C.1 SOLUTIONS OF DIFFERENTIAL EQUATIONS

Find general solutions of differential equations. • Find particular solutions of differential equations.

General Solution of a Differential Equation

A differential equation is an equation involving a differentiable function and one or more of its derivatives. For instance,

\[ y' + 2y = 0 \]

is a differential equation. A function \( y = f(x) \) is a solution of a differential equation if the equation is satisfied when \( y \) and its derivatives are replaced by \( f(x) \) and its derivatives. For instance,

\[ y = e^{-2x} \]

is a solution of the differential equation shown above. To see this, substitute for \( y \) and \( y' = -2e^{-2x} \) in the original equation.

\[
\begin{align*}
  y' + 2y &= -2e^{-2x} + 2(e^{-2x}) \\
  &= 0
\end{align*}
\]

In the same way, you can show that \( y = 2e^{-2x}, y = -3e^{-2x}, \) and \( y = \frac{1}{2}e^{-2x} \) are also solutions of the differential equation. In fact, each function given by

\[ y = Ce^{-2x} \]

where \( C \) is a real number, is a solution of the equation. This family of solutions is called the general solution of the differential equation.

EXAMPLE 1 Checking Solutions

Show that

(a) \( y = Ce^x \) and (b) \( y = Ce^{-x} \)

are solutions of the differential equation \( y'' - y = 0 \).

SOLUTION

(a) Because \( y' = Ce^x \) and \( y'' = Ce^x \), it follows that

\[
\begin{align*}
  y'' - y &= Ce^x - Ce^x \\
  &= 0.
\end{align*}
\]

So, \( y = Ce^x \) is a solution.

(b) Because \( y' = -Ce^{-x} \) and \( y'' = Ce^{-x} \), it follows that

\[
\begin{align*}
  y'' - y &= Ce^{-x} - Ce^{-x} \\
  &= 0.
\end{align*}
\]

So, \( y = Ce^{-x} \) is also a solution.
particular Solutions and Initial Conditions

A particular solution of a differential equation is any solution that is obtained by assigning specific values to the constants in the general equation. Geometrically, the general solution of a differential equation is a family of graphs called solution curves. For instance, the general solution of the differential equation $xy' - 2y = 0$ is

$$y = Cx^2.$$  
General solution

Figure A.7 shows several solution curves of this differential equation.

Particular solutions of a differential equation are obtained from initial conditions placed on the unknown function and its derivatives. For instance, in Figure A.7, suppose you want to find the particular solution whose graph passes through the point $(1, 3)$. This initial condition can be written as

$$y = 3 \quad \text{when} \quad x = 1.$$  
Initial condition

Substituting these values into the general solution produces $3 = C(1)^2$, which implies that $C = 3$. So, the particular solution is

$$y = 3x^2.$$  
Particular solution

**Example 2:** Finding a Particular Solution

Verify that $y = Cx^3$ is a solution of the differential equation $xy' - 3y = 0$ for any value of $C$. Then find the particular solution determined by the initial condition

$$y = 2 \quad \text{when} \quad x = -3.$$  
Initial condition

**Solution** The derivative of $y = Cx^3$ is $y' = 3Cx^2$. Substituting into the differential equation produces

$$xy' - 3y = x(3Cx^2) - 3(Cx^3) = 0.$$  

So, $y = Cx^3$ is a solution for any value of $C$. To find the particular solution, substitute $x = -3$ and $y = 2$ into the general solution to obtain

$$2 = C(-3)^3 \quad \text{or} \quad C = -\frac{2}{27}.$$  

This implies that the particular solution is

$$y = -\frac{2}{27}x^3.$$  
Particular solution

Some differential equations have solutions other than those given by their general solutions. These are called singular solutions. In this brief discussion of differential equations, singular solutions will not be discussed.
EXAMPLE 3 Finding a Particular Solution

You are working in the marketing department of a company that is producing a new cereal product to be sold nationally. You determine that a maximum of 10 million units of the product could be sold in a year. You hypothesize that the rate of growth of the sales \( x \) (in millions of units) is proportional to the difference between the maximum sales and the current sales. As a differential equation, this hypothesis can be written as

\[
\frac{dx}{dt} = k(10 - x), \quad 0 \leq x \leq 10.
\]

The general solution of this differential equation is

\[
x = 10 - Ce^{-kt}
\]

where \( t \) is the time in years. After 1 year, 250,000 units have been sold. Sketch the graph of the sales function over a 10-year period.

SOLUTION Because the product is new, you can assume that \( x = 0 \) when \( t = 0 \). So, you have two initial conditions.

\[
\begin{align*}
x &= 0 \quad \text{when} \quad t = 0 \quad \text{First initial condition} \\
x &= 0.25 \quad \text{when} \quad t = 1 \quad \text{Second initial condition}
\end{align*}
\]

Substituting the first initial condition into the general solution produces

\[
0 = 10 - Ce^{-k(0)}
\]

which implies that \( C = 10 \). Substituting the second initial condition into the general solution produces

\[
0.25 = 10 - 10e^{-k(1)}
\]

which implies that \( k = \ln \frac{10}{9} \approx 0.0253 \). So, the particular solution is

\[
x = 10 - 10e^{-0.0253t}.
\]

The table shows the annual sales during the first 10 years, and the graph of the solution is shown in Figure A.8.

<table>
<thead>
<tr>
<th>( t )</th>
<th>0.25</th>
<th>0.49</th>
<th>0.73</th>
<th>0.96</th>
<th>1.19</th>
<th>1.41</th>
<th>1.62</th>
<th>1.83</th>
<th>2.04</th>
<th>2.24</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.25</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>0.49</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td>0.73</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td>0.96</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1.19</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1.41</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1.62</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1.83</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>2.04</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>2.24</td>
</tr>
</tbody>
</table>

FIGURE A.8

In the first three examples in this section, each solution was given in explicit form, such as \( y = f(x) \). Sometimes you will encounter solutions for which it is more convenient to write the solution in implicit form, as shown in Example 4.
EXAMPLE 4: Sketching Graphs of Solutions

(a) Verify that
\[ 2y^2 - x^2 = C \]

is a general solution of the differential equation
\[ 2yy' - x = 0. \]

Then sketch the particular solutions represented by \( C = 0, C = \pm 1, \) and \( C = \pm 4. \)

SOLUTION To verify the given solution, differentiate each side with respect to \( x. \)

\[ 2yy' - x = 0 \]
\[ 4yy' - 2x = 0 \]
\[ 2yy' - x = 0 \]

Because the third equation is the given differential equation, you can conclude that
\[ 2y^2 - x^2 = C \]

is a solution. The particular solutions represented by \( C = 0, C = \pm 1, \) and \( C = \pm 4 \)
are shown in Figure A.9.

\[ \text{Figure A.9 Graphs of Five Particular Solutions} \]

TAKE ANOTHER LOOK

Writing a Differential Equation

Write a differential equation that has the family of circles
\[ x^2 + y^2 = C \]
as a general solution.
APPENDIX C  Differential Equations

PREQUISITE REVIEW C.1

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section.

In Exercises 1–4, find the first and second derivatives of the function.
1. \( y = 3x^2 + 2x + 1 \)
2. \( y = -2x^3 - 8x + 4 \)
3. \( y = -3e^{2x} \)
4. \( y = -3e^{2x} \)

In Exercises 5–8, use implicit differentiation to find \( \frac{dy}{dx} \).
5. \( x^2 + y^2 = 2x \)
6. \( 2x - y^3 = 4y \)
7. \( xy^2 = 3 \)
8. \( 3xy + x^2y^2 = 10 \)

In Exercises 9 and 10, solve for \( k \).
9. \( 0.5 = 9 - 9e^{-k} \)
10. \( 14.75 = 25 - 25e^{-2k} \)

EXERCISES C.1

In Exercises 1–10, verify that the function is a solution of the differential equation.

