

2.8 FUNCTIONS AND MATHEMATICAL MODELS

At one time [Conway] would be making constant appeals to give him a year, and he would immediately respond with the date of Easter, or to give him a date, so that he could tell you the day of the week or the age of the moon.

Richard K. Guy

In this section our discussion will be limited to a few different types of problems that involve familiar functions. Applications that require other kinds of functions (such as exponential, logarithmic, or trigonometric functions) will be discussed in later chapters.

First we consider a widely used mathematical model for motion due to gravitational attraction. Theoretically our model applies to objects under the sole influence of gravity, which really implies that the object is in a vacuum, and not affected by air resistance. Although we do not live in a vacuum, for many practical applications this model closely approximates what actually occurs when an object falls. Unless we make an explicit exception, we will assume that all falling-body problems are unaffected by air resistance.

This kind of analysis of motion dates back to the time of Isaac Newton and before. We really need only two types of functions for motion due to gravity, one function of time to give the location of the object at time t (the height, usually measured from the earth's surface), and one to give the velocity as a function of t .

Begin with some terminology. *Speed* indicates how fast an object is moving, while *velocity* includes both speed and the direction of motion. Except for this distinction, we use the words *speed* and *velocity* interchangeably. The problems in this section will suppose an object moving vertically either upward or downward, so its motion is one-dimensional. *Positive speed* means the body is moving upward, while *negative speed* means the body is moving downward (toward the surface of the earth).

Formulas for Objects Moving Under the Influence of Gravity

When an object is launched, thrown, or dropped vertically at an initial speed and is then subject only to gravity, we speak of a *freely falling body*. The position of any falling body is determined by its initial velocity and initial height. The same formulas for velocity and height apply to any such body. These formulas are stated in terms of feet and seconds.

Height and speed formulas for falling bodies

The height and velocity of a falling body with initial height s_0 (feet) and initial velocity v_0 (feet per second) after t seconds are given by:

$$s(t) = s_0 + v_0 t - 16t^2 \quad (1)$$

$$v(t) = v_0 - 32t \quad (2)$$

► **EXAMPLE 1** *Ball thrown vertically* A ball is thrown vertically upward from the top of a 320-foot high building at a speed of 64 feet per second.

- How far above the ground is the ball at its highest point?
- What is the total distance traveled by the ball in the first 5 seconds?
- When does the ball hit the ground?
- What is the velocity of the ball when t is 1? When t is 4?

Strategy: (a) First get a formula for s as a function of t by substituting 320 for s_0 and 64 for v_0 in Equation (1). Draw a graph of the quadratic function and find its maximum value. (c) To find when the ball hits the ground, set $s = 0$ and solve the equation $320 + 64t - 16t^2 = 0$.

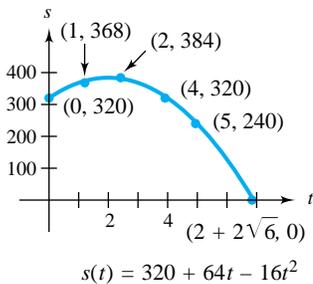


FIGURE 55

Solution

Follow the strategy.

$$s(t) = 320 + 64t - 16t^2.$$

- (a) The graph (Figure 55) is part of a parabola that opens downward and has its highest point (vertex) where

$$t = \frac{-b}{2a} = \frac{-64}{2(-16)} = 2, \quad s(2) = 320 + 64(2) - 16(2^2) = 384.$$

Note that this partial parabola is *not* the path of the ball (which goes straight up and down), but we can use it to easily read off the value of s for a given time t . For instance, at the end of 1 second, the height of the ball is 368 feet above the ground; in 2 seconds the ball is 384 feet above the ground, its maximum height. At the end of 4 seconds, $s = 320$, and so on.

Alternate Solution A physical consideration provides a different approach to finding the time when the ball reaches its highest point. At the highest point, the velocity must be zero since at that instant the ball is going neither up nor down. For this problem, $v = 64 - 32t$, so we want to find the value of t for which v is 0. Solve the equation $64 - 32t = 0$, from which t is 2, as we found above.

