Symmetry classification of general rank-3 Pfaffian systems in 5-dimension.

Involutive Systems, Monge Equations & Darboux Integrability

Francesco Strazzullo

Department of Mathematics and Statistics,
Utah State University.
francesco.strazzullo@aggiemail.usu.edu

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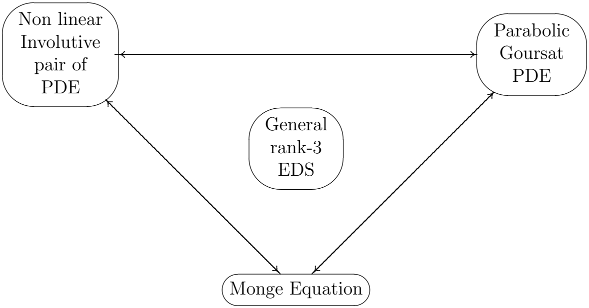
Cartan 1910 and today.



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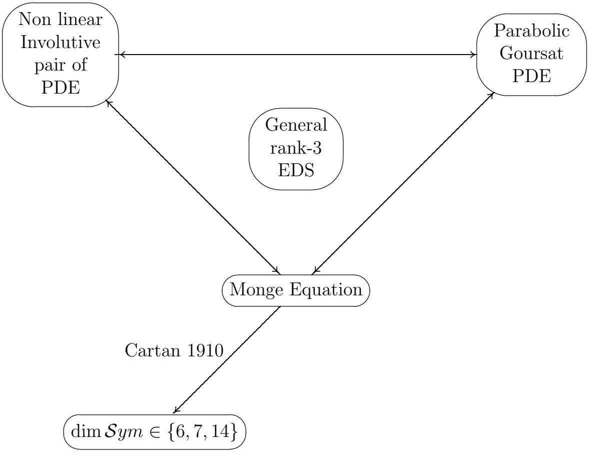

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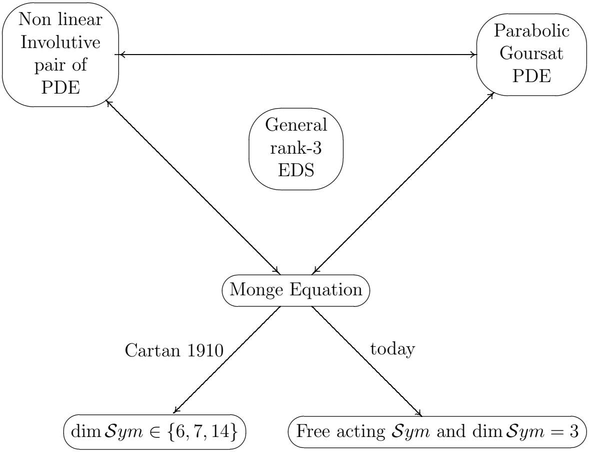

Contents

General rank-3 Pfaffian systems in 5-dimension

A rank-3 Pfaffian system I defined on a 5-manifold is called general (and denoted GR_3D_5) if $\dim I^{(1)} = 2$ and $\dim I^{(2)} = 0$.

Involutive systems

Any pair of PDE in the plane can be written as

$$r = R(x, y, z, p, q, t), \quad s = S(x, y, z, p, q, t). \tag{1}$$

The system (1) defines the rank-3 Pfaffian system on a 6-manifold $K = \{dz - p\,dx - q\,dy,\ dp - R\,dx - S\,dy,\ dq - S\,dx - t\,dx\}$.

- K is non-linear involutive $\Leftrightarrow \mathrm{Cau}(K) = \{C\}$ and $S_{tt} \neq 0$.

Theorem

Every non-linear involutive system K is reduced by C to a GR_3D_5 Pfaffian system.

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Monge equations

A second order Monge equation in one variable X and two unknown functions $Y = f(X)$ and $Z = g(X)$ is an under-detemined ODE

$$Z' = H(X, Y, Z, Y', Y''). \tag{2}$$

This defines the rank-3 Pfaffian system on a 5-manifold

$$J = \{dY - Y'\,dX,\ dY' - Y''\,dX,\ dZ - H\,dX\}.$$

Then

$$J \text{ is } GR_3D_5 \Leftrightarrow \frac{\partial^2 H}{\partial Y''^2} \neq 0$$

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Monge normal form

Theorem (2)

Every GR_3D_5 Pfaffian system is equivalent to a Monge system

$$J = \{d\,Y - Y'\,d\,X,\ d\,Y' - Y''\,d\,X,\ d\,Z - H\,d\,X\}$$

such that $\dfrac{\partial^2 H}{\partial Y''^2} \neq 0$.

