9.3 SYSTEMS OF NONLINEAR EQUATIONS

> In economics and psychology, linear least squares or constant input–output matrices are often used, and indeed, sometimes used automatically by means of canned programs, when fundamental force interactions are nonlinear. Unfortunately, linear models may be very poor approximations for nonlinear models.
>
> *Donald Greenspan*

> *I would read about the history of the subject I was taking . . . so I had a more comprehensive view of the subject than my fellow students. Many times I would walk up to the exam with some classmate and say, "I will review you for the exam, I'll ask you some questions," and he would give me the answers he had studied. I'd go in, take the exam, and get 20 percent more than he did. He'd be so full of the subject, he couldn't see the woods for the trees.*
>
> I. I. Rabi

Gaussian elimination and matrix methods are well-suited for systems of linear equations. However, we often must deal with systems that include nonlinear equations. Such systems can sometimes be difficult to solve, even with the aid of technology. Several of the ideas we have used to solve systems of linear equations have applicability to nonlinear systems. In particular, we often use the **method of substitution,** which is a special case of **eliminating a variable.** If we can graph the equations, we can make use of technology as well.

►EXAMPLE 1 ***A nonlinear system*** For the system $\begin{cases} x^2 + y^2 = 5 \\ x = y^2 - 3 \end{cases}$, **(a)** describe three different strategies for solving, and **(b)** show that two of your strategies in part **(a)** yield the same solution set.

Solution

(a) **(i)** Since the second equation is already solved for x, we can *substitute* $y^2 - 3$ for x in the first equation and solve the resulting equation for y.

(ii) Writing the system in the form $\begin{cases} x^2 + y^2 = 5 \\ x - y^2 = -3 \end{cases}$, we can *eliminate* y^2 by adding equations (replace E_2 by $E_2 + E_1$).

(iii) We can solve both equations for y and graph four equations: $y = \pm\sqrt{5 - x^2}$, $y = \pm\sqrt{x + 3}$. Then use graphical methods.

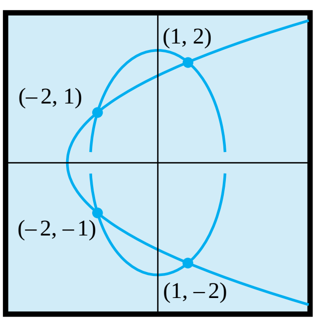
(b) **(i)** Substituting for x, the first equation becomes

$$(y^2 - 3)^2 + y^2 = 5, \quad \text{or} \quad y^4 - 5y^2 + 4 = 0.$$

Factoring, $(y^2 - 1)(y^2 - 4) = 0$, so $y = \pm 1$ or $y = \pm 2$. The corresponding x-values are -2 and 1. The solution set consists of the four points $\{(-2, 1), (-2, -1), (1, 2), (1, -2)\}$.

(ii) Adding equations gives $x^2 + x = 2$, with solutions $x = -2, 1$. Substituting each x-value into either of the original equations and solving for y gives the same four points as the solution set.

(iii) Graphing the four equations $y = \pm\sqrt{5 - x^2}$, $y = \pm\sqrt{x + 3}$ on the same screen gives something like Figure 5. In a decimal window we can read the coordinates of the same four points exactly, but in general we would have to settle for approximations. ◄



[−5, 5] by [−3, 3]

FIGURE 5

►EXAMPLE 2 ***Solving with graphs*** Solve the system of equations and show the solutions graphically.

$$x + 2y = -4$$
$$y = x^2 - 2x - 3$$

Strategy: Eliminate either x or y by substituting from the first equation into the second, and then solving a quadratic.

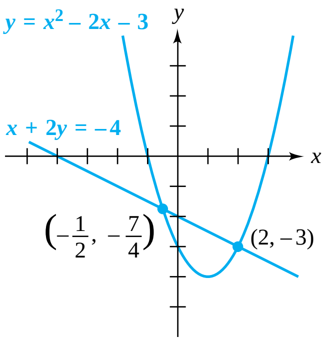


FIGURE 6

Solution

Follow the strategy. Solving the first equation for x gives $x = -2y - 4$. Substitute into the second equation and solve for y.

$$\begin{aligned} y &= (-2y-4)^2 - 2(-2y-4) - 3 \\ y &= 4y^2 + 16y + 16 + 4y + 8 - 3 \\ 4y^2 &+ 19y + 21 = 0 \\ (4y &+ 7)(y+3) = 0 \end{aligned}$$

Therefore, $y = -\frac{7}{4}$ or $y = -3$. Now use $x = -2y - 4$ to get the corresponding values of x. For $y = -\frac{7}{4}$, $x = -\frac{1}{2}$; for $y = -3$, $x = 2$.