- (b) To find the total distance traveled during the first 5 seconds, note that the ball travels upward a distance of $384 - 320 = 64$ feet during the first 2 seconds and then downward a distance of $384 - 240 = 144$ feet during the next 3 seconds. (Look at the graph.) Therefore, the total distance traveled during the first 5 seconds is $64 + 144$, or 208 feet.
- (c) When the ball hits the ground, the height is 0. Setting $s(t)$ equal to 0 and solving for t gives the time when the ball reaches ground level. Solving $320 + 64t - 16t^2 = 0$ yields two values, one positive ($2 + 2\sqrt{6}$) and one negative ($2 - 2\sqrt{6}$). Since only a positive time value has physical significance in this problem, the ball must hit the ground when t is $2 + 2\sqrt{6}$, or about 6.9 seconds after being thrown.

- (d) Replacing v_0 by 64 and substituting 1 for t in formula (2) gives

$$v_1 = 64 - 32 \cdot 1 = 32.$$

Hence when t is 1, the ball is moving upward at 32 feet per second. When t is 4,

$$v_4 = 64 - 32 \cdot 4 = 64 - 128 = -64.$$

The negative sign indicates that the ball is moving downward, so at the end of 4 seconds the ball is falling at a speed of 64 feet per second. ◀

In the next example we look at a slightly more involved problem.

► **EXAMPLE 2 Computing distance** A stone is dropped from the top of a building and falls past an office window below. Watchers carefully time the stone and determine that it takes 0.20 seconds to pass from the top to the bottom of the window, which measures 10 feet high. From what distance above the top of the window was the stone dropped? (That is, how far is it from the roof to the top of the window?)

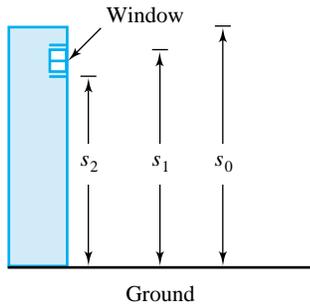


FIGURE 56

Solution

The diagram in Figure 56 identifies the distances s_0 , s_1 , and s_2 . Given that $s_1 - s_2 = 10$ (feet), let t_1 be the time it takes for the stone to reach the top of the window and t_2 be the time to reach the bottom of the window. The problem states that $t_2 - t_1 = 0.20$, and we wish to find $s_0 - s_1$.

Equation (1) applies, where $v_0 = 0$, so

$$s = s_0 - 16t^2$$

$$s_1 = s_0 - 16t_1^2$$

$$s_2 = s_0 - 16t_2^2 = s_0 - 16(t_1 + 0.20)^2.$$

Subtracting, and using the fact that $s_1 - s_2 = 10$, gives

$$s_1 - s_2 = -16t_1^2 + 16(t_1 + 0.20)^2$$

$$10 = 6.4t_1 + 0.64$$

$$t_1 = \frac{9.36}{6.4} = 1.4625.$$

Substituting this value of t_1 into $s_1 = s_0 - 16t_1^2$, we get

$$s_0 - s_1 = 16t_1^2 \approx 34.22.$$

Considering the precision of timing the fall past the window (two significant digits), the distance from the top of the window to the top of the building is about 34 feet. To minimize rounding error, carry out all intermediate calculations with full calculator accuracy and then round off the final result to be consistent with the accuracy of the data. ◀

Revenue Functions

We now look at a problem from the field of economics and business. The **revenue** R generated by selling x units of a product at p dollars per unit is given by the simple formula, $R = px$. The price per unit, p , is determined by a **demand function**, which is usually based on some sort of market analysis or, preferably, experience. Generally the number of units sold increases when the price goes down, and analysts often assume a linear demand function. We illustrate some of these concepts in the next example.

► **EXAMPLE 3 Revenue problem** The demand function for a certain product is given by

$$p = 12 - \frac{1}{2}x \quad \text{for} \quad 0 \leq x \leq 20,$$

where x is the number of units sold. As Figure 57 shows, the price decreases as more units are sold.

(a) Find a formula for the revenue R as a function of x .

(b) How many units should be sold to maximize revenue?

(c) What is the maximum revenue and what is the corresponding price per unit?

Solution

(a) $R(x) = px = (12 - \frac{1}{2}x)x = 12x - \frac{1}{2}x^2$.

