Problem Definition

Problem 29. Find the points of inflection of the graph of the given function.

\[ f(x) = (x - 1)^3(x - 5) \]

Solution Step 1:

The first step is to compute the derivative of the function.

\[ f'(x) = (3)(x - 1)^2(x - 5) + (x - 1)^3(1) \]
\[ = (x - 1)^2(3(x - 5) + (x - 1)) \]
\[ = (x - 1)^2(4x - 16) = 4(x - 1)^2(x - 4) \]

Solution Step 2:

The function is a polynomial and the only critical points are the zeros of the derivative. For this problem, the zeros are \( x = 1 \) (two copies) and \( x = 4 \).

Solution Step 3:

So, let’s try the second derivative test on the function. The second derivative is

\[ f''(x) = 8(x - 1)(x - 4) + 4(x - 1) \]
\[ = (x - 1)(8(x - 4) + 4(x - 1)) \]
\[ = (x - 1)(12x - 36) = 12(x - 1)(x - 3) \]

The product rule has been used and the result simplified.

Solution Step 4:

In this case, we need to find points where the second derivative changes sign. There are three intervals.

(a) The first interval is \((-\infty, 1)\). The second derivative at \( x = 0 \) is

\[ f''(0) = 12(0 - 1)(0 - 3) = 36 > 0 \]

So \( f(x) \) is concave upward on this interval.
(b) The second interval is (1, 3). The second derivative at \( x = 2 \) is
\[
\frac{d^2}{dx^2} f(x) = 12(2 - 1)(2 - 3) = -12 < 0
\]
So \( f(x) \) is concave downward on this interval.

(c) The last interval is (3, \( \infty \)). The second derivative at \( x = 4 \) is
\[
\frac{d^2}{dx^2} f(x) = 12(4 - 1)(4 - 3) = 36 > 0
\]
So \( f(x) \) is concave upward on this interval.

Since the concavity changes at both points, \( x = 1 \) and \( x = 3 \) are both points of inflection.

**Solution Step 5:**

To summarize, the points (1, 0) and (3, -36) on the graph of the function are inflection points.