Singular minimization problems

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1 Introduction

This paper is concerned with the existence of ground state solutions of degenerate elliptic equations on R^N of the following form :

$$-\mathrm{div}\,(A(x)\nabla u) = f(x,u), \quad x \in R^N.$$

where A is a nonnegative function that may have zeroes at some points and may be unbounded. Equations of this type comes from the consideration of standing waves in anisotropic Schrödinger equations. Two model equations we consider here are the following

$$-\mathrm{div}\,(|x|^{-2a}\nabla u) = |x|^{-bp}|u|^{p-2}u, \quad x \in R^N,$$

and

$$-\mathrm{div}\,(|x|^{-2a}\nabla u) + \lambda|x|^{-2(1+a)}u = |x|^{-bp}|u|^{p-2}u, \quad x \in R^N.$$

We are seeking solutions in $\mathcal{D}_a^{1,2}(R^N)$ which is defined as the completion of $\mathcal{D}(R^N)$ with respect to the inner product

$$(u,v) := \int_{R^N} |x|^{-2a}\nabla u \cdot \nabla v dx.$$

We shall consider the case $N \geq 3$, $0 \leq a \leq \frac{N-2}{2}$, $a \leq b \leq a+1$ and

$$p = p(a,b) := \frac{2N}{N-2+2(b-a)}.$$

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Moreover λ is a real parameter.

Using an inequality established by Caffarelli, Kohn and Nirenberg [4], these problems allow variational formulations for the above parameter range, and especially we can formulate the following minimization problems with constraints. Consider

$$S(a,b) := \inf_{\substack{u\in\mathcal{D}_a^{1,2}(R^N)\\ ||x|^{-b}u|_p=1}} ||x|^{-a}\nabla u|_2^2$$

and

$$S(a,b,\lambda) := \inf_{\substack{u\in\mathcal{D}_a^{1,2}(R^N)\\ ||x|^{-b}u|_p=1}} ||x|^{-a}\nabla u|_2^2 + \lambda||x|^{-(1+a)}u|_2^2.$$

Then the inequality by Caffarelli, Kohn and Nirenberg asserts that $S(a,b)$ is positive under the above assumptions. This inequality has the Sobolev inequality and the Hardy inequality embedded as particular cases. We are interested in the compactness of minimizing sequences of these problems.

The existence of a minimizer was proved by Lieb ([8]) when $a = 0$, $0 < b < 1$ and by Chou and Chu [5] when $a \le b < a+1$. On the other hand, $S(a, a+1)$ is never achieved ([5]). The particular case $S(0,1)$ corresponds to the Hardy inequality :

$$\Big||x|^{-1}u\Big|_2^2 \le S(0,1)^{-1}|\nabla u|_2^2.$$

In section 2, we solve completely the problem of compactness of minimizing sequences for $S(a,b)$. We prove that minimizing sequences are relatively compact up to a dilation when $a \le b < a+1$, $a+b > 0$. The case $a = b$ depends on a delicate estimate, since p is then equal to the critical exponent $2^* = 2N/(N-2)$ and there is a double dilation invariance in this case. The case $a = 0$, $0 < b < 1$ was solved by P.-L. Lions ([10], theorem 2.4). However our method is different, since we give a quantitative evaluation of the non-compactness of minimizing sequences. Moreover our approach depends only on the weak compactness of bounded sets of bounded measures (see e.g. [13]), so that it is applicable to many other problems (see e.g. [11], [12]).

In section 3, we prove that minimizing sequences for $S(a,b,\lambda)$ are relatively compact up to a dilation when $a \le b < a+1$ and $-S(a,a+1) < \lambda < 0$. The case $a = b = 0$ and $-S(0,1) < \lambda < 0$ was solved by P.-L. Lions([9], theorem 1.3). We consider also the case $\lambda > 0$. Unlike problem $S(a,b)$, for

$S(a,b,\lambda)$ with $\lambda \neq 0$ it is not clear whether there are explicit solutions of the problems. Our method gives a uniform approach to both problems.