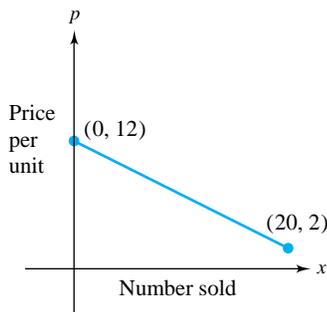
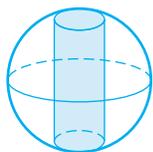
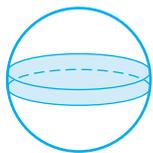


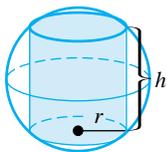
FIGURE 57



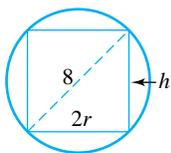
(a) Tall, skinny cylinder



(b) Wide, flat cylinder



(c) Cylinder with height h and radius r



(d) Cross section

FIGURE 58

- (b) The revenue function is a quadratic function whose graph is a parabola that opens downward. To find the maximum value, locate the vertex of the parabola, which occurs when $x = \frac{-b}{2a} = \frac{-(12)}{2(-\frac{1}{2})} = 12$. Therefore the maximum revenue will be produced when 12 units are sold.
- (c) The maximum revenue, which corresponds to $x = 12$, occurs when the unit price is $p(12) = 12 - \frac{1}{2}(12) = 6$ dollars per unit. The revenue is given by

$$R(12) = 12(12) - \frac{1}{2}(12)^2 = 144 - 72 = 72 \text{ dollars} \quad \blacktriangleleft$$

In the next example we look at a common type of problem in calculus.

► **EXAMPLE 4** *Maximum volume of a cylinder* What are the dimensions of the cylinder with the greatest volume that can be contained in a sphere of diameter 8?

Solution

First, get a feeling for the problem by trying to visualize various cylinders in the sphere, as in Figure 58. A tall cylinder is too skinny to have a large volume; at the other extreme, a wide flat cylinder also has a small volume. From one extreme to the other the cylinder volume first increases and then decreases, so the one with maximum volume must be somewhere in between.

Set up the problem mathematically by expressing the volume V of the cylinder as a function of its radius r . The formula for the volume of a cylinder with radius r and height h (see inside front cover) is $V = \pi r^2 h$.

It may be easier to see in cross section, as in diagram (d). A right triangle has a hypotenuse of 8 (the diameter of the sphere) and legs of h and $2r$. By the Pythagorean theorem,

$$h^2 + (2r)^2 = 8^2 \quad \text{or} \quad h = 2\sqrt{16 - r^2}.$$

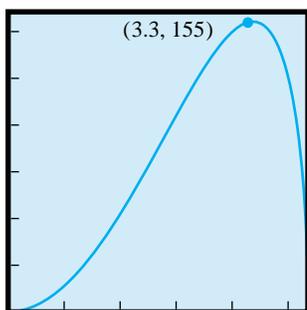
Substituting into the formula for the volume of the sphere,

$$V = \pi r^2 h = 2\pi r^2 \sqrt{16 - r^2}$$

This is another problem that requires calculus for an answer in exact form, but where technology can give a very acceptable approximation. We graph the volume as $Y = 2\pi X^2 \sqrt{16 - X^2}$ and look for the maximum value in an appropriate window.

From the diagram in Figure 58d, the radius must be a positive number less than 4, so we can take $[0, 4]$ for an x -range. When $r = 3$, the volume is nearly 150, so we try a y -range of $[0, 160]$. The calculator graph is shown in Figure 59. Tracing to find the maximum, we find a volume of 154.75 near $r = 3.28$, $h \approx 4.62$. If we zoom in a couple of times, we can locate the high point more precisely at about $(3.266, 154.778)$.

For comparison, calculus techniques show that the maximum volume is $\frac{256\pi}{3\sqrt{3}}$ when $r = \frac{4\sqrt{6}}{3}$, which to four decimal place accuracy corresponds to the point $(3.2660, 154.7775)$. \blacktriangleleft



$[0, 4]$ by $[0, 160]$

FIGURE 59

$$y = 2\pi x^2 \sqrt{16 - x^2}$$

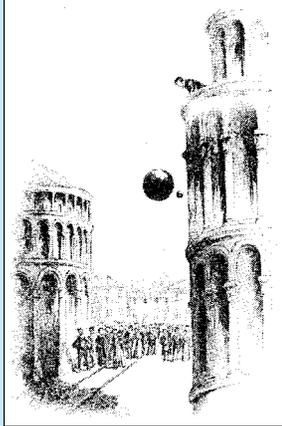
HISTORICAL NOTE

MATHEMATICAL MODELS AND GRAVITY

When we write an equation or function to describe a real-world situation, we almost always need to simplify. Einstein said this well: “Everything should be made as simple as possible, but not simpler.” The test of a mathematical model is its capacity to accurately describe and predict real events.