Algorithm to solve non-linear involutive systems

$$r = R(x, y, z, p, q, t)\,, \quad s = S(x, y, z, p, q, t)\,. \tag{3}$$

1. Write
$$K = \{dz - p\,dx - q\,dy,\ dp - R\,dx - S\,dy,\ dq - S\,dx - t\,dx\}.$$
2. Reduce K by its C.c. to a GR_3D_5 Pfaffian system I.
3. Write I in its Monge normal form
$$J = \{dY - Y'\,dX,\ dY' - Y''\,dX,\ dZ - H\,dX\}.$$
4. The integral manifolds of (3) are expressed in terms of two independent variables $\hat{x}$ and X, one generic function $Y = f(X)$, its derivatives Y' and Y'', and one integral expression.

Example HC

The involutive system

$$r = \frac{1}{3}t^3, \quad s = \frac{1}{2}t^2 \tag{4}$$

is reduced by its Cauchy characteristic to the Monge equation

$$Z' = Y''^2. \tag{5}$$

The integral manifolds of (4) are

$$x = \hat{x}, \quad y = \frac{1}{2}X - 2Y''\hat{x},$$
$$z = \frac{1}{2}Y + \left(Z - 2Y'Y''\right)\hat{x} + \frac{4}{3}Y''^3\hat{x}^2,$$

where $Z = \int Y''^2 \, dX$.

Normal forms in Cartan 1910

Root type	Symmetry Algebra Γ	Monge normal form
$[\infty]$	$\dim\Gamma = 14$ and $\Gamma \simeq \mathfrak{g}_2$	$Z' = Y''^2$.
$[4]$	Γ is solvable $\dim\Gamma = 7$ ($k = \text{const}$) or $\dim\Gamma = 6$	$Z' = -\frac{1}{2}[Y''^2 + \frac{10}{3}kY'^2 + (1 + k^2 - k'')Y^2]$ where $k = k(X)$.
$[2,2]$	$\dim\Gamma = 6$	§XI, three cases: only equations (10) and (13).
$[3,1]$, $[2,1,1]$, $[1,1,1,1]$	$\dim\Gamma \leq 5$	none.

Example of Cartan's integration method.

Theorem

The integral manifolds of a non-linear involutive system

$$r = R(x, y, z, p, q, t), \quad s = S(x, y, z, p, q, t),$$

with a 7-dimensional symmetry algebra are expressed in terms of two independent variables $\hat{x}$ *and* X*, a constant* k*, one generic function* $Y = f(X)$*, its derivatives* Y' *and* Y''*, and the integral expression* $-\frac{1}{2}\int Y''^2 + \frac{10}{3}kY'^2 + (1+k^2)Y^2\, dX.$

Main Results

I determined the normal forms for all GR_3D_5 Pfaffian systems with 3-dimensional symmetry algebra, acting transversely and freely. Here in Table 1, it is displayed the list of Monge equations with prescribed symmetry Γ of algebraic type $\mathfrak{g}$.

Table 1: solvable without parameter (1/3)

$$Z' = H(X, Y, Z, Y', Y'')$$

$\mathfrak{g}$	Γ	H
$3A_1$, Abelian	$\partial_X,\ \partial_Y,\ \partial_Z.$	$h(Y', Y'')$
$A_1 \oplus A_2$	$\partial_Z,\ -X\,\partial_X + Z\,\partial_Z,\ \partial_Y.$	$\frac{1}{X^2} h\big(XY', X^2 Y''\big)$
$A_{3,1}$, Heisenberg	$\partial_Z,\ \partial_X,\ \partial_Y + X\,\partial_Z.$	$Y + h(Y', Y'')$
$A_{3,2}$	$\partial_Z,\ X\,\partial_Y + \ln X\,\partial_Z,\ -X\,\partial_X + Z\,\partial_Z.$	$\frac{1}{X^2}[Y + h(XY' - Y, X^2 Y'')]$