<table>
<thead>
<tr>
<th>Solution</th>
<th>Differential Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( y = x^3 + 5 )</td>
<td>( y' = 3x^2 )</td>
</tr>
<tr>
<td>2. ( y = 2x^3 - x + 1 )</td>
<td>( y' = 6x^2 - 1 )</td>
</tr>
<tr>
<td>3. ( y = e^{-2x} )</td>
<td>( y' + 2y = 0 )</td>
</tr>
<tr>
<td>4. ( y = 3e^{x^2} )</td>
<td>( y' - 2xy = 0 )</td>
</tr>
<tr>
<td>5. ( y = 2x^3 )</td>
<td>( y' - \frac{3}{x}y = 0 )</td>
</tr>
<tr>
<td>6. ( y = 4x^2 )</td>
<td>( y' - \frac{2}{x}y = 0 )</td>
</tr>
<tr>
<td>7. ( y = x^2 )</td>
<td>( x^2y'' - 2y = 0 )</td>
</tr>
<tr>
<td>8. ( y = \frac{1}{x} )</td>
<td>( xy'' + 2y' = 0 )</td>
</tr>
<tr>
<td>9. ( y = 2e^{2x} )</td>
<td>( y'' - y' - 2y = 0 )</td>
</tr>
<tr>
<td>10. ( y = e^{x^2} )</td>
<td>( y'' - 3x^2y' - 6xy = 0 )</td>
</tr>
</tbody>
</table>

In Exercises 11–28, verify that the function is a solution of the differential equation for any value of \( C \).

<table>
<thead>
<tr>
<th>Solution</th>
<th>Differential Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>11. ( y = \frac{1}{x} + C )</td>
<td>( \frac{dy}{dx} = -\frac{1}{x^2} )</td>
</tr>
<tr>
<td>12. ( y = \sqrt{4 - x^2} + C )</td>
<td>( \frac{dy}{dx} = -\frac{x}{\sqrt{4 - x^2}} )</td>
</tr>
<tr>
<td>13. ( y = Ce^{4x} )</td>
<td>( \frac{dy}{dx} = 4y )</td>
</tr>
<tr>
<td>14. ( y = Ce^{-4x} )</td>
<td>( \frac{dy}{dx} = -4y )</td>
</tr>
<tr>
<td>15. ( y = Ce^{-x^3} + 7 )</td>
<td>( 3\frac{dy}{dt} + y - 7 = 0 )</td>
</tr>
<tr>
<td>16. ( y = Ce^{-x} + 10 )</td>
<td>( y' + y - 10 = 0 )</td>
</tr>
<tr>
<td>17. ( y = Cx^2 - 3x )</td>
<td>( xy' - 3x - 2y = 0 )</td>
</tr>
<tr>
<td>18. ( y = x \ln x^2 + 2x^{3/2} + Cx )</td>
<td>( y' - \frac{y}{x} = 2 + \sqrt{x} )</td>
</tr>
<tr>
<td>19. ( y = x^2 + 2x + \frac{C}{x} )</td>
<td>( xy' + y = x(3x + 4) )</td>
</tr>
<tr>
<td>20. ( y = C_1 + C_2e^x )</td>
<td>( y'' - y' = 0 )</td>
</tr>
</tbody>
</table>
### Solution

<table>
<thead>
<tr>
<th>Equation</th>
<th>Initial Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = C_1e^{x/2} + C_2e^{-2x} )</td>
<td>( 2y'' + 3y' - 2y = 0 )</td>
</tr>
<tr>
<td>( y = C_1 + C_2x )</td>
<td>( y'' - 3y' - 4y = 0 )</td>
</tr>
<tr>
<td>( y = \frac{bx^4}{4 - a} + Cx^a )</td>
<td>( y' - ay = bx^3 )</td>
</tr>
<tr>
<td>( y = \frac{x^3}{5} + x + C\sqrt{x} )</td>
<td>( 2xy' - y = x^3 - x )</td>
</tr>
<tr>
<td>( y = \frac{2}{1 + Ce^{x^2}} )</td>
<td>( y' + 2xy = xy^2 )</td>
</tr>
<tr>
<td>( y = Ce^{-x^2} )</td>
<td>( y' + (2x - 1)y = 0 )</td>
</tr>
<tr>
<td>( y = x \ln x + Cx + 4 )</td>
<td>( x(y' - 1) - (y - 4) = 0 )</td>
</tr>
<tr>
<td>( y = x(\ln x + C) )</td>
<td>( x + y - xy' = 0 )</td>
</tr>
</tbody>
</table>

In Exercises 29–32, use implicit differentiation to verify that the equation is a solution of the differential equation for any value of C.

<table>
<thead>
<tr>
<th>Solution</th>
<th>Differential Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^2 + y^2 = Cy )</td>
<td>( y' = \frac{2xy}{x^2 - y^2} )</td>
</tr>
<tr>
<td>( x^2 + 2xy - x^2 = C )</td>
<td>( (x + y)y' - x + y = 0 )</td>
</tr>
<tr>
<td>( x^2 + xy = C )</td>
<td>( x^3y'' - 2(x + y) = 0 )</td>
</tr>
<tr>
<td>( x^2 - y^2 = C )</td>
<td>( y^2y'' + x^2 - y^2 = 0 )</td>
</tr>
</tbody>
</table>

In Exercises 33–36, determine whether the function is a solution of the differential equation \( y'''' - 16y = 0 \).

| \( y = e^{-2x} \) | \( -4y \) |
| \( y = 3 \ln x \) | \( y = \frac{4}{x} \) |
| \( y = 4e^{2x} \) | \( y = x \ln x \) |

In Exercises 37–40, determine whether the function is a solution of the differential equation \( y''' - 3y'' + 2y = 0 \).

| \( y = \frac{2e^{-2x}}{x} \) | \( -y - 7 = 0 \) |
| \( y = 4e^x + \frac{2}{5}xe^{-2x} \) | \( x - 10 = 0 \) |
| \( y = xe^x \) | \( 3x - 2y = 0 \) |
| \( y = x \ln x \) | \( y = 2 + \sqrt{x} \) |
| \( y = x(3x + 4) \) | \( y = e^x(3x + 4) \) |

In Exercises 41–48, verify that the general solution satisfies the differential equation. Then find the particular solution that satisfies the initial condition.

<table>
<thead>
<tr>
<th>General Solution</th>
<th>Differential Equation</th>
<th>( C )-Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = Ce^{-2x} )</td>
<td>( y' + 2y = 0 )</td>
<td>( 1, 2, 4 )</td>
</tr>
<tr>
<td>( y = Ce^{-x} )</td>
<td>( y' + y = 0 )</td>
<td>( 0, \pm 1, \pm 4 )</td>
</tr>
<tr>
<td>( y = C(x + 2)^2 )</td>
<td>( (x + 2)y' - 2y = 0 )</td>
<td>( 0, \pm 1, \pm 2 )</td>
</tr>
<tr>
<td>( y = Ce^{-x} )</td>
<td>( y' + y = 0 )</td>
<td>( 0, \pm 1, \pm 2 )</td>
</tr>
</tbody>
</table>

In Exercises 53–60, use integration to find the general solution of the differential equation.

<table>
<thead>
<tr>
<th>General Solution</th>
<th>Differential Equation</th>
<th>( C )-Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{dy}{dx} = 3x^2 )</td>
<td>( \frac{dy}{dx} = \frac{1}{1 + x} )</td>
<td>( 0, \pm 1, \pm 4 )</td>
</tr>
<tr>
<td>( \frac{dy}{dx} = \frac{x + 3}{x} )</td>
<td>( \frac{dy}{dx} = \frac{x - 2}{x} )</td>
<td>( 0, \pm 1, \pm 2 )</td>
</tr>
<tr>
<td>( \frac{dy}{dx} = \frac{1}{x^2 - 1} )</td>
<td>( \frac{dy}{dx} = \frac{x}{1 + x^2} )</td>
<td>( 0, \pm 1, \pm 2 )</td>
</tr>
<tr>
<td>( \frac{dy}{dx} = \frac{x}{x^2 - 1} )</td>
<td>( \frac{dy}{dx} = \frac{x}{x^2 - 3} )</td>
<td>( 0, \pm 1, \pm 2 )</td>
</tr>
<tr>
<td>( \frac{dy}{dx} = xe^x )</td>
<td>( \frac{dy}{dx} = xe^x )</td>
<td>( 0, \pm 1, \pm 2 )</td>
</tr>
</tbody>
</table>
In Exercises 61–64, you are shown the graphs of some of the solutions of the differential equation. Find the particular solution whose graph passes through the indicated point.

61. \( y^2 = Cx^3 \)
62. \( 2x^2 - y^2 = C \)

\[
2xy' - 3y = 0
\]
\[
yy' - 2x = 0
\]

\[
(4, 4)
\]
\[
(3, 4)
\]

63. \( y = Ce^x \)
64. \( y^2 = 2Cx \)

\[
y' - y = 0
\]
\[
2xy' - y = 0
\]

\[
(0, 3)
\]
\[
(2, 1)
\]

65. **Biology** The limiting capacity of the habitat of a wildlife herd is 750. The growth rate \( dN/dt \) of the herd is proportional to the unutilized opportunity for growth, as described by the differential equation

\[
dN/dt = k(750 - N).
\]

The general solution of this differential equation is

\[
N = 750 - Ce^{-kt}.
\]

When \( t = 0 \), the population of the herd is 100. After 2 years, the population has grown to 160.