Finally, we remark that a better understanding of the ground state solutions in the whole space would be useful for the existence of positive solutions of the corresponding equations in bounded domains and unbounded domains other than the whole space. Our methods should work equally well for these problems and some future work is planned in this direction.

2 Minimizing sequences for $S(a,b)$

In order to prove that $S(a,b)$ is achieved, we can consider an arbitrary minimizing sequence $(u_n) \subset \mathcal{D}_a^{1,2}(R^N)$:

$$\left||x|^{-b}u_n\right|_p = 1, \quad \left||x|^{-a}\nabla u_n\right|_2^2 \to S(a,b), \quad n \to \infty. \tag{1}$$

Going if necessary to a subsequence, we may assume $u_n \rightharpoonup u$ in $\mathcal{D}_a^{1,2}(R^N)$, so that

$$\left||x|^{-a}\nabla u\right|_2^2 \leq \lim \left||x|^{-a}\nabla u_n\right|_2^2 = S(a,b).$$

Hence u is a minimizer provided $\left||x|^{-b}u\right|_p = 1$. But we know only that $\left||x|^{-b}u\right|_p \leq 1$. Indeed for any $v \in \mathcal{D}_a^{1,2}(R^N)$ and $t > 0$ the rescaled function

$$v^t(x) := t^{\frac{N-2a-2}{2}} v(tx)$$

satisfies

$$\left||x|^{-a}\nabla v^t\right|_2 = \left||x|^{-a}\nabla v\right|_2, \quad \left||x|^{-b}v^t\right|_p = \left||x|^{-b}v\right|_p.$$

Thus the problem is invariant by dilation. In order to prove that (u_n) is relatively compact up to a dilation, we need some preliminary results.

Let $u(x) := (1+|x|^2)^{-\frac{N-2}{2}}$. It is well know (see e.g. [12]) that

$$\frac{\int_{R^N} |\nabla u|^2}{\left(\int_{R^N} |u|^{2^*}\right)^{2/2^*}} = S$$

the best Sobolev constant.

Lemma 1. *For $0 \leq a < \frac{N-2}{2}$ define*

$$g(a) := \frac{\int_{R^N} |x|^{-2a}|\nabla u|^2}{\left(\int_{R^N} |x|^{-2^*a}u^{2^*}\right)^{2/2^*}}.$$

Then $g'(a) < 0$ *for* $a \in [0, \frac{N-2}{2}[$.

First, using the following formulas for the Gamma function Γ (see e.g. [2]) :

$$\int_0^\infty (1+\gamma)^{-(y+z)} \gamma^{y-1}\, d\gamma = \Gamma(y)\Gamma(z)/\Gamma(y+z),$$
$$\Gamma(z+1) = z\Gamma(z),$$

we have with a straightforward computation

$$g(a) = \frac{(N-2)^2 w \int_0^\infty (1+\gamma^2)^N \gamma^{-2a+2+N-1}\, d\gamma}{\left(w \int_0^\infty (1+\gamma^2)^N \gamma^{-2^* a+N-1}\, d\gamma\right)^{2/2^*}}$$

$$= \frac{(N-2)^2 (\frac{w}{2})^{2/N}}{(\Gamma(N))^{2/N}} \frac{\Gamma\left(\frac{-2a+N+2}{2}\right)\Gamma\left(\frac{2a+N-2}{2}\right)}{\left[\Gamma\left(\frac{-2^* a+N}{2}\right)\Gamma\left(\frac{2^* a+N}{2}\right)\right]^{2/2^*}}$$

$$= (N-2)^2 \left(\tfrac{w}{2}\right)^{\frac{2}{N}} (\Gamma(N))^{-\frac{2}{N}} \frac{N-2a}{N-2+2a} \frac{\Gamma(\frac{N}{2}-a)\Gamma(\frac{N}{2}+a)}{\left[\Gamma(\frac{N}{2}-\frac{aN}{N-2})\Gamma(\frac{N}{2}+\frac{aN}{N-2})\right]^{\frac{N-2}{N}}}$$

where w is the volume of S^{N-1}.

