Math 1210 Test #2 Name: Solutions

October 15, 2009 A-Number: A00000000

Directions: You must show all work to receive full credit for problems. Partial credit will only be given if the work is essentially correct with a minor error like a sign error.

Problem 1. Compute derivatives of the following functions using the method specified in parentheses. You may need to use other methods of differentiation along the way. However, you must use the indicated methods at least once in the calculation.

$$y = \frac{x^2+2}{1-3x} \quad \text{(quotient rule)}$$

$$y' = \frac{\left(\frac{d}{dx}(x^2+2)\right)(1-3x) - (x^2+2)\frac{d}{dx}(1-3x)}{(1-3x)^2}$$

$$= \frac{2x(1-3x) - (x^2+2)(-3)}{(1-3x)^2}$$

b. What is the domain of the resulting function?

$$y = x^2 log_2(2x+3) \quad \text{(product rule)}$$

$$y' = \left(\frac{d}{dx}x^2\right)\log_2(2x+3) + x^2 \cdot \frac{d}{dx}\left(\log_2(2x+3)\right)$$

$$= 2x\log_2(2x+3) + x^2 \cdot \frac{1}{2x+3} \cdot \frac{1}{\ln(2)} \cdot \frac{d}{dx}(2x+3)$$

$$= 2x\log_2(2x+3) + \frac{x^2}{(2x+3)\ln(2)} \cdot (2)$$

Domain

$2x+3 > 0$

$\Rightarrow 2x > -3$

$\Rightarrow x > -3/2$

$x \in (-3/2, +\infty)$

c. What is the domain of the resulting function?

$$y = e^{sin(2x)} \quad \text{(chain rule)}$$

$$y' = \frac{d}{dx} e^{\sin(2x)}$$

$$= e^{\sin(2x)} \cdot \frac{d}{dx}(\sin(2x))$$

$$= e^{\sin(2x)} \cdot \cos(2x) \cdot \frac{d}{dx}(2x)$$

$$= e^{\sin(2x)} \cos(2x) \cdot (2)$$

Domain

$x \in (-\infty, +\infty)$

Problem 2. Compute derivatives of the following functions.

a. $f(x) = 3x + x^2 - 2x^3$

$$f'(x) = 3 + 2x - 2(3x^2)$$
$$= 3 + 2x - 6x^2$$

b. $h(z) = \dfrac{sin(z)}{z^2+4}$

$$h'(z) = \frac{\left(\frac{d}{dz}(\sin(z))\right)(z^2+4) - \sin(z)\cdot\left(\frac{d}{dz}(z^2+4)\right)}{(z^2+4)^2}$$
$$= \frac{\cos(z)(z^2+4) - \sin(z)(2z)}{(z^2+4)^2}$$

c. $g(x) = cos(4x+3)$

$$g'(x) = -\sin(4x+3)\cdot\frac{d}{dx}(4x+3)$$
$$= -4\sin(4x+3)$$

d. $f(x) = sec^3(x)$

$$f'(x) = 3\sec^2(x)\cdot\frac{d}{dx}(\sec(x))$$
$$= 3\sec^2(x)\cdot\sec(x)\tan(x)$$
$$= 3\sec^3(x)\tan(x)$$

e. $y = tan^{-1}(2x)$

$$y' = \frac{1}{(2x)^2+1}\cdot\frac{d}{dx}(2x)$$
$$= \frac{1}{4x^2+1}\cdot(2) = \frac{2}{4x^2+1}$$

f. $h(x) = 10^{\pi x}$

$$h'(x) = 10^{\pi x} \cdot \ln(10) \cdot \frac{d}{dx}(\pi x)$$
$$= 10^{\pi x} \cdot \ln(10) \cdot \pi$$
$$= \pi \ln(10) \cdot 10^{\pi x}$$

g. $f(x) = \sqrt{x}\, tan(x)$

$$f'(x) = \frac{1}{2} x^{-1/2} \tan(x) + x^{1/2} \sec^2(x)$$
$$= \frac{1}{2\sqrt{x}} \tan(x) + \sqrt{x} \sec^2(x)$$

h. $y = sin^{-1}(5x)$

$$y' = \frac{1}{\sqrt{1-(5x)^2}} \cdot \frac{d}{dx}(5x)$$
$$= \frac{5}{\sqrt{1-25x^2}}$$

Problem 3a. Given the implicit relationship

$$xy^2 + x^3y + x\ sin(y) = 2^x y$$

compute $y' = \frac{dy}{dx}$

$$\frac{d}{dx}\left(xy^2 + x^3y + x\sin(y)\right) = \frac{d}{dx}(2^x y)$$

$$\Rightarrow\ y^2 + x\left(2y\frac{dy}{dx}\right) + 3x^2y + x^3\frac{dy}{dx} + \sin(y) + x\cos(y)\frac{dy}{dx} = \ln(2)2^x y + 2^x\frac{dy}{dx}$$

$$\Rightarrow\ y^2 + 3x^2y + \sin(y) - \ln(2)\cdot 2^x y = \left(2^x - 2xy - x^3 - x\cos(y)\right)\frac{dy}{dx}$$

$$\Rightarrow\ \frac{dy}{dx} = \frac{y^2 + 3x^2y + \sin(y) - \ln(2)\, 2^x y}{2^x - 2xy - x^3 - x\cos(y)}$$

b. Compute the equation of the tangent line at the point $(x_0, y_0) = (2, 0)$

For $x_0 = 2,\ y_0 = 0$

$$\frac{dy}{dx} = \frac{0^2 + 3(2)(0) + \sin(0) - \ln(2)\, 2^2 \cdot 0}{2^2 - 2(2)(0) - 2^3 - 2\cos(0)} = \frac{0}{\#} = 0$$