(a) Write the population function \( N \) as a function of \( t \).
(b) Use a graphing utility to graph the population function.
(c) What is the population of the herd after 4 years?

66. **Investment** The rate of growth of an investment is proportional to the amount in the investment at any time \( t \). That is,

\[
dA/dt = kA.
\]

The initial investment is $1000, and after 10 years the balance is $3320.12. The general solution is

\[
A = Ce^{kt}.
\]

What is the particular solution?

67. **Marketing** You are working in the marketing department of a computer software company. Your marketing team determines that a maximum of 30,000 units of a new product can be sold in a year. You hypothesize that the rate of growth of the sales \( x \) is proportional to the difference between the maximum sales and the current sales. That is,

\[
\frac{dx}{dt} = k(30,000 - x).
\]

The general solution of this differential equation is

\[
x = 30,000 - Ce^{-kt},
\]

where \( t \) is the time in years. During the first year, 2000 units are sold. Complete the table showing the numbers of units sold in subsequent years.

<table>
<thead>
<tr>
<th>Year, ( t )</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Units, ( x )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

68. **Marketing** In Exercise 67, suppose that the maximum annual sales are 50,000 units. How does this change the sales shown in the table?

69. **Safety** Assume that the rate of change in the number of miles \( s \) of road cleared per hour by a snowplow is inversely proportional to the depth \( h \) of the snow. This rate of change is described by the differential equation

\[
\frac{ds}{dh} = \frac{k}{h}.
\]

Show that

\[
s = 25 - \frac{13}{\ln 3} \ln \frac{h}{2}
\]

is a solution of this differential equation.

70. Show that \( y = a + Ce^{kt} - k \) is a solution of the differential equation

\[
y = a + b(y - a) + \left(\frac{1}{k}\right)\left(\frac{dy}{dt}\right)
\]

where \( k \) is a constant.

71. The function \( y = Ce^{kt} \) is a solution of the differential equation

\[
\frac{dy}{dt} = 0.07y.
\]

Is it possible to determine \( C \) or \( k \) from the information given? If so, find its value.

**True or False?** In Exercises 72 and 73, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

72. A differential equation can have more than one solution.

73. If \( y = f(x) \) is a solution of a differential equation, then \( y = f(x) + C \) is also a solution.
C.2 SEPARATION OF VARIABLES

Use separation of variables to solve differential equations. Use differential equations to model and solve real-life problems.

Separation of Variables

The simplest type of differential equation is one of the form \( y' = f(x) \). You know that this type of equation can be solved by integration to obtain

\[
y = \int f(x) \, dx.
\]

In this section, you will learn how to use integration to solve another important family of differential equations—those in which the variables can be separated. This technique is called separation of variables.

Separation of Variables

If \( f \) and \( g \) are continuous functions, then the differential equation

\[
\frac{dy}{dx} = f(x)g(y)
\]

has a general solution of

\[
\int \frac{1}{g(y)} \, dy = \int f(x) \, dx + C.
\]

Essentially, the technique of separation of variables is just what its name implies. For a differential equation involving \( x \) and \( y \), you separate the \( x \) variables to one side and the \( y \) variables to the other. After separating variables, integrate each side to obtain the general solution. Here is an example.

EXAMPLE 1 Solving a Differential Equation

Find the general solution of

\[
\frac{dy}{dx} = \frac{x}{y^2 + 1}.
\]

SOLUTION Begin by separating variables, then integrate each side.

\[
\frac{dy}{y^2 + 1} = \frac{x}{dx},
\]

Differential equation

\[
(y^2 + 1) \, dy = x \, dx
\]

Separate variables.

\[
\int (y^2 + 1) \, dy = \int x \, dx
\]

Integrate each side.

\[
\frac{y^3}{3} + y = \frac{x^2}{2} + C
\]

General solution

You can use a symbolic integration utility to solve a separable variables differential equation. Use a symbolic integration utility to solve the differential equation

\[
y' = \frac{x}{y^2 + 1}.
\]
EXAMPLE 2 | Solving a Differential Equation

Find the general solution of

\[ \frac{dy}{dx} = \frac{x}{y}. \]

**SOLUTION** Begin by separating variables, then integrate each side.

\[ \frac{dy}{y} = \frac{x}{y} \quad \text{Differential equation} \]

\[ y \, dy = x \, dx \quad \text{Separate variables}. \]

\[ \int y \, dy = \int x \, dx \quad \text{Integrate each side}. \]

\[ \frac{y^2}{2} = \frac{x^2}{2} + C_1 \quad \text{Find antiderivatives}. \]

\[ y^2 = x^2 + C \quad \text{Multiply each side by 2}. \]

So, the general solution is \( y^2 = x^2 + C \). Note that \( C_1 \) is used as a temporary constant of integration in anticipation of multiplying each side of the equation by 2 to produce the constant \( C \).

**STUDY TIP**

After finding the general solution of a differential equation, you should use the techniques demonstrated in Section C.1 to check the solution. For instance, in Example 2 you can check the solution by differentiating the equation \( y^2 = x^2 + C \) to obtain \( 2yy' = 2x \) or \( y' = x/y \).

EXAMPLE 3 | Solving a Differential Equation

Find the general solution of

\[ e^y \frac{dy}{dx} = 2x. \]

Use a graphing utility to graph several solutions.

**SOLUTION** Begin by separating variables, then integrate each side.

\[ e^y \frac{dy}{dx} = 2x \quad \text{Differential equation} \]

\[ e^y \, dy = 2x \, dx \quad \text{Separate variables}. \]

\[ \int e^y \, dy = \int 2x \, dx \quad \text{Integrate each side}. \]

\[ e^y = x^2 + C \quad \text{Find antiderivatives}. \]

By taking the natural logarithm of each side, you can write the general solution as \( y = \ln(x^2 + C) \).

The graphs of the particular solutions given by \( C = 0, C = 5, C = 10, \) and \( C = 15 \) are shown in Figure A.10.
**Example 4** Finding a Particular Solution

Solve the differential equation
\[ xe^x + yy' = 0 \]
subject to the initial condition \( y = 1 \) when \( x = 0 \).

**SOLUTION**

\[ xe^x + yy' = 0 \]

Differential equation

\[ y \frac{dy}{dx} = -xe^x \]

Subtract \( xe^x \) from each side.

\[ y \ dy = -xe^x \ dx \]

Separate variables.

\[ \int y \ dy = \int -xe^x \ dx \]

Integrate each side.

\[ \frac{y^2}{2} = -\frac{1}{2}e^{-x^2} + C_1 \]

Find antiderivatives.

\[ y^2 = -e^{-x^2} + C \]

Multiply each side by 2.

To find the particular solution, substitute the initial condition values to obtain

\[ 1^2 = -e^{0^0} + C. \]

This implies that \( 1 = -1 + C \), or \( C = 2 \). So, the particular solution that satisfies the initial condition is

\[ y^2 = -e^{-x^2} + 2. \]

Particular solution.

---

**Example 5** Solving a Differential Equation

Example 3 in Section C.1 uses the differential equation

\[ \frac{dx}{dt} = k(10 - x) \]

to model the sales of a new product. Solve this differential equation.

**SOLUTION**

\[ \frac{dx}{dt} = k(10 - x) \]

Differential equation

\[ \frac{1}{10 - x} \ dx = k \ dt \]

Separate variables.

\[ \int \frac{1}{10 - x} \ dx = \int k \ dt \]

Integrate each side.

\[ -\ln(10 - x) = kt + C_1 \]

Find antiderivatives.

\[ \ln(10 - x) = -kt - C_1 \]

Multiply each side by \(-1\).

\[ 10 - x = e^{-kt - C_1} \]

Exponentiate each side.

\[ x = 10 - Ce^{-kt} \]

Solve for \( x \).

---

**Study Tip**

In Example 5, the context of the original model indicates that \((10 - x)\) is positive. So, when you integrate \(1/(10 - x)\), you can write \(-\ln(10 - x)\), rather than \(-\ln|10 - x|\).

Also note in Example 5 that the solution agrees with the one that was given in Example 3 in Section C.1.
Corporate profits in the United States are closely monitored by New York City's Wall Street executives. Corporate profits, however, represent only about 10.5% of the national income. In 2003, the national income was more than $9.5 trillion. Of this, about 65% was employee compensation.