Next, letting $g_1(a) := \frac{N-2a}{N-2+2a}$,

$$g_+(a) := \Gamma\left(\frac{N}{2}+a\right)\left[\Gamma(\frac{N}{2}+\frac{aN}{N-2})\right]^{-\frac{N-2}{N}}$$
$$g_-(a) := \Gamma\left(\frac{N}{2}-a\right)\left[\Gamma(\frac{N}{2}-\frac{aN}{N-2})\right]^{-\frac{N-2}{N}}$$

we claim

(i) $g_1'(a) < 0, \quad a \in [0, \frac{N-2}{2}[$;

(ii) $g_\pm'(0) = 0, \quad g_\pm'(a) < 0$, for $a \in]0, \frac{N-2}{2}[$.

Then our lemma follows from these claims.

Claim (i) is obvious. To prove claim (ii) we use the following formula for the Gamma function :

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} + \sum_{k=1}^\infty \left(\frac{1}{k} - \frac{1}{z+k}\right)$$

for $z \neq 0, -1, -2, \ldots$. Here $\gamma = -\Gamma'(1) > 0$ is a constant. This formula implies that $\frac{\Gamma'(z)}{\Gamma(z)}$ is a strictly increasing function of z in $]0, \infty[$. Now it is easy to verify the following

$$g'_+(a) = \Gamma\left(\frac{N}{2}+\frac{aN}{N-2}\right)^{-\frac{(N-2)}{N}}\Gamma\left(\frac{N}{2}+a\right)\left\{\frac{\Gamma'(\frac{N}{2}+a)}{\Gamma(\frac{N}{2}+a)} - \frac{\Gamma'(\frac{N}{2}+\frac{aN}{N-2})}{\Gamma(\frac{N}{2}+\frac{aN}{N-2})}\right\}$$

$$g'_-(a) = -\Gamma\left(\frac{N}{2}-\frac{aN}{N-2}\right)^{-\frac{(N-2)}{N}}\Gamma\left(\frac{N}{2}-a\right)\left\{\frac{\Gamma'(\frac{N}{2}-a)}{\Gamma(\frac{N}{2}-a)} - \frac{\Gamma'(\frac{N}{2}-\frac{aN}{N-2})}{\Gamma(\frac{N}{2}-\frac{aN}{N-2})}\right\}.$$

Hence $g'_\pm(0)=0$ and $g'_\pm(a)<0$ for $a\in]0,\frac{N-2}{2}[$.

Lemma 2. *Let $N\geq 3$ and $0\leq a<(N-2)/2$. If $u_n \rightharpoonup u$ in $\mathcal{D}^{1,2}_a(R^N)$ then $|x|^{-a}u_n \to |x|^{-a}u$ in $L^2_{\mathrm{loc}}(R^N)$.*

By definition, we have

$$\begin{aligned}\int_{B(0,R)}|x|^{-2a}|u|^2\,dx &\leq R^2\int_{B(0,R)}|x|^{-2(a+1)}|u|^2\,dx\\ &\leq R^2 S(a,a+1)\int_{R^N}|x|^{-2a}|\nabla u|^2\,dx.\end{aligned}$$

We can assume that $u_n \rightharpoonup 0$ in $\mathcal{D}^{1,2}_a(R^N)$. By the preceding inequality for every $\varepsilon>0$ there exists $R>0$ such that, for all $n\geq 0$,

$$\int_{B(0,R)}|x|^{-2a}|u_n|^2\,dx\leq\varepsilon.$$

The Rellich theorem implies that $|x|^{-a}u_n\to 0$ in $L^2_{\mathrm{loc}}(R^N\backslash B(0,R))$. It is thus easy to conclude.