$y - 0 = 0(x - 2)$

$\Rightarrow \boxed{y = 0}$

c. Find $\frac{dx}{dy}$

$$\frac{dx}{dy} = \frac{1}{dy/dx} = \frac{2^x - 2xy - x^3 - x\cos(y)}{y^2 + 3x^2 y\ \sin(y) - \ln(2)\, 2^x\, y}$$

Problem 4. For this problem, define the function

$$f(x) = 3x^4 - 2x^3$$

Analyze the function through the following questions.

a. Determine the critical points for $f(x)$.

$f'(x) = 12x^3 - 6x^2$
$= 6x^2(2x - 1)$ $\Rightarrow$ $x = 0$, $x = 1/2$

$f'(-1) < 0$ | $f'(1/4) < 0$ | $f'(1) > 0$

$\searrow$ $\quad 0 \quad$ $\searrow$ $\quad 1/2 \quad$ $\nearrow$

b. Determine the intervals where $f(x)$ is increasing.

$f'(-1) = 6(-1)^2(-2-1) < 0$
$f'(1/4) = 6(1/4)^2(+\frac{1}{2} - 1) < 0$

$f'(1) = 6(1)(2-1)$
$= 6 > 0$

f is increasing on $(1/2, +\infty)$

c. Determine the intervals where $f(x)$ is decreasing.

From b) f is decreasing on
$(-\infty, 0)$ and $(0, 1/2)$

d. Determine the locations of local minimums of $f(x)$.

Using the First Derivative Test
$x = 1/2$ is the location of a local min

$f(1/2) = 3(\frac{1}{16}) - 2(\frac{1}{8})$
$= \frac{3}{16} - \frac{1}{4} = -\frac{1}{16}$

e. Determine the locations of local maximums of $f(x)$.

The are no local maximims by the first derivative test

f. Determine all inflection points for the function $f(x)$.

$f'' > 0 \;|\; f'' < 0 \;|\; f'' > 0$

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$0 \qquad \tfrac{1}{3}$

$f''(-1) = -12(-4) > 0$

$f''(\tfrac{1}{6}) = 2(-\tfrac{5}{6}) < 0$

$f''(1) = 12(3\cdot 1) = 36 > 0$

$$f''(x) = 36x^2 - 12x$$
$$= 12x(3x-1)$$

$x=0$, $x=\tfrac{1}{3}$ are both inflection points

Problem 5. Determine the absolute maximum and absolute minimum value of

$$y = x^{2/3} - x$$

on the interval $[-1, 1]$.

All we need to do is identify critical points in $(-1,1)$ and compare. So,

① $y' = \tfrac{2}{3}x^{-1/3} - 1$

$= 0 \Rightarrow \tfrac{2}{3}x^{-1/3} - 1 = 0$

$\Rightarrow x^{1/3} = \tfrac{2}{3}$

$\Rightarrow x = \tfrac{8}{27} \qquad -1 < \tfrac{8}{27} < 1$ ✓

② y' DNE at $x = 0 \in (-1,1)$.

There are two critical points: $x = \tfrac{8}{27}$, $x = 0$.

$y(-1) = (-1)^{2/3} + 1 = (1)^{1/3} + 1 = 2$ (abs. max value)

$y(0) = 0 - 0 = 0$ (abs min)

$y(\tfrac{8}{27}) = (\tfrac{8}{27})^{2/3} - \tfrac{8}{27} = (\tfrac{2}{3})^2 - \tfrac{8}{27} = \tfrac{4}{9} - \tfrac{8}{27} = \tfrac{12-8}{27} = \tfrac{4}{27}$

$y(1) = (1)^{2/3} - 1 = 1 - 1 = 0$ (OR abs min)

Problem 6. A snowball is melting at a rate of 10 cubic centimeters per minute. Determine the rate at which the radius is changing when the radius of the snowball is 20 centimeters.

So, the volume of a sphere is

$$V = \frac{4}{3}\pi r^3$$

and computing the rate of change gives

$$\frac{dV}{dt} = \frac{4}{3}\pi \, (3r^2 \frac{dr}{dt}$$

Since

$$\frac{dV}{dt} = -10 \text{ cm}^3/\text{min}$$

$$\Rightarrow \quad -10 = 4\pi \, (20)^2 \frac{dr}{dt} \qquad r = 20$$

$$\Rightarrow \quad \frac{-10}{1600\pi} = \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{-1}{160\pi} \text{ cm/min}.$$