**Applications**

**EXAMPLE 6 Modeling National Income**

Let \( y \) represent the national income, let \( a \) represent the income spent on necessities, and let \( b \) represent the percent of the remaining income spent on luxuries. A commonly used economic model that relates these three quantities is

\[
\frac{dy}{dt} = k(1 - b)(y - a)
\]

where \( t \) is the time in years. Assume that \( b \) is 75%, and solve the resulting differential equation.

**SOLUTION** Because \( b \) is 75%, it follows that \( (1 - b) \) is 0.25. So, you can solve the differential equation as shown.

\[
\frac{dy}{dt} = k(0.25)(y - a)
\]

Differential equation

\[
\frac{1}{y - a} \frac{dy}{dt} = 0.25k \frac{dt}{y - a}
\]

Separate variables.

\[
\int \frac{1}{y - a} \frac{dy}{dt} = \int 0.25k \frac{dt}{y - a}
\]

Integrate each side.

\[
\ln(y - a) = 0.25kt + C_1
\]

Find antiderivatives, given \( y - a > 0 \).

\[
y - a = Ce^{0.25kt}
\]

Exponentiate each side.

\[
y = a + Ce^{0.25kt}
\]

Add \( a \) to each side.

The graph of this solution is shown in Figure A.11. In the figure, note that the national income is spent in three ways.

\[
\text{National income} = \text{(necessities)} + \text{(luxuries)} + \text{(capital investment)}
\]
EXAMPLE 7 Using Graphical Information

Find the equation of the graph that has the characteristics listed below.

1. At each point \((x, y)\) on the graph, the slope is \(-x/2y\).
2. The graph passes through the point \((2, 1)\).

SOLUTION Using the information about the slope of the graph, you can write the differential equation

\[
\frac{dy}{dx} = -\frac{x}{2y}
\]

Using the point on the graph, you can determine the initial condition \(y = 1\) when \(x = 2\).

\[
\int 2y\,dy = \int -x\,dx
\]

Separate variables.

\[
y^2 = -\frac{x^2}{2} + C_1
\]

Find antiderivatives.

\[
2y^2 = -x^2 + C
\]

Multiply each side by 2.

\[
x^2 + 2y^2 = C
\]

Simplify.

Applying the initial condition yields

\[(2)^2 + 2(1)^2 = C\]

which implies that \(C = 6\). So, the equation that satisfies the two given conditions is

\[x^2 + 2y^2 = 6.\]

As shown in Figure A.12, the graph of this equation is an ellipse.

FIGURE A.12

TAKE ANOTHER LOOK

Classifying Differential Equations

In which of the differential equations can the variables be separated?

\[
a. \quad \frac{dy}{dx} = \frac{3x}{y}
\]

\[
b. \quad \frac{dy}{dx} = \frac{3x}{y} + 1
\]

\[
c. \quad x^2\frac{dy}{dx} = \frac{3x}{y}
\]

\[
d. \quad \frac{dy}{dx} = \frac{3x + y}{y}
\]
In Exercises 1–6, find the indefinite integral and check your result by differentiating.

1. \[ \int x^{3/2} \, dx \]
2. \[ \int (t^2 - t^{3/2}) \, dt \]
3. \[ \int \frac{2}{x - 5} \, dx \]
4. \[ \int \frac{y}{2y^2 + 1} \, dy \]
5. \[ \int e^y \, dy \]
6. \[ \int xe^{-x^2} \, dx \]

In Exercises 7–10, solve the equation for C or k.

7. \[ (3)^2 - 6(3) = 1 + C \]
8. \[ (-1)^2 + (-2)^2 = C \]
9. \[ 10 = 2e^k \]
10. \[ (6)^2 - 3(6) = e^{-2} \]

**EXERCISES C.2**

In Exercises 1–6, decide whether the variables in the differential equation can be separated.

1. \[ \frac{dy}{dx} = \frac{x}{y + 3} \]
2. \[ \frac{dy}{dx} = \frac{x + 1}{x} \]
3. \[ \frac{dy}{dx} = \frac{1}{x + y} \]
4. \[ \frac{dy}{dx} = \frac{x}{x + y} \]
5. \[ \frac{dy}{dx} = x - y \]
6. \[ \frac{dy}{dx} = 1 \]

In Exercises 7–26, use separation of variables to find the general solution of the differential equation.

7. \[ \frac{dy}{dx} = 2x \]
8. \[ \frac{dy}{dx} = \frac{1}{x} \]
9. \[ 3y^2 \frac{dy}{dx} = 1 \]
10. \[ \frac{dy}{dx} = x^2y \]
11. \[ \frac{(y + 1) \, dy}{dx} = 2x \]
12. \[ (1 + y) \frac{dy}{dx} = 4x \]
13. \[ y' - xy = 0 \]
14. \[ y' - y = 5 \]
15. \[ \frac{dy}{dt} = \frac{e^t}{4y} \]
16. \[ e^t \frac{dy}{dt} = 3t^2 + 1 \]
17. \[ \frac{dy}{dx} = \sqrt{1 - y} \]
18. \[ \frac{dy}{dx} = \frac{\sqrt{x}}{y} \]
19. \[ (2 + x)y' = 2y \]
20. \[ y' = (2x - 1)(y + 3) \]
21. \[ xy' = y \]
22. \[ y' - y(x + 1) = 0 \]
23. \[ y' = \frac{x}{y} - \frac{x}{1 + y} \]
24. \[ \frac{dy}{dx} = \frac{x^2 + 2}{3y^2} \]
25. \[ e^2(y' + 1) = 1 \]
26. \[ yy' - 2xe^x = 0 \]

In Exercises 27–32, use the initial condition to find the particular solution of the differential equation.

<table>
<thead>
<tr>
<th>Differential Equation</th>
<th>Initial Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>27. [ yy' - e^x = 0 ]</td>
<td>( y = 4 ) when ( x = 0 )</td>
</tr>
<tr>
<td>28. [ \sqrt{x} + \sqrt{y}y' = 0 ]</td>
<td>( y = 4 ) when ( x = 1 )</td>
</tr>
<tr>
<td>29. [ x(y + 4) + y' = 0 ]</td>
<td>( y = -5 ) when ( x = 0 )</td>
</tr>
<tr>
<td>30. [ \frac{dy}{dx} = x^2(1 + y) ]</td>
<td>( y = 3 ) when ( x = 0 )</td>
</tr>
<tr>
<td>31. [ dP - 6P , dt = 0 ]</td>
<td>( P = 5 ) when ( t = 0 )</td>
</tr>
<tr>
<td>32. [ dT + k(T - 70) , dt = 0 ]</td>
<td>( T = 140 ) when ( t = 0 )</td>
</tr>
</tbody>
</table>
In Exercises 33 and 34, find an equation for the graph that passes through the point and has the specified slope. Then graph the equation.

33. Point: (-1, 1)
   Slope: \( y' = \frac{6x}{5y} \)

34. Point: (8, 2)
   Slope: \( y' = \frac{2y}{3x} \)

**Velocity** In Exercises 35 and 36, solve the differential equation to find velocity \( v \) as a function of time \( t \) if \( v = 0 \) when \( t = 0 \). The differential equation models the motion of two people on a toboggan after consideration of the force of gravity, friction, and air resistance.

35. \( 12.5 \frac{dv}{dt} = 43.2 - 1.25v \)
36. \( 12.5 \frac{dv}{dt} = 43.2 - 1.75v \)

**Chemistry: Newton's Law of Cooling** In Exercises 37–39, use Newton's Law of Cooling, which states that the rate of change in the temperature \( T \) of an object is proportional to the difference between the temperature \( T \) of the object and the temperature \( T_0 \) of the surrounding environment. This is described by the differential equation \( \frac{dT}{dt} = k(T - T_0) \).

37. A steel ingot whose temperature is 1500°F is placed in a room whose temperature is a constant 90°F. One hour later, the temperature of the ingot is 1120°F. What is the ingot's temperature 5 hours after it is placed in the room?

38. A room is kept at a constant temperature of 70°F. An object placed in the room cools from 350°F to 150°F in 45 minutes. How long will it take for the object to cool to a temperature of 80°F?

39. Food at a temperature of 70°F is placed in a freezer that is set at 0°F. After 1 hour, the temperature of the food is 48°F. (a) Find the temperature of the food after it has been in the freezer 6 hours. (b) How long will it take the food to cool to a temperature of 10°F?

