Linear Approximation Theorem:

Suppose $f$ is defined in an open interval containing $x_0$. Then $f$ is differentiable at $x_0$ if and only if there exists a linear (affine) function

$$A(x) = a(x-x_0) + f(x_0)$$

that approximates $f$ near $x_0$ in the sense that

$$\lim_{x \to x_0} \frac{f(x) - A(x)}{|x-x_0|} = 0 .$$

Proof.

I. Suppose $f$ is differentiable at $x = x_0$. Let $A(x) = f'(x_0)(x-x_0) + f(x_0)$.

$$\lim_{x \to x_0} \frac{f(x) - A(x)}{|x-x_0|} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x-x_0} - f'(x_0) = 0 .$$

So,

$$\lim_{x \to x_0} \frac{f(x) - A(x)}{|x-x_0|} = 0 .$$

II. Suppose there exists $A(x) = a(x-x_0) + f(x_0)$ such that

$$\lim_{x \to x_0} \frac{f(x) - A(x)}{|x-x_0|} = 0 .$$

Then,

$$\lim_{x \to x_0} \left| \frac{f(x) - A(x)}{x-x_0} \right| = 0$$

and

$$\lim_{x \to x_0} \left| \frac{f(x) - f(x_0) - a(x-x_0)}{x-x_0} \right| = 0 ,$$

$$\lim_{x \to x_0} \left| \frac{f(x) - f(x_0)}{x-x_0} - a \right| = 0 .$$

This implies that

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x-x_0} = a.$$