Math 4200, Honors Paper
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"God made natural numbers; all else is the work of man."

-- Leopold Kronecker

The above quote states that the natural numbers are what we were given, and all other numbers, such as integers and rational numbers, were created from them. This implies that all other sets of numbers can be formed out of the natural numbers. The object of this paper is to show the connection between each set of numbers by showing how we can form each set from the natural numbers.

We will begin by defining all our terms. Let J represent the set of all natural numbers, W represent the set of all whole numbers, Z represent the set of all integers, Q represent the set of all rational numbers, and R represent the set of all real numbers. We will use Peano's Axioms to define the natural numbers in order to give us a starting point. Peano's Axioms are as follows:

Axiom 1: 1 is a natural number. That is, our set is not empty; it contains an object called "one."

Axiom 2: For each x, there exists exactly one natural number, called the successor of x, which will be denoted by x'. (think $x' = x + 1$)

Axiom 3: We always have $x' \neq 1$. That is, there is no number whose successor is 1.

Axiom 4: If $x'=y'$ then $x=y$. That is, for any given number, there exists either no number or exactly one number whose successor is the given number.

Axiom 5 (Axiom of Induction): Let there be given a set J of natural numbers, with the following properties:

- **I.** 1 belongs to J.
- **II.** If x belongs to J, then so does x'.

Then J contains all the natural numbers. Thus, if $1'=2$, $2'=3$, $3'=4$, …, then $J=\{1, 2, 3, 4, \ldots\}$.

From this definition of the natural numbers, we can easily create the set of all whole numbers by including zero. We will let W have the same properties as J, except for we will let 0 belong to W such that $0'=1$ and 0 is not the successor of any number. Thus, W = {0, 1, 2, 3, …). We will now define addition and multiplication as follows:

Addition (+): For all x, $x+1 = x'$. For all x and y, $x+y' = (x+y)'$.

Notice that $2=1'$, so we have the following: $4+2 = 4+1' = (4+1)' = (4')' = 6$.

Multiplication (*): For all x, x*1 = x. For all x and y, x*y' = x*y + x.

For example, if y=1 and x=4, then x*y' = 4*1' = 4*1+4 = 4+4 = 8.

We will now define 0 as follows: 0+x=x and x*0=0 for all x. From here we can easily create the integers by letting –x denote the additive inverse of x. This means that x + (-x) = 0 for all x. The integers, will then be defined as follows:

Z = {x: x∈ J} ∪ {0} ∪ {-x: x∈ J}

Thus Z = {…, -2, -1, 0, 1, 2, …}. (To see more about building the integers, see http://en.wikipedia.org/wiki/Integer#Construction).

We will now construct the rational numbers from the integers by constructing equivalence classes using the integers. These equivalence classes will be pairs of integers put together as follows: {(a,b): a,b∈Z, b≠0}. Also, (a,b) is equivalent to (c,d) if and only if ad=bc (think (a,b) and (c,d) as $\frac{a}{b}$ and $\frac{c}{d}$ respectively.) For example, let [(3,7)] denote the equivalence defined in the pair (3,7). Then, [(3,7)] = {(a,b)∈ S: (a,b) ≈ (3,7)}. For example, (6,14) ≈ (3,7). We can represent [(3,7)] by the pairs (3,7) or (6,14), or more formerly as $\frac{3}{7}$ or $\frac{6}{14}$. The set of all rational numbers Q is built from these equivalence classes. (Further information about constructing the rationals: http://en.wikipedia.org/wiki/Rational_number#Formal_construction). Addition and multiplication for the rational numbers are as follows:

Addition: (a,b)+(c,d) = (ad+bc,bd)

Multiplication: (a,b)*(c,d) = (ac,bd)

Now we are ready to construct the real numbers. Let T = {$\{r_n\}$: r_n is rational and $\{r_n\}$ is a Cauchy sequence.} We will let two sequences $\{r_n\}$ and $\{s_n\}$ in T be equivalent to each other if and only if $\lim_{n\to\infty}(r_n - s_n) = 0$. From here, it is easy to show that this relation is reflexive, transitive, and symmetric. The set T is partitioned into disjoint equivalence classes which we will call real numbers. For example, let a_n = 2 for all n. The equivalence class $[\{a_n\}]$ is the real number 2. The real number 2 is also represented by $\{2 + \frac{1}{n}\}$. Since $\lim_{n\to\infty} 2 = 2$ and $\lim_{n\to\infty}\left(2 + \frac{1}{n}\right) = 2$, then $\lim_{n\to\infty}(2 - \left(2 + \frac{1}{n}\right)) = 2 - 2 = 0$ since the limit of the sum is the sum of the limits.