**Biology: Cell Growth** The rate of growth of a spherical cell with volume \( V \) is proportional to its surface area \( S \). For a sphere, the surface area and volume are related by \( S = 4\pi r^2 \) and \( V = \frac{4}{3}\pi r^3 \). So, a model for the cell's growth is

\[
\frac{dV}{dt} = kV^{2/3}.
\]

Solve this differential equation.

**APPENDIX C.2 Separation of Variables**

41. **Learning Theory** The management of a factory has found that a worker can produce at most 30 units per day. The number of units \( N \) per day produced by a new employee will increase at a rate proportional to the difference between 30 and \( N \). This is described by the differential equation

\[
\frac{dN}{dt} = k(30 - N)
\]

where \( t \) is the time in days. Solve this differential equation.

42. **Sales** The rate of increase in sales \( S \) (in thousands of units) of a product is proportional to the current level of sales and inversely proportional to the square of the time \( t \). This is described by the differential equation

\[
\frac{dS}{dt} = \frac{ks}{t^2}
\]

where \( t \) is the time in years. The saturation point for the market is 50,000 units. That is, the limit of \( S \) as \( t \to \infty \) is 50. After 1 year, 10,000 units have been sold. Find \( S \) as a function of the time \( t \).

43. **Economics: Pareto's Law** According to the economist Vilfredo Pareto (1848–1923), the rate of decrease of the number of people \( y \) in a stable economy having an income of at least \( x \) dollars is directly proportional to the number of such people and inversely proportional to their income \( x \). This is modeled by the differential equation

\[
\frac{dy}{dx} = -\frac{k}{x^2}
\]

Solve this differential equation.

44. **Economics: Pareto's Law** In 2001, 15.2 million people in the United States earned more than $75,000 and 90.9 million people earned more than $25,000 (see figure). Assume that Pareto's Law holds and use the result of Exercise 43 to determine the number of people (in millions) who earned (a) more than $20,000 and (b) more than $100,000. (Source: U.S. Census Bureau)
C.3 FIRST-ORDER LINEAR DIFFERENTIAL EQUATIONS

Solve first-order linear differential equations. • Use first-order linear differential equations to model and solve real-life problems.

First-Order Linear Differential Equations

Definition of a First-Order Linear Differential Equation

A first-order linear differential equation is an equation of the form
\[ y' + P(x)y = Q(x) \]
where \( P \) and \( Q \) are functions of \( x \). An equation that is written in this form is said to be in standard form.

To solve a linear differential equation, write it in standard form to identify the functions \( P(x) \) and \( Q(x) \). Then integrate \( P(x) \) and form the expression
\[ u(x) = e^{\int P(x) \, dx} \]
which is called an integrating factor. The general solution of the equation is
\[ y = \frac{1}{u(x)} \int Q(x)u(x) \, dx. \]
General solution

EXAMPLE 1 Solving a Linear Differential Equation

Find the general solution of
\[ y' + y = e^x. \]
SOLUTION For this equation, \( P(x) = 1 \) and \( Q(x) = e^x \). So, the integrating factor is
\[ u(x) = e^{\int 1 \, dx} \]
\[ = e^x. \]
Integrating factor

This implies that the general solution is
\[ y = \frac{1}{e^x} \int e^x e^x \, dx \]
\[ = e^{-x} \left( \frac{1}{2} e^{2x} + C \right) \]
\[ = \frac{1}{2} e^x + Ce^{-x}. \]
General solution

In Example 1, the differential equation was given in standard form. For equations that are not written in standard form, you should first convert to standard form so that you can identify the functions \( P(x) \) and \( Q(x) \).
EXAMPLE 2 Solving a Linear Differential Equation

Find the general solution of
\[ xy' - 2y = x^2. \]
Assume \( x > 0 \).

SOLUTION Begin by writing the equation in standard form.
\[ y' - \left( \frac{2}{x} \right) y = x \]
In this form, you can see that \( P(x) = -\frac{2}{x} \) and \( Q(x) = x \). So,
\[ \int P(x) \, dx = -\int \frac{2}{x} \, dx \]
\[ = -2 \ln x \]
\[ = -\ln x^2 \]
which implies that the integrating factor is
\[ u(x) = e^{\int P(x) \, dx} \]
\[ = e^{-\ln x^2} \]
\[ = \frac{1}{x^2} \]
Integrating factor
This implies that the general solution is
\[ y = \frac{1}{u(x)} \int Q(x)u(x) \, dx \]
\[ = \frac{1}{x^2} \int \frac{x}{x^2} \, dx \]
\[ = \int \frac{1}{x} \, dx \]
\[ = \ln x + C. \]
General solution

Guidelines for Solving a Linear Differential Equation

1. Write the equation in standard form.
\[ y' + P(x)y = Q(x). \]
2. Find the integrating factor
\[ u(x) = e^{\int P(x) \, dx}. \]
3. Evaluate the integral below to find the general solution.
\[ y = \frac{1}{u(x)} \int Q(x)u(x) \, dx. \]
Application

**EXAMPLE 3** Finding a Balance

You are setting up a "continuous annuity" trust fund. For 20 years, money is continuously transferred from your checking account to the trust fund at the rate of $1000 per year (about $2.74 per day). The account earns 8% interest, compounded continuously. What is the balance in the account after 20 years?

**SOLUTION** Let $A$ represent the balance after $t$ years. The balance increases in two ways: with interest and with additional deposits. The rate at which the balance is changing can be modeled by

$$\frac{dA}{dt} = 0.08A + 1000.$$  

In standard form, this linear differential equation is

$$\frac{dA}{dt} - 0.08A = 1000$$  

which implies that $P(t) = -0.08$ and $Q(t) = 1000$. The general solution is

$$A = -12,500 + Ce^{0.08t}.$$  

Because $A = 0$ when $t = 0$, you can determine that $C = 12,500$. So, the revenue after 20 years is

$$A = -12,500 + 12,500e^{0.08(20)} 
= -12,500 + 61,912.91 
= 49,412.91.$$  

**TAKE ANOTHER LOOK**

Why an Integrating Factor Works

When both sides of the first-order linear differential equation

$$y' + P(x)y = Q(x)$$

are multiplied by the integrating factor $e^{\int P(x)\,dx}$, you obtain

$$y'e^{\int P(x)\,dx} + P(x)e^{\int P(x)\,dx}y = Q(x)e^{\int P(x)\,dx}.$$  

Show that the left side is the derivative of $ye^{\int P(x)\,dx}$, which implies that the general solution is given by

$$ye^{\int P(x)\,dx} = \int Q(x)e^{\int P(x)\,dx} \,dx.$$
In Exercises 1–4, simplify the expression.
1. $e^{-t}(e^{2t} + e^t)$
2. $e^{-t}(e^{-x} + e^{2x})$
3. $e^{x^2}$
4. $e^{2x} + e^x$

In Exercises 5–10, find the indefinite integral.
5. $\int e^{(2 + x^{-2})} \, dx$
6. $\int e^{(x-1)} (x^2 + 1) \, dx$
7. $\int \frac{1}{2x + 5} \, dx$
8. $\int \frac{x + 1}{x^2 + 2x + 3} \, dx$
9. $\int (4x - 3)^2 \, dx$
10. $\int (1 - x^2)^2 \, dx$

In Exercises 19–22, solve for $y$ in two ways.
19. $y' + y = 4$
20. $y' + 10y = 5$
21. $y' - 2xy = 2x$
22. $y' + 4xy = x$

In Exercises 23–26, match the differential equation with its solution.

<table>
<thead>
<tr>
<th>Differential Equation</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y' - 2x = 0$</td>
<td>(a) $y = Ce^x$</td>
</tr>
<tr>
<td>$y' - 2y = 0$</td>
<td>(b) $y = -\frac{1}{2} + Ce^x$</td>
</tr>
<tr>
<td>$y' - 2xy = 0$</td>
<td>(c) $y = x^2 + C$</td>
</tr>
<tr>
<td>$y' - 2y = x$</td>
<td>(d) $y = Ce^{2x}$</td>
</tr>
</tbody>
</table>

In Exercises 27–34, find the particular solution that satisfies the initial condition.