Galileo measured falling bodies and decided that the distance fallen is proportional to the square of the time (in modern terms, $f(t) = 16t^2$). His timing instrument was his pulse! We may wonder what his results might have been if his pulse had been less steady.

How good is his simple model? For heavy bodies near the earth it works beautifully. For objects like feathers or paper airplanes, the model is too simple.



Galileo's experiments with falling objects led to a mathematical model of the force of gravity.

Another example occurs in Newton's account of his discovery of the inverse square law of the force of gravity. Newton, born in 1642, the year of Galileo's death, took refuge at age 24 on an isolated farm to avoid the plague, which was then ravaging London. He devoted himself to study and within a year he had his model for the gravitational attraction between two bodies, $F = g\left(\frac{mM}{r^2}\right)$. To test it, he

“compared the force requisite to keep the moon in her orb with the force of gravity at the surface of the earth, and found the answer fits ‘pretty nearly.’” Newton's model

was good enough to analyze the motion of the planets.

Strategy: The longest mirror is the shortest segment L touching the inside corner and both walls, forming two similar right triangles, with sides 5 and y , x and 4, as in the diagram. In the similar triangles, $\frac{y}{5} = \frac{4}{x}$.

► **EXAMPLE 5 Getting around a corner** Plate glass mirrors must be replaced in a dance studio. Unfortunately, the only way into the studio is down a hallway 5 feet wide and then around a corner into a hallway 4 feet wide. See the diagram in Figure 60. The question is what length mirror can be carried (vertically) around the corner. Can a 10-foot mirror be installed? 12-foot? 15-foot?

- (a) Use the observation in the strategy to express L in terms of x and y , and from the relation of x and y , write L as a function of x .
 (b) Use a graph to find the minimum possible length of L .

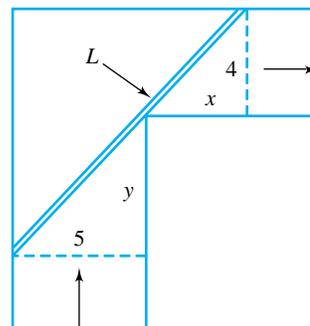


FIGURE 60

Solution

- (a) Follow the strategy.
- L
- is the sum of the lengths of each hypotenuse:

$$L = \sqrt{5^2 + y^2} + \sqrt{x^2 + 4^2}.$$

From the relation of x and y in the Strategy, we can solve for y : $y = \frac{20}{x}$, and substitute into the expression for L to get a function of x alone.

$$\begin{aligned} L &= \sqrt{25 + \frac{400}{x^2}} + \sqrt{x^2 + 16} \\ &= \sqrt{\frac{25(x^2 + 16)}{x^2}} + \sqrt{x^2 + 16} \\ &= \frac{5}{x} \sqrt{x^2 + 16} + \sqrt{x^2 + 16} = \left(\frac{5}{x} + 1\right) \sqrt{x^2 + 16}. \end{aligned}$$

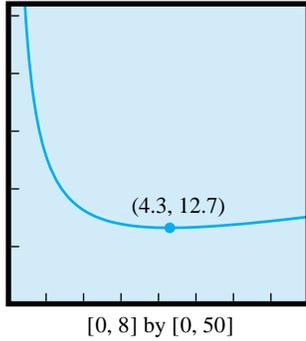


FIGURE 61

- (b) We want to graph $Y = (5/X + 1)\sqrt{X^2 + 16}$ in an appropriate window. It seems as if $[0, 8]$ should be adequate for an x -range, and we sample a few values for y . In a $[0, 8] \times [0, 30]$ window we get a graph like that in Figure 61. Tracing, we find the low point near the point indicated in the figure. Thus the minimum length is about 12.7 feet, when $x \approx 4.3$ and $y \approx 4.7$. Clearly the 10-foot mirror and the 12-foot mirror can be carried around the corner (with room for fingers) but the 15-foot mirror cannot. ◀

EXERCISES 2.8**Check Your Understanding**

Exercises 1–10 True or False. Give reasons.