The graph of the first equation is a line and that of the second is a parabola that opens upward with vertex at $(1, -4)$. See Figure 6. The points of intersection are $(-\frac{1}{2}, -\frac{7}{4})$ and $(2, -3)$, which correspond to the two solutions. ◀

▶EXAMPLE 3 ***Complex solution*** Solve the system of equations and interpret the solution graphically.

$$\begin{aligned} 2x + y &= 3 \\ x^2 + y &= 1 \end{aligned}$$

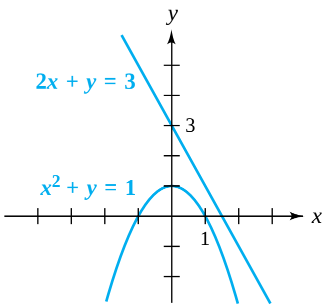


FIGURE 7

Solution

Solve the second equation for y to get $y = 1 - x^2$. Substitute into the first equation and solve for x.

$$\begin{aligned} 2x + (1 - x^2) &= 3 \\ x^2 - 2x + 2 &= 0 \end{aligned}$$

To solve the quadratic equation, apply the quadratic formula.

$$x = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i.$$

We find the coordinates of the points of intersection, if any, of the parabola and the line by solving the system for real number solutions only. Since we have imaginary number solutions, there are no points of intersection, as we see from the graphs in Figure 7. ◀

▶EXAMPLE 4 ***Eliminate variable*** Find the solution set for the system of equations

$$\begin{aligned} x^2 - y^2 &= 3 \\ 2x^2 + y^2 &= 9. \end{aligned}$$

Solution

Eliminate y by adding the two equations and then solve for x.

$$3x^2 = 12 \qquad x^2 = 4 \qquad x = \pm 2.$$

To get the corresponding values of y, substitute 2 or -2 into either of the given equations, say the first.

$$4 - y^2 = 3 \qquad y^2 = 1 \qquad y = \pm 1.$$

This gives four solutions. The solution set is $\{(2, 1), (2, -1), (-2, 1), (-2, -1)\}$. ◀

Strategy: The domain of $y = 2 \ln x$ is $(0, \infty)$ while that of $y = \ln(4 - x)$ is $(-\infty, 4)$. Thus any solution will be such that x is in $(0, 4)$. To draw graphs, use properties of the ln function from Chapter 4.

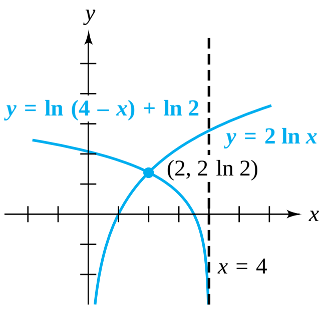


FIGURE 8

►EXAMPLE 5 ***Graph intersections*** Find the points of intersection of the graphs of $y = 2 \ln x$ and $y = \ln(4 - x) + \ln 2$. Draw the graphs.

Solution

Follow the strategy. Eliminate y from the equations.

$$2 \ln x = \ln(4 - x) + \ln 2$$

Now use properties of logarithms from Chapter 4 and then solve for x.

$$\begin{aligned} \ln x^2 &= \ln[2(4 - x)] \\ x^2 &= 2(4 - x) \\ x^2 + 2x - 8 &= 0 \\ (x - 2)(x + 4) &= 0 \end{aligned}$$

Therefore, we get two possible solutions for x: $x = 2$ or $x = -4$, but -4 is not a solution. See the Strategy. To find the corresponding value of y use either of the given equations, say $y = 2 \ln x$. For $x = 2$, $y = 2 \ln 2 = \ln 4$. The graphs of the two equations intersect at only one point $(2, \ln 4) \approx (2, 1.39)$. See Figure 8. ◄

EXERCISES 9.3

Check Your Understanding

Exercise 1–6 True or False. Give reasons.

1. The system of equations
$$\begin{aligned} \sqrt{2}x - \sqrt{5}y &= 4 \\ \sqrt{3}x + \sqrt{7}y &= 3 \end{aligned}$$
is a nonlinear system.

2. The graphs of $y = x^2$ and $y = -x - 1$ intersect at two points.

3. The graphs of $y = |x|$ and $y = x$ have in common only one point, $(0, 0)$.

4. The system of equations
$$\begin{aligned} x - y &= 0 \\ x^2 + y^2 &= 8 \end{aligned}$$
has exactly two solutions.

5. The system
$$\begin{aligned} x - 2y - 1 &= 0 \\ x^2 + y^2 + 1 &= 0 \end{aligned}$$
has no real solutions.

6. The system
$$\begin{aligned} x^2 + y &= 0 \\ x^2 - y &= 0 \end{aligned}$$
has no real solutions.

Exercises 7–10 Fill in the blank so that the resulting statement is true.