<table>
<thead>
<tr>
<th>Differential Equation</th>
<th>Initial Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y' + y = 6e^x$</td>
<td>$y = 3$ when $x = 0$</td>
</tr>
<tr>
<td>$y' + 2y = e^{-2x}$</td>
<td>$y = 4$ when $x = 1$</td>
</tr>
<tr>
<td>$xy' + y = 0$</td>
<td>$y = 2$ when $x = 2$</td>
</tr>
<tr>
<td>$y' + y = x$</td>
<td>$y = 4$ when $x = 0$</td>
</tr>
<tr>
<td>$y' + 3x^2y = 3x^2$</td>
<td>$y = 6$ when $x = 0$</td>
</tr>
<tr>
<td>$y' + (2x - 1)y = 0$</td>
<td>$y = 2$ when $x = 1$</td>
</tr>
<tr>
<td>$xy' - 2y = -x^2$</td>
<td>$y = 5$ when $x = 1$</td>
</tr>
<tr>
<td>$x^2y' - 4xy = 10$</td>
<td>$y = 10$ when $x = 1$</td>
</tr>
</tbody>
</table>
35. **Sales** The rate of change (in thousands of units) in sales $S$ is modeled by

\[
\frac{dS}{dt} = 0.2(100 - S) + 0.2t
\]

where $t$ is the time in years. Solve this differential equation and use the result to complete the table.

<table>
<thead>
<tr>
<th>$t$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

36. **Sales** The rate of change in sales $S$ is modeled by

\[
\frac{dS}{dt} = k_s(L - S) + k_t t
\]

where $t$ is the time in years and $S = 0$ when $t = 0$. Solve this differential equation for $S$ as a function of $t$.

**Elasticity of Demand** In Exercises 37 and 38, find the demand function $p = f(x)$. Recall from Section 3.5 that the price elasticity of demand was defined as $\eta = (p/x)/(dp/dx)$.

37. $\eta = 1 - \frac{400}{3x}$, \( p = 340 \) when $x = 20$
38. $\eta = 1 - \frac{500}{3x}$, \( p = 2 \) when $x = 100$

**Supply and Demand** In Exercises 39 and 40, use the demand and supply functions to find the price $p$ as a function of time $t$. Begin by setting $D(t)$ equal to $S(t)$ and solving the resulting differential equation. Find the general solution, and then use the initial condition to find the particular solution.

39. $D(t) = 480 + 5p(t) - 2p'(t)$  Demand function
   $S(t) = 300 + 8p(t) + p'(t)$  Supply function
   $p(0) = 75.00$  Initial condition
40. $D(t) = 4000 + 5p(t) - 4p'(t)$  Demand function
   $S(t) = 2800 + 7p(t) + 2p'(t)$  Supply function
   $p(0) = 1000.00$  Initial condition

41. **Investment** A brokerage firm opens a new real estate investment plan for which the earnings are equivalent to continuous compounding at the rate of $r$. The firm estimates that deposits from investors will create a net cash flow of $Pt$ dollars, where $t$ is the time in years. The rate of change in the total investment $A$ is modeled by

\[
\frac{dA}{dt} = rA + Pt.
\]

(a) Solve the differential equation and find the total investment $A$ as a function of $t$. Assume that $A = 0$ when $t = 0$.

(b) Find the total investment $A$ after 10 years given that $P = 500,000$ and $r = 9\%$.

42. **Investment** Let $A(t)$ be the amount in a fund earning interest at the annual rate of $r$, compounded continuously. If a continuous cash flow of $P$ dollars per year is withdrawn from the fund, then the rate of decrease of $A$ is given by the differential equation

\[
\frac{dA}{dt} = rA - P
\]

where $A = A_0$ when $t = 0$.

(a) Solve this equation for $A$ as a function of $t$.

(b) Use the result of part (a) to find $A$ when $A_0 = 2,000,000$, $r = 7\%$, $P = 250,000$, and $t = 5$ years.

(c) Find $A_0$ if a retired person wants a continuous cash flow of $40,000$ per year for 20 years. Assume that the person's investment will earn $8\%$, compounded continuously.

43. **Velocity** A booster rocket carrying an observation satellite is launched into space. The rocket and satellite have mass $m$ and are subject to air resistance proportional to the velocity $v$ at any time $t$. A differential equation that models the velocity of the rocket and satellite is

\[
m\frac{dv}{dt} = -mg - kv
\]

where $g$ is the acceleration due to gravity. Solve the differential equation for $v$ as a function of $t$.

44. **Health** An infectious disease spreads through a large population according to the model

\[
\frac{dy}{dt} = 1 - y
\]

where $y$ is the percent of the population exposed to the disease, and $t$ is the time in years.

(a) Solve this differential equation, assuming $y(0) = 0$.

(b) Find the number of years it takes for half of the population to have been exposed to the disease.

(c) Find the percentage of the population that has been exposed to the disease after 4 years.

45. **Research Project** Use your school's library, the Internet, or other reference source to find an article in a scientific or business journal that uses a differential equation to model a real-life situation. Write a short paper describing the situation. If possible, describe the solution of the differential equation.
C.4 APPLICATIONS OF DIFFERENTIAL EQUATIONS

Use differential equations to model and solve real-life problems.

EXAMPLE 1 Modeling Advertising Awareness

The new cereal product from Example 3 in Section C.1 is introduced through an advertising campaign to a population of 1 million potential customers. The rate at which the population hears about the product is assumed to be proportional to the number of people who are not yet aware of the product. By the end of 1 year, half of the population has heard of the product. How many will have heard of it by the end of 2 years?

SOLUTION Let $y$ be the number (in millions) of people at time $t$ who have heard of the product. This means that $(1 - y)$ is the number of people who have not heard of it, and $\frac{dy}{dt}$ is the rate at which the population hears about the product. From the given assumption, you can write the differential equation as shown:

$$\frac{dy}{dt} = k(1 - y)$$

Using separation of variables or a symbolic integration utility, you can find the general solution to be

$$y = 1 - Ce^{-kt}.$$  

To solve for the constants $C$ and $k$, use the initial conditions. That is, because $y = 0$ when $t = 0$, you can determine that $C = 1$. Similarly, because $y = 0.5$ when $t = 1$, it follows that $0.5 = 1 - e^{-k}$, which implies that

$$k = \ln 2 \approx 0.693.$$  

So, the particular solution is

$$y = 1 - e^{-0.693t}.$$  

This model is shown graphically in Figure A.13. Using the model, you can determine that the number of people who have heard of the product after 2 years is

$$1 - e^{-0.693(2)} \approx 0.75 \text{ or } 750,000 \text{ people.}$$

FIGURE A.13
**EXAMPLE 2** Modeling a Chemical Reaction

During a chemical reaction, substance A is converted into substance B at a rate that is proportional to the square of the amount of A. When \( t = 0 \), 60 grams of A is present, and after 1 hour \( (t = 1) \), only 10 grams of A remains unconverted. How much of A is present after 2 hours?

**SOLUTION** Let \( y \) be the unconverted amount of substance A at any time \( t \). From the given assumption about the conversion rate, you can write the differential equation as shown.

\[
\frac{dy}{dt} = ky^2
\]

Rate of change of \( y \) is proportional to the square of \( y \).

Using separation of variables or a symbolic integration utility, you can find the general solution to be

\[
y = \frac{-1}{kt + C}
\]

To solve for the constants \( C \) and \( k \), use the initial conditions. That is, because \( y = 60 \) when \( t = 0 \), you can determine that \( C = -\frac{1}{60} \). Similarly, because \( y = 10 \) when \( t = 1 \), it follows that

\[
10 = \frac{-1}{k - (1/60)}
\]

which implies that \( k = -\frac{1}{12} \). So, the particular solution is

\[
y = \frac{-1}{(-1/12)t - (1/60)}
\]

Using the model, you can determine that the unconverted amount of substance A after 2 hours is

\[
y = \frac{60}{5(2) + 1} = 5.45 \text{ grams.}
\]

In Figure A.14, note that the chemical conversion is occurring rapidly during the first hour. Then, as more and more of substance A is converted, the conversion rate slows down.

**STUDY TIP**

In Example 2, the rate of conversion was assumed to be proportional to the square of the unconverted amount. How would the result change if the rate of conversion were assumed to be proportional to the unconverted amount?
Earlier in the text, you studied two models for population growth: exponential growth, which assumes that the rate of change of $y$ is proportional to $y$, and logistic growth, which assumes that the rate of change of $y$ is proportional to $y$ and $L - y$, where $L$ is the population limit.

The next example describes a third type of growth model called a Gompertz growth model. This model assumes that the rate of change of $y$ is proportional to the natural log of $L/y$, where $L$ is the population limit.

**Example 3** Modeling Population Growth

A population of 20 wolves has been introduced into a national park. The forest service estimates that the maximum population the park can sustain is 200 wolves. After 3 years, the population is estimated to be 40 wolves. If the population follows a Gompertz growth model, how many wolves will there be 10 years after their introduction?

