Problem Definition

Problem 41. Investment A brokerage firm opens a new real estate investment plan for which the earnings are equivalent to continuous compounding at the rate of \( r \). The firm estimates that deposits from investors will create a net cash flow of \( Pt \) dollars, where \( t \) is the time in years. The rate of change in the total investment \( A \) is modeled by

\[
\frac{dA}{dt} = rA + Pt
\]

(a) Solve the differential equation and find the total investment \( A \) as a function of time, \( t \). Assume that \( A = 0 \) when \( t = 0 \).

(b) Find the total investment \( A \) after 10 years given that \( P = 500,000 \) and \( r = 9\% \).

Solution Step 1:

The first step is to rewrite the differential equation in the form of a first order linear differential equation. That is, rewrite the equation as

\[
\frac{dA}{dt} - rA = Pt
\]

In terms of the definition of a first order linear equation we can identify

\[
P(t) = -r
\]

and

\[
Q(t) = Pt
\]

It is unfortunate notation, but the \( P(t) \) in the integrating factor above is not the same as the \( P \) in the differential equation. However, if you keep that in mind we should be ok.

Solution Step 2:

The integrating factor is defined by

\[
u(t) = e^{\int -rt \, dt} = e^{-rt}
\]
Solution Step 3:

With the integrating factor computed, the solution is

\[
A(t) = \frac{1}{u(t)} \int Ptu(t)dt
\]

\[
= \frac{1}{e^{-rt}} \int Pte^{-rt}dt
\]

\[
= Pe^{rt} \int te^{-rt}dt
\]

At this point we need to compute the integral via an integration by parts.

Solution Step 4:

To compute the integral necessary for the solution we will need to apply our knowledge of integration by parts. Use \( u = t \) and \( dv = e^{-rt}dt \).

\[
\int te^{-rt}dt = t \left( \frac{1}{-r} e^{-rt} \right) - \int \left( -\frac{1}{r} e^{-rt} \right) dt
\]

\[
= \frac{-t}{r} e^{-rt} + \frac{1}{r} \int e^{-rt}dt
\]

\[
= \frac{-t}{r} e^{-rt} - \frac{1}{r^2} e^{-rt}
\]

\[
= \frac{-e^{-rt}}{r^2} (1 + rt)
\]

Solution Step 5:

The solution is of the form

\[
A(t) = Pe^{rt} \left( \frac{-e^{-rt}}{r^2} (1 + rt) + C \right)
\]

\[
= P \left( \frac{-1}{r^2} (1 + rt) + Ce^t \right)
\]

Solution Step 6:
The initial value that we have to apply is

\[
A(0) = P \left( -\frac{1}{r^2} (1 + r(0)) + Ce^{r(0)} \right) \\
= P \left( -\frac{1}{r^2} + C \right) = 0
\]

Then

\[
P \left( \frac{1}{r^2} + C \right) = 0
\]

which implies \( C = r^{-2} \). This means the solution can be written as

\[
A(t) = \frac{P}{r^2} \left( -(1 + rt) + e^{rt} \right) \\
= \frac{P}{r^2} \left( e^{rt} - (1 + rt) \right)
\]

**Solution Step 7:**

The last step is to approximate the amount in the investment given when \( P = 500,000 \) and \( r = 9\% \). In this case,

\[
A(t) = \frac{500000}{(0.09)^2} \left( e^{(0.09)t} - 1 - (0.09)t \right)
\]

and

\[
A(10) = \frac{500000}{(0.09)^2} \left( e^{(0.09)(10)} - 1 - (0.09)(10) \right) \approx 34,543,402
\]