1. A ball dropped from a height of 256 feet takes 4 seconds to hit the ground.
2. It takes twice as long for a ball to fall to the ground from a height of 64 feet than from a height of 32 feet.
3. If a ball is dropped from a height of 256 feet and at the same instant a second ball is thrown upward from ground level at a speed of 128 feet per second, the two balls will meet at a point 192 feet above the ground.
4. In Exercise 3, the two balls will meet in 2 seconds.
5. A ball rolls down a long inclined plane. It takes longer to roll down the first 10 feet than it does to roll down the next 10 feet.
6. It takes the same amount of time to travel 240 miles at 55 mph as it takes to travel the first 120 miles at 50 mph and the final 120 miles at 60 mph.
7. If a square and an equilateral triangle are inscribed in the same circle, then the square has greater area than the triangle.
8. For any rectangle with a perimeter of 16, the length of one side must be at least 4.

9. No triangle can have sides of lengths 3, 4, and 8.
10. If a sphere has diameter d , then its volume V is given by $\frac{\pi d^3}{12}$.

Develop Mastery

Exercises 1–23 Apply the formulas for motion due to gravitational attraction.

1. A stone is dropped from the top of a cliff that is 160 feet tall. How long will the stone take to hit the ground?
2. A stone is dropped from the top of a building and hits the ground 3.5 seconds later. How tall is the building?
3. A helicopter is ascending vertically at a speed of 25 feet per second. At a height of 480 feet, the pilot drops a box.
 - (a) How long will it take for the box to reach the ground?
 - (b) At what speed does the box hit the ground?
4. A helicopter is climbing vertically at a speed of 24 feet per second when it drops a pump near a leaking boat. The pump reaches the water 4 seconds after being dropped.

- (a) How high is the helicopter when the pump is dropped?
 (b) How high is the helicopter when the pump reaches the water?
5. A baseball is thrown vertically upward. When it leaves the player's fingers it is 6 feet off the ground and traveling at a speed of 48 feet per second.
 (a) How high will it go?
 (b) How many seconds after the ball is thrown will it hit the ground?
6. A rock is dropped from the top of a cliff 360 feet directly above a lake.
 (a) State a formula that gives the height s as a function of t .
 (b) What is the domain of this function?
 (c) How far above the lake is the rock 2 seconds after being dropped?
 (d) How far does the rock fall during the third second?
7. A rock is blasted vertically upward from the ground at a speed of 128 feet per second (about 80 mph).
 (a) Find a formula that relates s and t .
 (b) How far from the ground is the rock 2 seconds after the blast?
 (c) How high will the rock go?
8. A vertical cliff 160 feet tall stands at the edge of a lake. A car is pushed over the edge. How many seconds will it take to hear the sound of the splash at the top of the cliff? Assume that sound travels at a speed of 1080 feet per second.
9. A rock is dropped into a deep well. It takes 4.5 seconds before the sound of the splash is heard. Assume that sound travels at a speed of 1080 feet per second. Determine s_0 , the distance from the top of the well to the water level. If you measure the height s above water level, then the formula for motion due to gravity applies.
 (a) Show that the rock takes $t_1 = \frac{\sqrt{s_0}}{4}$ seconds to reach the water.
 (b) Show that the sound of the splash takes $t_2 = \frac{s_0}{1080}$ seconds to be heard.
 (c) Since the total time that elapses before hearing the splash is 4.5 seconds, you have the equation

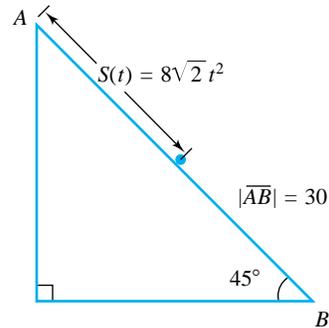
$$\frac{s_0}{1080} + \frac{\sqrt{s_0}}{4} = 4.5.$$

Clear the fractions and simplify to get

$$s_0 + 270\sqrt{s_0} - 4860 = 0,$$

which is a quadratic equation in $\sqrt{s_0}$. Use the quadratic formula to solve for $\sqrt{s_0}$ and then find s_0 .