**Solution** Let $y$ be the number of wolves at any time $t$. From the given assumption about the rate of growth of the population, you can write the differential equation as shown.

\[
\frac{dy}{dt} = ky \ln \left( \frac{200}{y} \right)
\]

where $k$ is the rate of change of $y$ is proportional to the product of $y$ and the log of the ratio of 200 and $y$.

Using separation of variables or a symbolic integration utility, you can find the general solution to be

\[
y = 200e^{-Ce^{-kt}}.
\]

To solve for the constants $C$ and $k$, use the initial conditions. That is, because $y = 20$ when $t = 0$, you can determine that

\[
C = \ln 10 = 2.3026.
\]

Similarly, because $y = 40$ when $t = 3$, it follows that

\[
k = \ln \left( \frac{40}{20} \right) = 0.1194.
\]

Using the model, you can estimate the wolf population after 10 years to be

\[
y = 200e^{-2.3026e^{-0.1194(10)}} = 100\text{ wolves}.
\]

During the second half of the twentieth century, wolves disappeared from most of the middle and northern areas of the United States. Recently, however, wolf populations have been reappearing in several northern national parks.
In genetics, a commonly used hybrid selection model is based on the differential equation
\[
\frac{dy}{dt} = ky(1 - y)(a - by).
\]

In this model, \( y \) represents the portion of the population that has a certain characteristic and \( t \) represents the time (measured in generations). The numbers \( a, b, \) and \( k \) are constants that depend on the genetic characteristic that is being studied.

**EXAMPLE 4** Modeling Hybrid Selection

You are studying a population of beetles to determine how quickly characteristic D will pass from one generation to the next. At the beginning of your study \((t = 0)\), you find that half the population has characteristic D. After four generations \((t = 4)\), you find that 80% of the population has characteristic D. Use the hybrid selection model above with \( a = 2 \) and \( b = 1 \) to find the percent of the population that will have characteristic D after 10 generations.

**SOLUTION** Using \( a = 2 \) and \( b = 1 \), the differential equation for the hybrid selection model is
\[
\frac{dy}{dt} = ky(1 - y)(2 - y).
\]

Using separation of variables or a symbolic integration utility, you can find the general solution to be
\[
y(2 - y) \over (1 - y)^2 = Ce^{2kt}.
\]

To solve for the constants \( C \) and \( k \), use the initial conditions. That is, because \( y = 0.5 \) when \( t = 0 \), you can determine that \( C = 3 \). Similarly, because \( y = 0.8 \) when \( t = 4 \), it follows that
\[
\frac{0.8(1.2)}{(0.2)^2} = 3e^{8k}
\]

which implies that
\[
k = \frac{1}{8} \ln 8 \approx 0.2599.
\]

So, the particular solution is
\[
y(2 - y) \over (1 - y)^2 = 3e^{0.5199t}.
\]

Using the model, you can estimate the percent of the population that will have characteristic D after 10 generations to be given by
\[
y(2 - y) \over (1 - y)^2 = 3e^{0.5199(10)}.
\]

Using a symbolic algebra utility, you can solve this equation for \( y \) to obtain \( y \approx 0.96 \). The graph of the model is shown in Figure A.16.

---

**FIGURE A.16**

- **Hybrid Selection**
  - **Percent of population**
  - **Time (in generations)**
  - **Rate of change of y**
  - **Sketch the curve**
  - **Explain**

**EXAMP**

- **Tank containing 2 gal of solution at 10%**
- **Rate of change of 4 gallons per second**
- **Sketch the graph**
- **Explain**

**SOLUTION**

- **The percent at 4 g**
- **is**
  - **because 4 gallons is**
EXAMPLE 5 Modeling a Chemical Mixture

A tank contains 40 gallons of a solution composed of 90% water and 10% alcohol. A second solution containing half water and half alcohol is added to the tank at the rate of 4 gallons per minute. At the same time, the tank is being drained at the rate of 4 gallons per minute, as shown in Figure A.17. Assuming that the solution is stirred constantly, how much alcohol will be in the tank after 10 minutes?

SOLUTION Let \( y \) be the number of gallons of alcohol in the tank at any time \( t \). The percent of alcohol in the 40-gallon tank at any time is \( \frac{y}{40} \). Moreover, because 4 gallons of solution is being drained each minute, the rate of change of alcohol is

\[
\frac{dy}{dt} = -4 \left( \frac{y}{40} \right) + 2
\]

where 2 represents the number of gallons of alcohol entering each minute in the 50% solution. In standard form, this linear differential equation is

\[
y' + \frac{1}{10} y = 2.
\]

Using an integrating factor or a symbolic integration utility, you can find the general solution to be

\[
y = 20 + Ce^{-t/10}.
\]

Because \( y = 4 \) when \( t = 0 \), you can conclude that \( C = -16 \). So, the particular solution is

\[
y = 20 - 16e^{-t/10}.
\]

Using this model, you can determine that the amount of alcohol in the tank when \( t = 10 \) is

\[
y = 20 - 16e^{-(10)/10}
\]

\( \approx 14.1 \) gallons.

TAKE ANOTHER LOOK

Chemical Mixture

Sketch the particular solution obtained in Example 5. Describe the rate of change of the amount of alcohol in the tank. Does the amount approach 0 as \( t \) increases? Explain your reasoning.
In Exercises 1–4, use separation of variables to find the general solution of the differential equation.
1. \( \frac{dy}{dx} = 3x \)
2. \( 2y \frac{dy}{dx} = 3 \)
3. \( 3y \frac{dy}{dx} = 2xy \)
4. \( \frac{dy}{dx} = \frac{x - 4}{4y^3} \)

In Exercises 5–8, use an integrating factor to solve the first-order linear differential equation.
5. \( y' + 2y = 4 \)
6. \( y' + 2y = e^{-2x} \)
7. \( y' + xy = x \)
8. \( xy' + 2y = x^2 \)

In Exercises 9 and 10, write the equation that models the statement.
9. The rate of change of \( y \) with respect to \( x \) is proportional to the square of \( x \).
10. The rate of change of \( x \) with respect to \( t \) is proportional to the difference of \( x \) and \( t \).

In Exercises 11–14, the rate of change of \( y \) is proportional to the product of \( y \) and the difference of \( L \) and \( y \). Solve the resulting differential equation \( \frac{dy}{dx} = ky(L - y) \) and find the particular solution that passes through the points.
11. \( L = 20; (0, 1), (5, 10) \)
12. \( L = 100; (0, 10), (5, 30) \)
13. \( L = 1000; (0, 250), (25, 2000) \)
14. \( L = 1000; (0, 100), (4, 750) \)

9. **Sales Growth** The rate of change in sales \( S \) (in thousands of units) of a new product is proportional to the difference between \( L \) and \( S \) (in thousands of units) at any time \( t \). When \( t = 0 \), \( S = 0 \). Write and solve the differential equation for this sales model.

10. **Sales Growth** Use the result of Exercise 9 to write \( S \) as a function of \( t \) if (a) \( L = 100 \), \( S = 25 \) when \( t = 2 \), and (b) \( L = 500 \), \( S = 50 \) when \( t = 1 \).

In Exercises 11–14, the rate of change of \( y \) is proportional to the product of \( y \) and the difference of \( L \) and \( y \). Solve the resulting differential equation \( \frac{dy}{dx} = ky(L - y) \) and find the particular solution that passes through the points for the given value of \( L 

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In Exercises 11–14, the rate of change of \( y \) is proportional to the product of \( y \) and the difference of \( L \) and \( y \). Solve the resulting differential equation \( \frac{dy}{dx} = ky(L - y) \) and find the particular solution that passes through the points for the given value of \( L 

11. \( L = 20; (0, 1), (5, 10) \)
12. \( L = 100; (0, 10), (5, 30) \)
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In Exercises 11–14, the rate of change of \( y \) is proportional to the product of \( y \) and the difference of \( L \) and \( y \). Solve the resulting differential equation \( \frac{dy}{dx} = ky(L - y) \) and find the particular solution that passes through the points for the given value of \( L 

11. \( L = 20; (0, 1), (5, 10) \)
12. \( L = 100; (0, 10), (5, 30) \)
13. \( L = 1000; (0, 250), (25, 2000) \)
14. \( L = 1000; (0, 100), (4, 750) \)
15. Biology At any time $t$, the rate of growth of the population $N$ of deer in a state park is proportional to the product of $N$ and $L - N$, where $L = 500$ is the maximum number of deer the park can maintain. When $t = 0$, $N = 100$, and when $t = 4$, $N = 200$. Write $N$ as a function of $t$.

