Math 4200

Theorem. Between any two real numbers there exists an irrational number.

Proof:

Case 1. Suppose x is positive and $x < y$. There exists a natural number k such that $\frac{\sqrt{2}}{k} < y - x$. Let $w = \frac{\sqrt{2}}{k}$. Notice that w is an irrational number. Starting at $x = 0$, we will walk along the positive real line with a step size equal to w. We will prove that we must step into the interval (x, y) since its width is greater than w.

Let $S = \{j : jw \geq y\}$. It follows from the Archimedean property that S is not empty. Let m be the least element of S. Then, $mw \geq y$ and $(m-1)w < y$.

We must show that $x < (m-1)w$. Suppose $(m-1)w \leq x$. Then $mw - w \leq x$, and $mw \leq x + w < x + y - x = y$. This implies that $mw < y$, a contradiction. Thus, $x < (m-1)w < y$. Since w is irrational, $(m-1)w$ is also an irrational number.

Case 2. Suppose $x \leq 0$. Choose a positive integer n such that $x + n > 0$. Consider the positive numbers $x + n$ and $y + n$. From case 1, there exists an irrational number u such that $x + n < u < y + n$. Subtracting n from both sides of this inequality gives $x < u - n < y$. Now, $u - n$ is irrational since u is irrational, and n us rational.

Theorem. Between any two real numbers there exists a rational number.

Proof:

Modify the proof above by replacing $\frac{\sqrt{2}}{k}$ with $\frac{1}{k}$.