10. A ball is released from rest at point A, the top of an inclined plane 30 feet long (see the diagram). If $S(t)$ denotes the number of feet the ball rolls down the incline in t seconds after its release, then $S(t) = 8\sqrt{2}t^2$.



- (a) How long does it take for the ball to reach the end of the plane?
 (b) How far does the ball roll during the first 1.5 seconds?
 (c) How long does it take for the ball to roll down the final 12 feet of the plane?
11. A stone is dropped from the top of New York's Empire State Building, which is 1476 feet tall.
 (a) How long does it take for the stone to reach the ground?
 (b) What is the speed of the stone when it hits the ground?
12. If a kangaroo jumps 8 feet vertically, how long is it in the air during the jump?
13. With what minimum vertical speed must a salmon leave the water to jump to the top of a waterfall that is 2.4 feet high?
14. A rock is thrown upward at an initial speed of 16 feet per second from the edge of a cliff 160 feet above a lake. One second later a second rock is dropped from the edge of the cliff. Which rock will hit the water first? By how many seconds?
15. **What's Wrong?** Major league pitches are often clocked at more than 90 miles per hour. How fast a ball can a catcher be expected to handle? In 1946, a backup catcher for the St. Louis Browns named Hank Helf caught a ball dropped from 52 stories up (a 701-foot drop to his glove). The speed of the ball was measured at 138 mph.

Use Equations (1) and (2) from this section to find (a) how long it takes for a ball to drop 701 feet, and (b) the speed of the ball when it hits the glove (in feet per second and in miles per hour). (c) Write a brief paragraph to discuss why the measured speed is not the same as the speed predicted by Equation (2). (*Hint:* Read the Historical Note.)

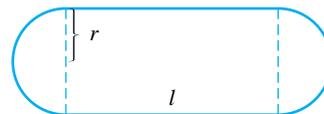
16. A diver in Acapulco leaps horizontally from a point 112 feet above the sea.
- How long does it take for the diver to reach the water?
 - At what speed does the diver enter the water?
17. A ball player catches a ball 5 seconds after throwing it vertically upward.
- At what initial speed was the ball thrown?
 - What was the speed of the ball when it was caught?
18. A stone is thrown vertically upward with a speed of 32 feet per second from the edge of a cliff that is 240 feet high.
- How many seconds later will it reach the bottom of the cliff?
 - What is its speed when it hits the ground?
 - What is its speed when it is 120 feet above the bottom of the cliff?
 - What is the total distance traveled by the stone?
19. Robin, a skydiver, leaves the plane at an altitude of 1000 feet above the ground and accidentally drops her binoculars. If she descends at a constant speed of 20 feet per second, how much time elapses between the arrival of the binoculars on the ground and the time when Robin lands?
20. Frank is ballooning at an altitude of 480 feet when he turns on the burner and accidentally knocks his lunch out of the balloon. If he immediately starts to ascend at a constant speed of 4.8 feet per second, how high will he be when his lunch hits the ground?
21. A stone is dropped from the top of a building 240 feet high. It is observed to take 0.20 seconds to go past an office floor-to-ceiling window that is 12 feet high. How far is it from the bottom of the window down to the street? (*Hint:* See Example 2.)
22. A toy rocket is fired upward from ground level near an office building. Its initial velocity is 80 feet per second. An observer in one of the offices determines that the rocket takes 0.32 seconds to pass by the office window, which is 16 feet tall. How far is it from the ground to the bottom of the window?
23. The acceleration of gravity on the moon is about one-sixth of what it is on earth. The formula for a freely falling object on the moon is given by $s = s_0 + v_0t - \frac{8}{3}t^2$. If an object is thrown upward on the moon, how much higher will it go than it would have on the earth, assuming the same initial velocity of 64 feet per second?
24. **Maximum Revenue** The manager of a store estimates that the demand function for calculators (see Example 3) is given by

$$p = 36 - \frac{1}{3}x \quad 0 \leq x \leq 96,$$

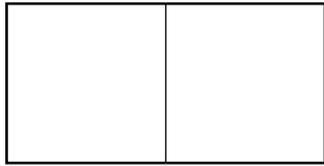
where x is the number of calculators sold and p is the price of each calculator. The revenue R is given by $R = px$.

