different if there were no minimum acreage to be alotted to corn?

63. What would be the optimal planting scheme for Example 5 if the expected return on soybeans were **(a)** \$100 per acre? **(b)** \$110 per acre?

64. In Example 5 how many acres of corn should be planted and how many of soybeans if the return of corn were to drop to \$25 per acre?

65. A commercial gardener wants to feed plants a very specific mix of nitrates and phosphates. Two kinds of fertilizer, Brand A and Brand B, are available, each sold in 50 pound bags, with the following quantities of each mineral per bag:

	Phosphate	*Nitrate*
Brand A	2.5 lbs	10 lbs
Brand B	5.0	5

The gardener wants to put at least 30 lbs of nitrates and 15 lbs of phosphates on the gardens and not more than 250 lbs of fertilizer altogether. If Brand A costs \$8.50 a bag and Brand B costs \$3.50 a bag, how many bags of each would minimize fertilizer costs?

66. Repeat Exercise 65 if the cost of Brand B fertilizer increases to \$6.00 a bag.

9.5 DETERMINANTS

> . . . A staggering paradox hits us in the teeth. For abstract mathematics happens to work. It is the tool that physicists employ in working with the nuts and bolts of the universe! There are many examples from the history of science of a branch of pure mathematics which, decades after its invention, suddenly finds a use in physics.
>
> *F. David Peat*

> *From childhood on, Shannon was fascinated by both the particulars of hardware and the generalities of mathematics. (He) tinkered with erector sets and radios given him by his father . . . and solved mathematical puzzles supplied by his older sister, Catherine, who became a professor of mathematics.*
>
> Claude Shannon

In Section 9.2 we introduced matrices as convenient tools for keeping track of coefficients and handling the arithmetic required to solve systems of linear equations. Matrices are being used today in more and more applications. A matrix presents a great deal of information in compact, readable form. Finding optimal solutions to large linear programming problems requires extensive use of matrices. The properties and applications of matrices are studied in *linear algebra,* a discipline that includes much of the material of this chapter. In this section we introduce the determinant of a square matrix as another tool to help solve systems of linear equations.

Dimension (Size) of a Matrix and Matrix Notation

A matrix is a rectangular array arranged in horizontal **rows** and vertical **columns.** The number of rows and columns give the dimension, or size, of the matrix. A matrix with m rows and n columns is called an $\boldsymbol{m}$ **by** $\boldsymbol{n}$ **(**$\boldsymbol{m \times n}$**) matrix.** Double subscripts provide a convenient system of notation for labeling or locating matrix entries.

Here are some matrices of various sizes:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} \\ b_{21} \\ b_{33} \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Matrix A is 3×3, B is 3×1, and C is 2×2. A and B show the use of double subscripts: a_{ij} is the entry in the ith row and the jth column. The first subscript identifies the row, the second tells the column; virtually all references to matrices are given in the same order, row first and then column. A matrix with the same number of rows and columns is a **square matrix.**