Consider the real number e. We can represent this number by the limit of the binomial expansion: $\lim_{n\to\infty}(1 + \frac{1}{n})^n$. Consider the binomial expansion for n=1,000,000. We then have $(1+\frac{1}{1000000})^{1000000}$=2.71828046… which we see is very close to the value of e. Another way to approximate the number e is through the Taylor series expansion for e^x=$\sum_{n=1}^{\infty}\frac{x^n}{n!}$ for x=1. This gives us $\sum_{n=1}^{\infty}\frac{1}{n!} = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$ By adding the terms of this series through n=15, we get 2.71828182… which is also rather close. These continue to get closer and closer to e as n approaches infinity.

Now consider the real number π. Just like e, there are various ways that we can approximate π as well. For example, we have the Gregory-Leibniz series as follows: $\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 4(\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\cdots)$. This approximation, however, converges too slowly for it to be a practical approximation. Another approximation was given by Newton:
$\pi = 2\sum_{n=0}^{\infty}\frac{2^n n!^2}{(2n+1)!} = 2(1+\frac{1}{3}\left(1+\frac{2}{5}\left(1+\frac{3}{7}(1+\cdots)\right)\right))$.

We will now look at a series to represent $\sqrt{2}$. Let r_n be the sequence of largest rational numbers with denominator less than or equal to n such that $(r_n)^2 < 2$. This gives us the following sequence: $\{1,1,\frac{4}{3},\frac{4}{3},\frac{7}{5},\ldots\}$ which is a Cauchy sequence such that $\lim_{n\to\infty} r_n = \sqrt{2}$. Given that $\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$ and $\sin(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$, then we get that $\sqrt{2} = 2\sin(\frac{\pi}{4}) = 2\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n+1}}{4^{2n+1}(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n+1}}{2^{4n+1}(2n+1)!}$. Thus, $\sqrt{2} = \frac{\pi}{2} - \frac{\pi^3}{2^5 3!} + \frac{\pi^5}{2^9 5!} - \frac{\pi^7}{2^{13} 7!} + \cdots$. In a like manner, by using cos(x) in place of sin(x), we can show that $\sqrt{2} = \sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}}{2^{4n-1}(2n)!} = 2 - \frac{\pi^2}{2^3 2!} + \frac{\pi^4}{2^7 4!} - \frac{\pi^6}{2^{11} 6!} + \frac{\pi^8}{2^{15} 8!} - \cdots$. Thus we see that we can find various representations for various real numbers including the irrational numbers like π, e, and $\sqrt{2}$.

We will now take a look at the number Φ (phi) which represents the *golden ratio*. This number is interesting because many artists and architects have proportioned their works to approximate the ratio because the proportion is aesthetically pleasing. Two quantities a and b are said to be in the golden ratio Φ if $\frac{a+b}{a} = \frac{a}{b} = \Phi$. This results in $\Phi = \frac{1+\sqrt{5}}{2}$ (more information can be found at http://en.wikipedia.org/wiki/Golden_ratio). The most interesting thing about this number comes from one method of its construction. From the Fibonacci sequence defined as $f_n = f_{n-1} + f_{n-2}$ where $f_0=0$ and $f_1=1$, we can create a sequence that converges to Φ. Consider $\lim_{n\to\infty}\frac{f_n}{f_{n-1}}$. One closed formula for $f_n = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n$. It is easy to show from this closed formula that $\frac{f_n}{f_{n-1}} = \frac{1+\sqrt{5}}{2} = \Phi$. Thus, the sequence of the ratios of successive pairs of numbers taken from the Fibonacci sequence will converge to Φ.