7. The graphs of $y = x^2 - 1$ and $y = 1 - x^2$ intersect at ______.

8. The graphs of $x^2 + y^2 = 25$ and $4x + 3y = 0$ intersect at ______.

9. The graphs of $|x| + y = 0$ and $y + 2 = 0$ intersect at ______.

10. The graphs of $y = |x| - 1$ and $y = 1 - |x|$ intersect at ______.

Develop Mastery

Exercises 1–16 ***Solve, Draw Graphs*** *Find all pairs of real numbers x, y that satisfy the system of equations. Draw graphs and show points of intersection (if any).*

1. $y = 3x + 4$, $y = x^2$

2. $2x - y + 2 = 0$, $x^2 + y^2 = 169$

3. $3x + y = 0$, $2x^2 + 4x + y = 0$

4. $2x - y = -2$, $xy = 4$

5. $2x + 3y = -3$, $xy = -3$

6. $5x - y = 10$, $x^2 + x - y = 6$

7. $2x - y = 0$, $x^2 - y = -3$

8. $x + y = 2$, $x^2 + y^2 = 2$

9. $3x - y = 5$, $x^2 + y^2 = 25$

10. $y = x^2 - 4x + 4$, $y = -2x^2 + x + 16$

11. $x - y = 2$
$x^2 + y = 2$

12. $y = \sqrt{x}$
$y = 2x - 6$

13. $x - y = 2$
$\sqrt{x} - y = 0$

14. $2x - y = 0$
$xy - y = 2$

15. $x^2 - y^2 = 0$
$x^2 + y^2 = 8$

16. $x - y = 0$
$x^3 - 3x + y = 0$

Exercises 17–30 ***Nonlinear Systems*** *Solve the system of equations. If results involve irrational numbers, give approximations rounded off to two decimal places.*

17. $y = \ln x$
$y = \ln(2 - x)$

18. $y = e^x$
$x + \ln y = 0$

19. $y = 2 \ln x$
$y = \ln(3 - x) + \ln 4$

20. $y = \ln x^2$
$y = \ln(3 - x) + \ln 4$

21. $x - \ln y = 2$
$x - \ln(y - 3) = 3$

22. $x + \ln(y + 1) = 2$
$x - \ln y = 1$

23. $2^x + y = 16$
$2^{x+1} - y = 8$

24. $3^x + 3y = 10$
$3^{x-1} - y = 8$

25. $x^2 y = 2$
$y = 2x^2$

26. $xy = 2$
$y = \sqrt{x} + 1$

27. $x^2 + 2y^2 = 6$
$xy = 2$

28. $x^2 y = 1$
$y = -x^2 + 2$

29. $x^2 + y^2 - xy = 3$
$x^2 + y^2 = 5$

30. $2x^2 + 5xy + 3y^2 = 4$
$xy = -2$

Exercises 31–34 ***Trigonometric Functions*** *Solve the system of equations. Assume that* $0 \leq x \leq 2\pi$; *for Exercises* 33 *and* 34, $0 \leq y \leq 2\pi$.

31. $\sin x - y = 0$
$\cos x - y = 0$

32. $\sin x + y = 0$
$\sin 2x - y = 0$

33. $2 \sin x + \cos y = 2$
$\sin x - \cos y = -0.5$

34. $\sin x + \cos y = 0$
$2 \sin x - 4 \cos y = 3\sqrt{2}$

Exercises 35–36 ***Absolute Values*** *Solve the system of equations.* (*Hint: How could you graph an equation involving* $|y|$?)

35. $3|x| - 2|y| = -2$
$|x| + 3|y| = 14$

36. $2|x| - 3|y| = 0$
$4|x| + 3|y| = 18$

Exercises 37–42 ***Nonlinear Systems***

37. $6e^x - e^y = 1$
$3e^x + e^y = 8$

38. $e^x + 2e^y = 8$
$2e^x - e^y = 1$

39. $\ln x + \ln y = 0$
$2 \ln x + \ln y = 1$

40. $\ln x + \ln y = 0$
$3 \ln x + 4 \ln y = 2$

41. $x + y + |x| = 9$
$x - y + |y| = 12$
(*Hint:* What are the possible values of $x + |x|$?)

42. $x + y + \sqrt{x^2} = 6$
$x + \sqrt{y^2} - y = 8$

Exercises 43–46 ***Rectangles***

43. The perimeter of a rectangle is 40 cm and the area is 96 cm^2. Find the dimensions of the rectangle.

44. Find the dimensions of a rectangle that has a diagonal of length 13 cm and a perimeter of 34 cm.

45. One side of a rectangle is 3 cm longer than twice the shorter side, and the area is 230 cm^2. Find the perimeter of the rectangle.

46. A rectangle is inscribed in a circle of radius $\sqrt{10}$. If the area of the rectangle is 16, find its dimensions.

Exercises 47–48 ***Line Through Intersections***

47. Find an equation for the line that passes through the points of intersection of the graphs of $y = x^2 + 2x$ and $y = -x^2$.

48. Find an equation for the line that passes through the points of intersection of the graphs of $y = x^2 - 4x - 5$ and $y = -x^2 + 2x + 3$.

49. An altitude of a triangle is twice as long as the corresponding base and the area of the triangle is 36 cm^2. Find the altitude and the base. Does the given information determine a unique triangle? Suppose the problem states that one of the other sides is $4\sqrt{10}$. What is the perimeter of the triangle?

50. ***Explore*** Find all pairs of real numbers (if any) such that
(a) their difference is 1 and their product is 1.
(b) their sum is 1 and their product is 1.
(c) their difference is 1 and their quotient is 1.
(d) their sum is 1 and their quotient is 1.