Table 1: solvable with parameter (2/3)

$$Z' = H(X, Y, Z, Y', Y'')$$

$\mathfrak{g}$	Γ	H
$A^{\epsilon}_{3,5}$ $\epsilon \neq 0$	$X^{1/\epsilon}\,\partial_Y,\ \partial_Z,$ $-\epsilon X\,\partial_X + \epsilon Z\,\partial_Z.$	$\frac{1}{\epsilon X^2}\,h(\epsilon XY' - Y,\ \epsilon^2 X^2 Y'' + \epsilon^2 XY' - Y)$
$A^{\epsilon}_{3,7}$ $\epsilon \geq 0$	$e^{\epsilon X}\sin X\,\partial_Y,$ $-e^{\epsilon X}\cos X\,\partial_Y,$ $-\partial_X.$	$h(Z,\ Y'' - 2\epsilon Y' + \epsilon^2 Y + Y)$

$$Z' = H(X, Y, Z, Y', Y'')$$

$\mathfrak{g}$	Γ	H
$A_{3,8,1}$	$2X\,\partial_X - 2\partial_Y$, $-X^2\,\partial_X + 2X\,\partial_Y$, ∂_X.	$e^Y\,h\left(Z, \dfrac{2Y'' - Y'^2}{2e^{2Y}}\right)$
$A_{3,8,2}$	$2X\,\partial_X - 2Z\,\partial_Z$, $(1 + 2XZ)\,\partial_Z - X^2\,\partial_X$, ∂_X.	$Z^2 + Y'^2\,h\left(Y, \dfrac{Y'' - 2Y'Z}{Y'^2}\right)$
$A_{3,9}$	$-\alpha\cos X\,(Y'\,\partial_X + \partial_Y)$, $\alpha\sin X\,(Y'\,\partial_X + \partial_Y)$, ∂_X. $\alpha = -1/\sqrt{1 - Y'^2}$	$\beta\,h\left(Z,\ Y + \arctan\dfrac{Y''}{1 - Y'^2}\right)$, $\beta = \sqrt{1 - Y'^2 + \dfrac{Y''^2}{1 - Y'^2}}$

Three Applications

- Cartan integration method on a wider class of non-linear involutive systems.
- A broad classification of Darboux integrable hyperbolic PDE in the plane.
- Many new examples of inequivalent non-linear involutive systems.

Heisenberg non-linear involutive systems.

Theorem

Let I be the reduction by the Cauchy characteristic of a non-linear involutive system

$$r = R\,(x, y, z, p, q, t)\,, \quad s = S\,(x, y, z, p, q, t)\,. \tag{6}$$

Let the Heisenberg algebra be a symmetry algebra of I, acting transversely and freely. Then the integral manifolds of (6) are expressed in terms of two independent variables $\hat{x}$ and X, one generic function $Y = f(X)$, its derivatives Y' and Y'', a generic function $h(Y', Y'')$ and the integral expression $\int Y + h(Y', Y'')\, d\,X$.

Example: root type [1,1,1,1], Heisenberg.

The involutive system

$$r = -\frac{1}{4x}(2p + 2q + y\sin 2t - 2yt),\ s = \sqrt{\frac{y}{x}}\cos t, \tag{7}$$

has a 5-dimensional symmetry algebra, containing the Heisenberg algebra. Setting $A = 1/\sqrt[4]{96}$, the integral manifolds of (7) are

$$\begin{aligned} x &= A\hat{x}^2, \quad y = A(\hat{x} - X)^2 + 6AY''^2, \\ z &= \frac{5}{4}(\hat{x} - X)\,Y''^3 - 3Y''^2 Y' + \frac{1}{8}(\hat{x} - X)^3\,Y'' + (\hat{x} - X)\,Y + Z \\ &\quad + \frac{1}{8\sqrt{6}}\left[6Y''^2 + (\hat{x} - X)^2\right]^2 \arctan\left(\frac{\hat{x} - X}{6Y''}\right), \end{aligned}$$

where $Z = \int Y + Y''^3\, dX$.

Theorem (Heisenberg Darboux Integrability)

Let

$$F(x, y, z, p, q, r, s, t) = 0 \tag{8}$$

be hyperbolic D.I. with Vessiot algebra the Heisenberg algebra. Then (8) *is the quotient of two copies of*

$$Z' = Y + h(Y', Y'')$$

by the diagonal action of

$$\Gamma = \{\partial_Z, \partial_X, \partial_Y + X\,\partial_Z\}.$$

Conversely, every such quotient gives rise to a D.I. hyperbolic PDE (8) *with Heisenberg Vessiot algebra.*

New examples: A Riccati equation with Solvable symmetry

The symmetry algebra of the Riccati equation

$$Z' = Z^2 + (Y + Y'')^2 \tag{9}$$

is

$$\Gamma = \{\sin X\,\partial_Y, -\cos X\,\partial_Y, -\partial_X\}\,.$$

This is solvable, of algebraic type $A^{\epsilon}_{3,7}$ with $\epsilon = 0$.
The Cartan 2-tensor of (9) has root type $[2, 1, 1]$.

Thanks!