Variance and Standard Deviation

- While the mean is a measure of the location, or center, of a distribution, the *variance* and *standard deviation* (*s.d.*) measure a distribution's *spread*.

- Definition: Let *X* be a random variable with mean μ. Then the *variance* of *X*, σ^2, is

$$\sigma^2 = \text{Var}(X) = E[(X-\mu)^2].$$

- Definition: The *standard deviation,* σ, of random variable *X* is the square root of the variance of *X*.

$$\sigma = \sqrt{\sigma^2} = \sqrt{\text{Var}(X)}$$

- Example: $X \sim$ Bernoulli(p). ($E(X) = p$.) What is Var(X)?

- Example: $X \sim f(x) = \begin{cases} \frac{x}{2} & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}.$ ($E(X) = 4/3$.) What is Var(X)?

- Theorem: If Var(X) exists, then
  $\text{Var}(X) = E(X^2) - [E(X)]^2$.

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  What is Var(X)?

- Example: $X \sim f(x) = \begin{cases} \frac{x}{2} & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$.
  ($E(X) = 4/3$.) What is Var(X)?

- Example: Die roll
  - $p(x) = 1/6 \quad x \in \{1, 2, \ldots, 6\}$
  - $E(X) = ?$
  - $E(X^2) = ?$
  - Var(X) = ?
  - $\sigma = ?$
- Example: $p(x) = 1/2 \quad x \in \{3, 4\}$
  - $E(X) = ?$
  - $E(X^2) = ?$
  - Var(X) = ?
  - $\sigma = ?$

- Example: $X \sim U(a,b)$. What is Var(X)?

- The variance and standard deviation measure the spread of a distribution. How?

1) Hard Limit (Theorem):

- *Chebyshev's Inequality*: If random variable X has mean μ, and variance σ^2, then

$$P(|X-\mu|>t)\le \tfrac{\sigma^2}{t^2}.$$

- Alternatively, putting $t = k\sigma$,

$$P(\mu-k\sigma\le X\le\mu+k\sigma)\ge 1-\tfrac{1}{k^2}.$$

- For example:

$$P(\mu-2\sigma\le X\le\mu+2\sigma)\ge 1-\tfrac{1}{2^2}=\tfrac{3}{4}.$$

- Chebyshev's Inequality holds for *any* distribution, but is usually very conservative.

2) Rule of thumb (from normal distribution):

- Usually,

$$P(\mu-\sigma\le X\le\mu+\sigma)\approx\tfrac{2}{3}$$

$$P(\mu-2\sigma\le X\le\mu+2\sigma)\approx 0.95$$

$$P(\mu-3\sigma\le X\le\mu+3\sigma)\approx 1$$

- More Examples:
- $X\sim N(\mu,\sigma^2)$. What is Var(X)?

- $X \sim$ gamma(α,λ). What is Var(X)?
- $X \sim$ Poisson(λ). What is Var(X)?

- Theorem: If Var(X) exists, and $Y = a + bX$, then

$$\text{Var}(Y) = b^2\,\text{Var}(X).$$

- Example: Let the variance of Celsius temperature X be Var(X) = 25.
  - What is the standard deviation of X?
  - What is the variance of Y = 9/5 X + 32?
  - What is the standard deviation of Y?

Common Discrete Distributions

Distribution	$E(X)$	Var(X)
Bernoulli(p)	p	$p(1-p)$
Binomial(n, p)	np	$np(1-p)$
Geometric(p)	$1/p$	$(1-p)/p^2$
Negative Binomial(p, r)	r/p	$r(1-p)/p^2$
Hypergeometric(n, r, m)	$\frac{mr}{n}$	$\left(\frac{n-m}{n-1}\right)\frac{mr}{n}\left(1-\frac{r}{n}\right)$
Poisson(λ)	λ	λ

Common Continuous Distributions

Distribution	$E(X)$	$Var(X)$
Uniform(a, b)	$(a+b)/2$	$(b-a)^2/12$
Exponential(λ)	$1/\lambda$	$1/\lambda^2$
Gamma(α, λ)	α/λ	α/λ^2
Beta(α, β)	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$
Normal(μ, σ^2)	μ	σ^2

- Definition: Let X be a random variable. The k^{th} *moment* of X is $E(X^k)$, providing the expectation exists.
  - Example: The mean of X is its first moment.
- Definition: Let X be a random variable. The k^{th} *central moment* of X is

  $$E\{[X-E(X)]^k\},$$

  providing the expectation exists.
  - Example: The variance of X is its second central moment.

- Definition: Let X be a random variable with variance σ^2 and third central moment μ_3. Then the *coefficient of skewness*, μ_3/σ^3, is a measure of symmetry. Positive values indicate long right tails, and negative values indicate long left tails.
- Definition: Let X be a random variable with variance σ^2 and fourth central moment μ_4. Then the *coefficient of kurtosis*, (μ_4/σ^4-3), is a measure of heaviness of tails.

- Example: $X \sim$ Exponential(λ). Find the skewness and kurtosis of X.

- Example: $X \sim$ Normal(μ, σ^2). Find the skewness and kurtosis of X.

Measurement Error

- Suppose random variable X is an estimate of unknown constant x_0. A common model for the error involved in this measurement is $X = x_0 + \beta + \varepsilon$, where
  - x_0 is the true value, the unknown constant
  - β is the *bias*, the systematic error, another unknown constant
  - ε is the *variability*, or *noise*, a random variable with $E(\varepsilon) = 0$, $\text{Var}(\varepsilon) = \sigma^2$.

- Under this model, what are the mean and variance of X?

- One measure of how well our estimate X performs is known as the *mean squared error.*

$$MSE = E[(X-x_0)^2]$$

- Theorem: $MSE = \beta^2 + \sigma^2$.

  (The mean squared error is equal to the variance plus the square of the bias.)

- Read Examples A and B of Section 4.2.1. in Rice.