The Fundamental Theorem of Algebra

The Factor Theorem: If c is a zero of a polynomial $p(x)$ then $x - c$ is a factor of $p(x)$.

The Factor Theorem: If p is a polynomial function, and $p(c) = 0$ then $x - c$ is a factor of $p(x)$.

$p(x) = (x - c)$(*polynomial 1 degree less than* $p(x)$)

Example: 5 is a zero of $x^3 - 4x^2 - 4x - 5$ therefore, $x - 5$ is a factor of $x^3 - 4x^2 - 4x - 5$.

$x^3 - 4x^2 - 4x - 5 = (x - 5)$(2nd degree poloynomial)

Example: $-\sqrt{5}$ is a zero of $x^4 - 3x^2 - 10$ therefore, $x - (-\sqrt{5})$ or $x + \sqrt{5}$ is a factor of $4x^4 - 12x^2 - 40$.

$4x^4 - 12x^2 - 40 = (x + \sqrt{5})$(3rd degree polynomial)

The Fundamental Theorem of Algebra

Fundamental Theorem of Algebra:

Any nth degree polynomial with leading coeficient of a_n can be written as a_n multiplied by n linear factors of the form $x - c$ where c is a zero (real or imaginary) of the polynomial.

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = a_n (x - c_1)(x - c_2)\ldots(x - c_n)$$

Example: $-\sqrt{5}$, $\sqrt{5}$, $-\sqrt{2}\,i$, and $\sqrt{2}\,i$ are zeros of $4x^4 - 12x^2 - 40$.

$$4x^4 - 12x^2 - 40 = 4(x + \sqrt{5})(x - \sqrt{5})(x + \sqrt{2}\,i)(x - \sqrt{2}\,i)$$

The Fundamental Theorem of Algebra

If $(x-c)^n$ is a factor then c is said to be a zero with ***multiplicity*** n.

$(x-c)(x-c)(x-c)\ldots(x-c)\ (x-c)$
multiplied by itself n times.

Example: The zeros of $9x^4-12x^3-23x^2+36x-12$ are $\mathbf{\frac{2}{3}}$, $\mathbf{\sqrt{3}}$, $-\mathbf{\sqrt{3}}$.

$$9x^4-12x^3-23x^2+36x-12=9(x-\tfrac{2}{3})(x-\tfrac{2}{3})(x-\sqrt{3})(x+\sqrt{3})$$

$$9x^4-12x^3-23x^2+36x-12=9(x-\tfrac{2}{3})^2(x-\sqrt{3})(x+\sqrt{3})$$

The Fundamental Theorem of Algebra

If P is a polynomial with rational coefficients and if $a+\sqrt{b}$ is a zero of P then $a-\sqrt{b}$ must also be a zero of P.

Example: The zeros of $9x^4-12x^3-23x^2+36x-12$ are $\frac{2}{3}$, $\sqrt{3}$, $-\sqrt{3}$. Therefore $\frac{2}{3}$ must be a zero with multiplicity of 2.

The Fundamental Theorem of Algebra

If P is a polynomial with real coefficients and if $a+bi$ is a zero of P then $a-bi$ must also be a zero of P.

Example: If P is a fifth degree polynomial and $2,\ 2i,$ and $1-3i$ are zeros of the polynomial what other properties of P can be determined from this information?

1) $-2i$ and $1+3i$ are also zeros of P

2) $2,\ 2i,\ -2i,\ 1-3i,$ and $1+3i$ are the only zeros of the polynoial and they all have multiplicity 1.

3) The graph of $y=P(x)$ will only have 1 x-intercept.

4) $P(x)=A(x-2)(x-2i)(x+2i)(x-1-3i)(x-1+3i)$