- Express R as a function of x .
 - How many calculators should be sold to get the maximum revenue?
25. Answer the questions in Exercise 24 if the demand function is given by
- $$p = 36 - (0.2)x \quad 0 \leq x \leq 160.$$
26. A car rental agency rents 400 cars a day at a rate of \$40 for each car. For every dollar increase in the rental rate, it rents 8 fewer cars per day.
- What is the agency's income if the rental rate is \$40? \$42? \$45?
 - What rental rate will give the greatest income? What is this maximum income?
27. A car rental agency rents 200 cars a day at a rate of \$30 for each car. For every dollar increase in the rental rate, it rents 4 fewer cars per day.
- What is the agency's income if the rental rate is \$30? \$35? \$40?
 - What rental rate will give the greatest income? What is the maximum income?

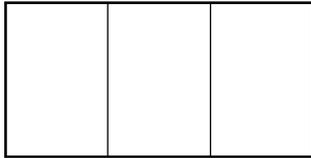
28. **Linear Depreciation** A computer is purchased for \$2000. After 5 years its salvage value (for tax purposes) is estimated to be \$400. Linear depreciation implies that the tax value V of the computer is a linear function of t , the number of years after purchase.
- Find a formula for the linear depreciation function.
 - In how many years after purchase will the tax value of the computer be zero?
29. Repeat Exercise 28 if the original cost of the computer is \$3000 and its tax value after 8 years is \$500.
30. An indoor gymnastics arena is to be built with a rectangular region and semicircular regions on each end (see the diagram). Around the outside is a running track whose inside length is to measure 220 yards (one-eighth of a mile).



- What dimensions for the rectangle will maximize the area of the rectangular region?
 - For the dimensions in part (a), what is the area of the entire region enclosed by the track?
31. A rancher has 240 feet of fencing to enclose two adjacent rectangular pens (see the diagram on the top of the next page). What dimensions will give a maximum total enclosed area?



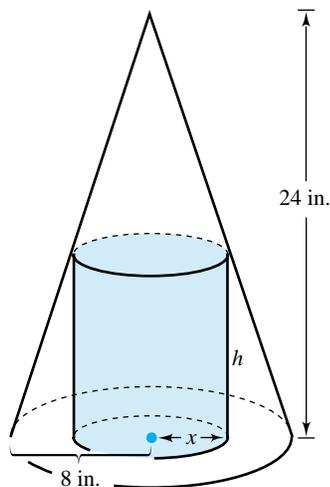
32. In Exercise 31 suppose that the rancher wants to make three adjacent pens (see the diagram). What dimensions will give a maximum total enclosed area?



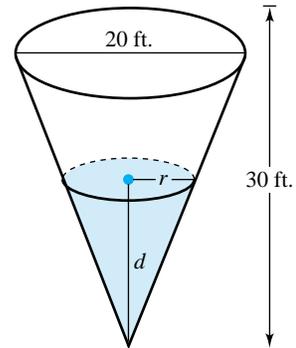
33. A pan is full of water when it springs a leak at the bottom. The volume V of water (in cubic inches) that remains in the pan t seconds after the leak occurs is given by

$$V = 1000 - 30t + 0.1t^2$$

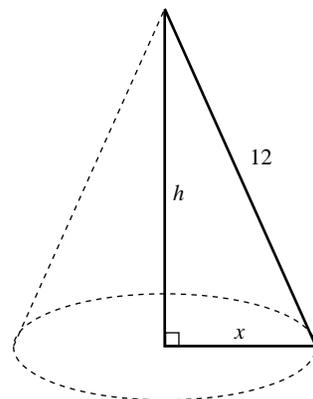
- (a) How much water is in the pan when the leak starts?
 (b) In how many seconds will the pan be empty?
 (c) What is the domain of the function?
 (d) How many seconds will it take for half of the water to leak out of the pan? How long for the final half?
34. **Looking Ahead to Calculus** A right circular cylinder is inscribed in a right circular cone that has a height of 24 inches and a radius of 8 inches (see the diagram). Let x denote the radius of the cylinder and h denote its height.
- (a) Express h as a function of x .
 (b) Express the volume V of the cylinder as a function of x .
 (c) Use a graph to find the value of x that gives the largest volume.



