Math 4200 Assignment

Definition. Suppose \( f \) is a function defined in a neighborhood of the point \( x = a \).
(This means that \( f \) is defined in some open interval containing \( x = a \), except perhaps at the point \( x = a \).) Then \( f \) has a limit as \( x \) approaches \( a \) provided there exists a number \( L \) such that

for each \( \epsilon > 0 \), there exists \( \delta > 0 \) such that

if \( 0 < |x - a| < \delta \) then \( |f(x) - L| < \epsilon \).

We write \( \lim_{x \to a} f(x) = L \).

Definition. Suppose \( f \) is a function defined in some open interval containing \( x = a \).
Then \( f \) is continuous at \( x = a \) provided

\[
\lim_{x \to a} f(x) = f(a).
\]

The function \( f \) is continuous on an open interval \((a, b)\) provided \( f \) is continuous at each point of \((a, b)\).

1. Suppose \( \lim_{x \to a} f(x) = L \) and \( \lim_{x \to a} g(x) = M \). Prove the following:

a) \( \lim_{x \to a} f(x) + g(x) = L + M \)

b) \( \lim_{x \to a} c \cdot f(x) = c \cdot L \) (\( c \) is a constant)

c) \( \lim_{x \to a} f(x) \cdot g(x) = L \cdot M \)

d) \( \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M} \) (Assume \( M \neq 0 \) and \( g(x) \neq 0 \) for all \( x \).)
2. Use the definition of the limit to decide whether the following limits exist at the indicated point.

a) \( \lim_{x \to 4} \sqrt{x} \)

b) \( \lim_{x \to 0} \frac{|x|}{x} \)

c) \( \lim_{x \to 1} \frac{x^3 - x^2}{x - 1} \)

3. Suppose \( \lim_{x \to a} f(x) = 0 \) and \( g \) is a bounded function.

Prove that \( \lim_{x \to a} f(x) \cdot g(x) = 0 \).

4. State and prove a "squeeze theorem" for limits of functions.

5. Let \( f \) be a function which is continuous at \( x = a \). Prove that there exist \( K > 0 \), \( \delta > 0 \) such that \( |f(x)| < K \) for all \( x \) in the interval \( (a - \delta, a + \delta) \).