Sequences of Real Numbers

Definition: A sequence of real numbers is simply a function \( f : J \to \mathbb{R} \). Sequences are usually denoted by \( f(n) = a_n \) for each \( n \) (the \( n \)-th term of the sequence), and we write \( \{a_n\} \) or \( \{a_n\}_{n=1}^{\infty} \) to represent the entire sequence.

Definition: The sequence \( \{a_n\} \) is said to converge to \( a \) if

for every \( \varepsilon > 0 \), there exists \( N > 0 \) such that if \( n > N \) then \( |a_n - a| < \varepsilon \).

The notation is \( \lim_{n \to \infty} a_n = a \) or \( a_n \to a \).

Theorem: Let \( \{a_n\} \) be a sequence of real numbers. Then \( a_n \to a \) if and only if every open interval containing \( a \), contains all but finitely many terms of the sequence.

Theorem: Every bounded monotone sequence converges.

Definition of \( e \):

For each \( n \), let \( a_n = \left( 1 + \frac{1}{n} \right)^n \). Since \( \{a_n\} \) is increasing and bounded above, it must converge and we define \( e = \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n \).

Basic Theorems on Sequences:

i) Every convergent sequence of real numbers is bounded.

ii) Suppose \( a_n \to a \) and \( b_n \to b \). Then \( a_n + b_n \to a + b \) and \( a_n \cdot b_n \to a \cdot b \).

iii) Suppose \( a_n \to a \), \( a \neq 0 \), and \( \forall n, \ a_n \neq 0 \). Then \( \frac{1}{a_n} \to \frac{1}{a} \).

iv) Squeeze Theorem: If \( a_n \to L \), \( b_n \to L \), and \( a_n \leq w_n \leq b_n \) for each \( n \), then \( w_n \to L \).
Nested Intervals Theorem: Suppose \( \{[a_n, b_n]\} \) is a sequence of closed intervals such that for each \( n \), \( [a_{n+1}, b_{n+1}] \subseteq [a_n, b_n] \). Then the intersection of all of these intervals is either a closed interval or a single point.

Theorem (Bolzano-Weierstrass): Every bounded sequence has a convergent subsequence.

Definition: A sequence \( \{a_n\} \) is said to be a Cauchy sequence if for each \( \varepsilon > 0 \), there exists \( M > 0 \) such that if \( i > M \), \( j > M \) then \( |a_i - a_j| < \varepsilon \).

Theorem: A sequence \( \{a_n\} \) converges if and only if it is a Cauchy sequence.