We denote by $\mathcal{M}(R^N)$ the space of bounded measures on R^N.

Lemma 3. *Let $N\geq 3$, $0\leq a<(N-2)/2$, $a\leq b\leq 1+a$ and $p=p(a,b)$. Let $(u_n)\subset\mathcal{D}^{1,2}_a(R^N)$ be a sequence such that*

$$\begin{array}{ll} u_n \rightharpoonup u & \text{in } \mathcal{D}^{1,2}_a(R^N),\\ \left||x|^{-a}\nabla(u_n-u)\right|^2 \rightharpoonup \mu & \text{in } \mathcal{M}(R^N),\\ \left||x|^{-b}(u_n-u)\right|^p \rightharpoonup \nu & \text{in } \mathcal{M}(R^N),\\ u_n\to u & \text{a.e. on } R^N\end{array}$$

and define

$$\mu_\infty := \lim_{R\to\infty}\overline{\lim_{n\to\infty}}\int_{|x|\geq R}|x|^{-2a}|\nabla u_n|^2\,dx, \tag{2}$$

$$\nu_\infty := \lim_{R\to\infty}\overline{\lim_{n\to\infty}}\int_{|x|\geq R}|x|^{-pb}|u_n|^p\,dx. \tag{3}$$

Then it follows that

$$\|\nu\|^{2/p}\leq S(a,b)^{-1}\|\mu\|, \tag{4}$$

$$\nu_\infty^{2/p} \le S(a,b)^{-1}\mu_\infty, \tag{5}$$

$$\overline{\lim_{n\to\infty}} \left||x|^{-a}\nabla u_n\right|_2^2 = \left||x|^{-a}\nabla u\right|_2^2 + \|\mu\| + \mu_\infty, \tag{6}$$

$$\overline{\lim_{n\to\infty}} \left||x|^{-b}u_n\right|_p^p = \left||x|^{-b}u\right|_p^p + \|\nu\| + \nu_\infty. \tag{7}$$

Moreover, if $b < 1 + a$ and $u = 0$ and $\|\nu\|^{2/p} = S(a,b)^{-1}\|\mu\|$, then ν and μ are concentrated at a single point.

1) Assume first $u = 0$. Choosing $h \in \mathcal{D}(R^N)$, we have, by definition,

$$\left(\int |x|^{-pb}|hu_n|^p\, dx\right)^{2/p} \le S(a,b)^{-1}\int |x|^{-2a}|\nabla(hu_n)|^2\, dx.$$

The preceding lemma implies that

$$\left(\int |h|^p\, d\nu\right)^{2/p} \le S(a,b)^{-1}\int |h|^2\, d\mu. \tag{8}$$

Inequality (4) follows.

2) For $R > 1$, let $\psi_R \in \mathcal{C}^\infty(R^N)$ be such that $\psi_R(x) = 1$ for $|x| > R+1$, $\psi_R(x) = 0$ for $|x| < R$ and $0 \le \psi_R \le 1$ on R^N. We have also by definition

$$(\int |x|^{-pb}|\psi_R u_n|^p\, dx)^{2/p} \le S(a,b)^{-1}\int |x|^{-2a}|\nabla(\psi_R u_n)|^2\, dx.$$

The preceding lemma implies that

$$\begin{aligned} &\overline{\lim}_{n\to\infty}(\textstyle\int |x|^{-pb}|\psi_R u_n|^p\, dx)^{2/p} \\ &\quad\le S(a,b)^{-1}\,\overline{\lim}_{n\to\infty}\textstyle\int |x|^{-2a}|\psi_R\nabla u_n|^2\, dx. \end{aligned} \tag{9}$$