35. A water tank in the shape of an inverted circular cone is initially full of water (see the diagram for dimensions). A control valve at the bottom of the tank allows water to drain from the tank. At any depth d , the water remaining in the tank is the shape of a cone with radius r .

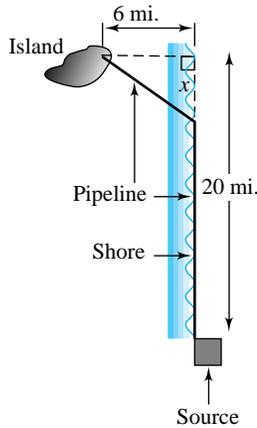


- (a) Express r as a function of d and then express the volume V of water remaining as a function of d .
 (b) If the depth of the water t minutes after starting to drain the tank is given by $d = 30 - 5\sqrt{t}$, then express V as a function of t .
 (c) What is the volume of water that remains at the end of 16 minutes?
 (d) How long will it take to empty the tank?
36. **Looking Ahead to Calculus** A right triangle has a fixed hypotenuse of length 12, but legs whose lengths can vary. The triangle is rotated about the vertical leg to generate a cone of radius x and height h , where x and h are the lengths of the legs of the triangle (see the diagram).

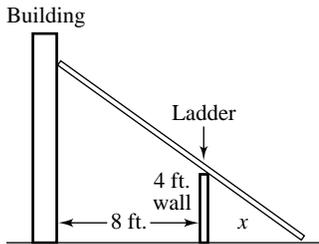


- (a) Express h as a function of x .
 (b) Express the volume V of the cone as a function of x .
 (c) Of all such possible cones, determine x and h for the one with the largest volume. Use a graph.

37. **Looking Ahead to Calculus** A freshwater pipeline is to be constructed from the shore to an island located as shown in the diagram. The cost of running the pipeline along the shore is \$8,000 per mile, but construction offshore costs \$12,000 per mile.



- (a) Express the construction cost C as a function of x .
 (b) What is the cost when x is 3, 5, 6, 8, 10, and 15?
 (c) Of all such possible pipelines, determine the value of x that will minimize the construction cost. What is the minimum cost? Use a graph.
38. **Looking Ahead to Calculus** A ladder of length L is placed so that it rests on the top of a 4 foot wall and leans against a building that is 8 feet from the wall. The ladder touches the ground x feet from the wall (see the diagram).



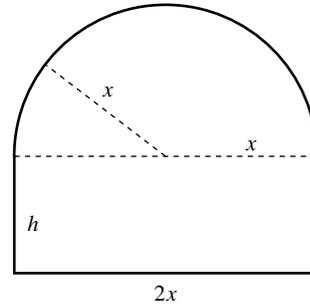
- (a) Show that L can be written as a function of x as follows:
- $$L = (x + 8)\sqrt{1 + \frac{16}{x^2}}$$
- (b) Evaluate L when x is 2, 3, 4, 5, 6, and 7.
 (c) Of all such possible ladders, determine the value of x that will require the shortest ladder. What is the length of the shortest ladder? Use a graph.

39. **Inscribed Rectangle**

- (a) Draw a graph of $y = 4x - x^2$ and inscribe a rectangle with base on the x axis and upper vertices on the graph.

- (b) Of all possible rectangles, find the dimensions of the one that has a maximum area. What is the area? (*Hint:* First get a formula giving the area A of any rectangle as a function of its height. Then use a graph.)

40. **Maximum Light** A so-called Norman window consists of a rectangle surmounted by a semicircle as shown in the diagram. The total perimeter of the window is to be 24 feet. What are the dimensions of the window that will admit the greatest amount of light? A graph will be helpful.



41. Solve the problem in Exercise 40 if the semicircular portion of the window is made of stained glass which admits one-half as much light as the rectangular portion.
42. **Around a Corner** Solve the problem in Example 5 if the hallways are 4 feet and 6 feet wide.
43. **Maximum Volume** Solve the problem in Example 4 if the diameter of the sphere is 12. Compare with the exact answer from calculus: The maximum volume is $96\pi\sqrt{3}$ when $r = 2\sqrt{6}$.
44. **Maximum Area** An isosceles triangle is inscribed in a circle of radius 8 cm. See diagrams where $|\overline{AB}| = |\overline{AC}|$, O is the center of the circle, $h = |\overline{AD}|$, and $b = |\overline{DC}|$.