16. Sales Growth The rate of change in sales $S$ (in thousands of units) of a new product is proportional to the product of $S$ and $L - S$, $L$ (in thousands of units) is the estimated maximum level of sales, and $S = 10$ when $t = 0$. Write and solve the differential equation for this sales model.

Biology In Exercises 25 and 26, use the hybrid selection model in Example 4 to find the percent of the population that has the indicated characteristic.

25. You are studying a population of mayflies to determine how quickly characteristic A will pass from one generation to the next. At the start of the study, half the population has characteristic A. After four generations, 75% of the population has characteristic A. Find the percent of the population that will have characteristic A after 10 generations. (Assume $a = 2$ and $b = 1$.)

26. A research team is studying a population of snails to determine how quickly characteristic B will pass from one generation to the next. At the start of the study, 40% of the snails have characteristic B. After five generations, 80% of the population has characteristic B. Find the percent of the population that will have characteristic B after eight generations. (Assume $a = 2$ and $b = 1$.)

17. Write and solve the differential equation for this learning theory model.

18. Use the solution of Exercise 17 to write $P$ as a function of $n$, and then use a graphing utility to graph the solution.

(a) $L = 1.00$
$P = 0.50$ when $n = 0$
$P = 0.85$ when $n = 4$
(b) $L = 0.80$
$P = 0.25$ when $n = 0$
$P = 0.60$ when $n = 10$

19. $y = 45$ grams when $t = 0$; $y = 4$ grams when $t = 2$
20. $y = 75$ grams when $t = 0$; $y = 12$ grams when $t = 1$

In Exercises 21 and 22, use the Gompertz growth model described in Example 3 to find the growth function, and sketch its graph.

21. $L = 500$; $y = 100$ when $t = 0$; $y = 150$ when $t = 2$
22. $L = 5000$; $y = 500$ when $t = 0$; $y = 625$ when $t = 1$

23. Biology A population of eight beavers has been introduced into a new wetlands area. Biologists estimate that the maximum population the wetlands can sustain is 60 beavers. After 3 years, the population is 15 beavers. If the population follows a Gompertz growth model, how many beavers will be in the wetlands after 10 years?

24. Biology A population of 30 rabbits has been introduced into a new region. It is estimated that the maximum population the region can sustain is 400 rabbits. After 1 year, the population is estimated to be 90 rabbits. If the population follows a Gompertz growth model, how many rabbits will be present after 3 years?

Biology In Exercises 25 and 26, use the hybrid selection model in Example 4 to find the percent of the population that has the indicated characteristic.

26. You are studying a population of mayflies to determine how quickly characteristic A will pass from one generation to the next. At the start of the study, half the population has characteristic A. After four generations, 75% of the population has characteristic A. Find the percent of the population that will have characteristic A after 10 generations. (Assume $a = 2$ and $b = 1$.)

27. Chemical Reaction In a chemical reaction, a compound changes into another compound at a rate proportional to the unchanged amount, according to the model $\frac{dy}{dt} = ky$.

(a) Solve the differential equation.
(b) If the initial amount of the original compound is 20 grams, and the amount remaining after 1 hour is 16 grams, when will 75% of the compound have been changed?

28. Chemical Mixture A 100-gallon tank is full of a solution containing 25 pounds of a concentrate. Starting at time $t = 0$, distilled water is admitted to the tank at the rate of 5 gallons per minute, and the well-stirred solution is withdrawn at the same rate.

(a) Find the amount $Q$ of the concentrate in the solution as a function of $t$. (Hint: $Q' + Q/20 = 0$)
(b) Find the time when the amount of concentrate in the tank reaches 15 pounds.

29. Chemical Mixture A 200-gallon tank is half full of distilled water. At time $t = 0$, a solution containing 0.5 pound of concentrate per gallon enters the tank at the rate of 5 gallons per minute, and the well-stirred mixture is withdrawn at the same rate.

(a) Find the amount $Q$ of concentrate in the tank after 30 minutes. (Hint: $Q' + Q/20 = \frac{1}{2}$)
(b) Find the time when the amount of concentrate in the tank reaches 15 pounds.

30. Safety Assume that the rate of change in the number of miles $s$ of road cleared per hour by a snowplow is inversely proportional to the depth $h$ of snow. That is,

$$\frac{ds}{dh} = \frac{k}{h}$$

Find $s$ as a function of $h$ if $s = 25$ miles when $h = 2$ inches and $s = 12$ miles when $h = 6$ inches ($2 \leq h \leq 15$).
31. **Chemistry** A wet towel hung from a clothesline to dry loses moisture through evaporation at a rate proportional to its moisture content. If after 1 hour the towel has lost 40% of its original moisture content, after how long will it have lost 80%?

32. **Biology** Let \( x \) and \( y \) be the sizes of two internal organs of a particular mammal at time \( t \). Empirical data indicate that the relative growth rates of these two organs are equal, and can be modeled by

\[
\frac{1}{x} \frac{dx}{dt} = \frac{1}{y} \frac{dy}{dt}.
\]

Use this differential equation to write \( y \) as a function of \( x \).

33. **Population Growth** When predicting population growth, demographers must consider birth and death rates as well as the net change caused by the difference between the rates of immigration and emigration. Let \( P \) be the population at time \( t \) and let \( N \) be the net increase per unit time due to the difference between immigration and emigration. So, the rate of growth of the population is given by

\[
\frac{dP}{dt} = kP + N, \quad N \text{ is constant}.
\]

Solve this differential equation to find \( P \) as a function of time.

34. **Meteorology** The barometric pressure \( y \) (in inches of mercury) at an altitude of \( x \) miles above sea level decreases at a rate proportional to the current pressure according to the model

\[
\frac{dy}{dx} = -0.2y
\]

where \( y = 29.92 \) inches when \( x = 0 \). Find the barometric pressure (a) at the top of Mt. St. Helens (8364 feet) and (b) at the top of Mt. McKinley (20,320 feet).

35. **Investment** A large corporation starts at time \( t = 0 \) to invest part of its receipts at a rate of \( P \) dollars per year in a fund for future corporate expansion. Assume that the fund earns \( r \) percent interest per year compounded continuously. So, the rate of growth of the amount \( A \) in the fund is given by

\[
\frac{dA}{dt} = rA + P
\]

where \( A = 0 \) when \( t = 0 \). Solve this differential equation for \( A \) as a function of \( t \).

**Investment** In Exercises 36–38, use the result of Exercise 35.

36. Find \( A \) for each situation.
   (a) \( P = $100,000, \quad r = 12\% \), and \( t = 5 \) years
   (b) \( P = $250,000, \quad r = 15\% \), and \( t = 10 \) years

37. Find \( P \) if the corporation needs \$120,000,000 in 8 years and the fund earns 16\% interest compounded continuously.

38. Find \( t \) if the corporation needs \$800,000 and it can invest \$75,000 per year in a fund earning 13\% interest compounded continuously.

**Medical Science** In Exercises 39–41, a medical researcher wants to determine the concentration \( C \) (in moles per liter) of a tracer drug injected into a moving fluid. Solve this problem by considering a single-compartment dilution model (see figure). Assume that the fluid is continuously mixed and that the volume of fluid in the compartment is constant.

**Figure for 39–41**

![Figure](image_url)

39. If the tracer is injected instantaneously at time \( t = 0 \), then the concentration of the fluid in the compartment begins diluting according to the differential equation

\[
\frac{dC}{dt} = -\frac{R}{V} C, \quad C = C_0 \text{ when } t = 0.
\]

(a) Solve this differential equation to find the concentration as a function of time.

(b) Find the limit of \( C \) as \( t \to \infty \).

40. Use the solution of the differential equation in Exercise 39 to find the concentration as a function of time, and use a graphing utility to graph the function.
   (a) \( V = 2 \text{ liters}, \quad R = 0.5 \text{ L/min}, \quad C_0 = 0.6 \text{ mol/L} \)
   (b) \( V = 2 \text{ liters}, \quad R = 1.5 \text{ L/min}, \quad C_0 = 0.6 \text{ mol/L} \)

41. In Exercises 39 and 40, it was assumed that there was a single initial injection of the tracer drug into the compartment. Now consider the case in which the tracer is continuously injected (beginning at \( t = 0 \)) at the rate of \( Q \) mol/min. Considering \( Q \) to be negligible compared with \( R \), use the differential equation

\[
\frac{dC}{dt} = \frac{Q}{V} - \frac{R}{V} C, \quad C = 0 \text{ when } t = 0.
\]

(a) Solve this differential equation to find the concentration as a function of time.

(b) Find the limit of \( C \) as \( t \to \infty \).