On the other hand, we have

$$\begin{aligned} \int_{|x|>R+1} |x|^{-2a}|\nabla u_n|^2\, dx &\le \int |x|^{-2a}\psi_R^2|\nabla u_n|^2\, dx \\ &\le \int_{|x|>R} |x|^{-2a}|\nabla u_n|^2\, dx \end{aligned}$$

and

$$\begin{aligned}\int_{|x|>R+1} |x|^{-pb}|u_n|^p\,dx &\le \int |x|^{-pb}\psi_R^p|u_n|^p\,dx \\ &\le \int_{|x|>R} |x|^{-pb}|u_n|^p\,dx.\end{aligned}$$

We obtain from (2) and (3)

$$\mu_\infty = \lim_{R\to\infty}\overline{\lim_{n\to\infty}}\int |x|^{-2a}|\nabla u_n|^2\psi_R^2\,dx,$$

$$\nu_\infty = \lim_{R\to\infty}\overline{\lim_{n\to\infty}}\int |x|^{-pb}|u_n|^p\psi_R^p\,dx,$$

Inequality (5) follows then from (9).

3) Assume moreover that $\|\nu\|^{2/p} = S(a,b)^{-1}\|\mu\|$. The Hölder inequality and (8) imply that, for $h \in \mathcal{D}(R^N)$,

$$\left(\int |h|^p\,d\nu\right)^{2/p} \le S(a,b)^{-1}\|\mu\|^{\frac{p-2}{p}}\left(\int |h|^p\,d\mu\right)^{2/p}.$$

We deduce $\nu = S(a,b)^{-p/2}\|\mu\|^{\frac{p-2}{2}}\mu$. It follows that, for $h \in \mathcal{D}(R^N)$,

$$\left(\int |h|^p\,d\nu\right)^{2/p}\|\nu\|^{\frac{p-2}{2}} \le \int |h|^2\,d\nu$$

and so, for each open set Ω,

$$(\nu(\Omega))^{2/p}\,\nu(R^N)^{\frac{p-2}{p}} \le \nu(\Omega).$$

Hence ν is concentrated at a single point.

4) Considering now the general case, we write $v_n := u_n - u$. Since

$$v_n \rightharpoonup 0 \quad \text{in} \quad \mathcal{D}_a^{1,2}(R^N),$$

we have

$$|x|^{-2a}|\nabla u_n|^2 \rightharpoonup \mu + |x|^{-2a}|\nabla u|^2 \quad \text{in} \quad \mathcal{M}(R^N),$$

According to the Brezis-Lieb lemma (see e.g. [13, Theorem 10.37] or [14, Lemma 1.32]),we have, for every nonnegative $h \in \mathcal{D}(R^N)$,

$$\begin{aligned} & \int h|x|^{-pb}|u|^p \, dx \\ = & \lim_{n\to\infty} \left(\int h|x|^{-pb}|u_n|^p \, dx - \int h|x|^{-pb}|v_n|^p \, dx \right). \end{aligned}$$

Hence we obtain

$$|x|^{-pb}|u_n|^p \rightharpoonup \nu + |x|^{-pb}|u|^p \quad \text{in} \quad \mathcal{M}(R^N).$$

Inequality (4) follows from the corresponding inequality for (v_n).

5) Since

$$\begin{aligned} & \overline{\lim}_{n\to\infty} \int_{|x|>R} |x|^{-2a}|\nabla v_n|^2 \, dx \\ = & \overline{\lim}_{n\to\infty} \int_{|x|>R} |x|^{-2a}|\nabla u_n|^2 \, dx - \int_{|x|>R} |x|^{-2a}|\nabla u| \, dx, \end{aligned}$$

we obtain

$$\lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{|x|>R} |x|^{-2a}|\nabla v_n|^2 \, dx = \mu_\infty.$$

By the Brezis-Lieb lemma, we have

$$\begin{aligned} & \int_{|x|>R} |x|^{-pb}|u|^p \, dx \\ = & \lim_{n\to\infty} \left(\int_{|x|>R} |x|^{-pb}|u_n|^p \, dx - \int_{|x|>R} |x|^{-pb}|v_n|^p \, dx \right) \end{aligned}$$

and so

$$\lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{|x|>R} |x|^{-pb}|v_n|^p \, dx = \nu_\infty.$$

Inequality (5) follows then from the corresponding inequality for (v_n).

6) For every $R > 1$, we have

$$\begin{aligned} & \overline{\lim_{n\to\infty}} \int |x|^{-2a}|\nabla u_n|^2 \, dx \\ = & \overline{\lim_{n\to\infty}} \int \psi_R|x|^{-2a}|\nabla u_n|^2 \, dx + \int (1-\psi_R)|x|^{-2a}|\nabla u_n|^2 \, dx \\ = & \overline{\lim_{n\to\infty}} \int \psi_R|x|^{-2a}|\nabla u_n|^2 \, dx + \int (1-\psi_R) \, d\mu \\ & + \int (1-\psi_R)|x|^{-2a}|\nabla u|^2 \, dx. \end{aligned}$$

When $R \to \infty$ we obtain, by Lebesgue Theorem,

$$\begin{aligned} \overline{\lim_{n\to\infty}} \int |x|^{-2a} |\nabla u_n|^2 \, dx &= \mu_\infty + \int d\mu + \int |x|^{-2a} |\nabla u|^2 \, dx \\ &= \mu_\infty + \|\mu\| + \left| |x|^{-a} \nabla u \right|_2^2 . \end{aligned}$$

The proof of (7) is similar.

<u>Remark</u>.

When $a = b = 0$, inequality (4) is due to Lions [9], inequality (5) to Bianchi, Chabrowski and Szulkin [3] and equalities (6) and (7) to Ben-Naoum, Troestler and Willem ([1] and [12]).

Theorem 4. *Let $N \geq 3$, $0 \leq a < (N-2)/2$, $a + b > 0$, $a \leq b < 1 + a$ and $p = p(a,b)$. Let $(u_n) \subset \mathcal{D}_a^{1,2}(R^N)$ be a minimizing sequence for $S(a,b)$ satisfying (1). Then there exists a sequence $(t_n) \subset]0, \infty[$ such that $(u_n^{t_n})$ contains a convergent subsequence. In particular there exists a minimizer for $S(a,b)$.*

For every n, there exists t_n such that

$$\int_{B(0,t_n)} |x|^{-pb} |u_n|^p \, dx = 1/2.$$

Let us define $v_n := u_n^{t_n}$. Hence $\left| |x|^{-b} v_n \right|_p = 1$, $\left| |x|^{-a} \nabla v_n \right|_2^2 \to S(a,b)$ and

$$\int_{B(0,1)} |x|^{-pb} |v_n|^p \, dx = \frac{1}{2}. \tag{10}$$

Since (v_n) is bounded in $\mathcal{D}_a^{1,2}(R^N)$, we may assume, going if necessary to a subsequence

$$\begin{array}{ll} v_n \rightharpoonup v & \text{in } \mathcal{D}_a^{1,2}(R^N), \\ \left| |x|^{-a} \nabla (v_n - v) \right|^2 \rightharpoonup \mu & \text{in } \mathcal{M}(R^N), \\ \left| |x|^{-b} (v_n - v) \right|^p \rightharpoonup \nu & \text{in } \mathcal{M}(R^N), \\ v_n \to v & \text{a.e. on } R^N. \end{array}$$

By the preceding lemma

$$S(a,b) = \left| |x|^{-a} \nabla v \right|_2^2 + \|\mu\| + \mu_\infty \tag{11}$$

$$1 = \left| |x|^{-b} v \right|_p^p + \|\nu\| + \nu_\infty. \tag{12}$$

We deduce from (4),(5),(11) and the definition of $S(a,b)$

$$S(a,b) \geq S(a,b)\left\{ (\left| |x|^{-b} v \right|_p)^{2/p} + \|\nu\|^{2/p} + \nu_\infty^{2/p} \right\}.$$

It follows from (12) that $\left| |x|^{-b} v \right|_p^p$, $\|\nu\|$ and ν_∞ are equal to 0 or to 1. By (10), $\nu_\infty \leq 1/2$, so that $\nu_\infty = 0$.

Assume by contradiction that $v = 0$. It follows from (4) and (11), that

$$1 = \|\nu\|^{2/p} = S(a,b)^{-1} \|\mu\|.$$

By the preceding lemma, ν end μ are concentrated at a single point x_0. By (10), x_0 is different from 0. If $a < b$, the Rellich theorem implies that

$$0 = \lim_{n \to \infty} \int_{B(x_0,r)} |x|^{-pb} |v_n|^p \, dx = \|\nu\| = 1$$

where $0 < r < |x_0|$. This is impossible. If $a = b > 0$, we obtain

$$A := \lim_{n \to \infty} \frac{\int_{B(x_0,r)} |x|^{-2a} |\nabla v_n|^2 \, dx}{\left(\int_{B(x_0,r)} |x|^{-2^* a} |v_n|^{2^*} \, dx \right)^{2/2^*}} = \|\mu\| = S(a,a).$$

Let $\eta \in \mathcal{D}(B(x_0,r))$ be such that $\eta = 1$ on $B(x_0, r/2)$. Since

$$A = \lim_{n \to \infty} \frac{\int_{B(x_0,r)} |\nabla(\eta v_n)|^2 \, dx}{\left(\int_{B(x_0,r)} |\eta v_n|^{2^*} \, dx \right)^{2/2^*}} \geq S,$$

it follows that $S \leq S(a,a)$. But, by lemma 1, $S(a,a) \leq g(a) < g(0) = S$.

Hence we have proved that $\left| |x|^{-b} v \right|_p = 1$ and so

$$\left| |x|^{-a} \nabla v \right|_2^2 = S(a,b) = \lim \left| |x|^{-a} \nabla v_n \right|_2^2.$$

3 Minimizing sequences for $S(a,b,\lambda)$

We consider now a minimizing sequence $(u_n) \subset \mathcal{D}_a^{1,2}(R^N)$ for $S(a,b,\lambda)$:

$$\left||x|^{-b}u_n\right| = 1, \quad ||x|^{-a}\nabla u_n|_2^2 + \lambda \left||x|^{-(1+a)}u_n\right|_2^2 \to S(a,b,\lambda), \tag{13}$$

$n \to \infty$.

The problem is also invariant by dilation.

Lemma 5. *Let $-S(a,a+1) < \lambda$ and let (u_n) be as in lemma 3 and such that*

$$|x|^{-2a}|\nabla(u_n - u)|^2 + \lambda|x|^{-2(1+a)}(u_n - u)^2 \rightharpoonup \gamma \text{ in } \mathcal{M}(R^N).$$

Define

$$\gamma_\infty := \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{|x|\geq R} |x|^{-2a}|\nabla u_n|^2 + \lambda|x|^{-2(1+a)}u_n^2\, dx.$$

Then it follows that

$$\|\nu\|^{2/p} \leq S(a,b,\lambda)^{-1}\|\gamma\|, \tag{14}$$

$$\nu_\infty^{2/p} \leq S(a,b,\lambda)^{-1}\gamma_\infty \tag{15}$$

$$\begin{aligned} &\overline{\lim}_{n\to\infty} ||x|^{-a}\nabla u_n|_2^2 + \lambda \left||x|^{-(a+1)}u_n\right|_2^2 \\ &\quad = ||x|^{-a}\nabla u|_2^2 + \lambda\, ||x|^{-a+1}u|_2^2 + \|\gamma\| + \gamma_\infty. \end{aligned} \tag{16}$$

Moreover if $b < 1 + a$ and $\|\nu\|^{2/p} = S(a,b,\lambda)^{-1}\|\gamma\|$, then ν and γ are concentrated at a single point.

The proof is similar to the proof of lemma 3.

Theorem 6. *Let $N \geq 3$, $0 \leq a < (N-2)/2$, $a \leq b < 1+a$, $p = p(a,b)$ and $-S(a,a+1) < \lambda < 0$. Let $(u_n) \subset \mathcal{D}_a^{1,2}(R^N)$ be a minimizing sequence for $S(a,b,\lambda)$ satisfying (13). Then there exists a sequence $(t_n) \subset]0,\infty[$ such that $(u_n^{t_n})$ contains a convergent subsequence. In particular, there exists a minimizer for $S(a,b,\lambda)$.*

As in the proof of theorem 4, there exists a sequence $(t_n) \subset]0,\infty[$ such that $v_n := u_n^{t_n}$ satisfies

$$\left||x|^{-b}v_n\right|_p = 1, ||x|^{-a}\nabla v_n|_2^2 + \lambda \left||x|^{-(a+1)}v_n\right|_2^2 \to S(a,b,\lambda)$$

$$\int_{B(0,1)} |x|^{-pb}|v_n|^p\, dx = 1/2. \tag{17}$$

We may also assume that

$$\begin{array}{ll} v_n \rightharpoonup v & \text{in } \mathcal{D}_a^{1,2}(R^N), \\ \left||x|^{-a}\nabla(u_n-u)\right|^2 + \lambda\left||x|^{-(a+1)}(v_n-v)\right|^2 \rightharpoonup \gamma & \text{in } \mathcal{M}(R^N), \\ \left||x|^{-b}(v_n-v)\right|^p \rightharpoonup \nu & \text{in } \mathcal{M}(R^N), \\ v_n \to v & \text{a.e. on } R^N. \end{array}$$

By the preceding lemma and lemma 3,

$$S(a,b,\lambda) = \left||x|^{-a}\nabla v\right|_2^2 + \lambda\left||x|^{-(a+1)}v\right|_2^2 + \|\gamma\| + \gamma_\infty, \tag{18}$$

$$1 = \left||x|^{-b}v\right|_p^p + \|\nu\| + \nu_\infty. \tag{19}$$

It follows from (14), (15), (18), (19) that $\left||x|^{-b}v\right|_p^p$, $\|\nu\|$ and ν_∞ are equal to 0 or to 1. By (17),$\nu_\infty = 0$.

Assume by contradiction that $v = 0$. It follows from (14) and (18) that

$$1 = \|\nu\|^{2/p} = S(a,b,\lambda)^{-1}\|\gamma\|.$$

By the preceding lemma, ν and γ are concentrated at a single point x_0. By (17), x_0 is different from 0. If $a < b$, the Rellich Theorem implies that $\nu = 0$. This is a contradiction, since $\|\nu\| = 1$. If $a = b$, we obtain, as in theorem 4,

$$S \leq \|\gamma\| = S(a,a,\lambda).$$

This is also a contradiction, since, by lemma 1, and theorem 4,

$$S(a,a,\lambda) < S(a,a) \leq S.$$

Thus $\left||x|^{-b}v\right|_p = 1$, v is a minimizer for $S(a,a,\lambda)$ and $v_n \to v$ in $\mathcal{D}_a^{1,2}(R^N)$.

Remark.

When R^N is replaced by a bounded domain Ω the situation is different (see [6], [7]). In particular, if Ω is starshaped with respect to 0, $a = b = 0$ and $\lambda < 0$, the corresponding infimum is never achieved [7, remark 3.8].

Theorem 7. *Let $N \geq 3$, $0 \leq a < (N-2)/2$, $p = p(a,b)$. Let (u_n) be a minimizing sequence for $S(a,b,\lambda)$ satisfying (13).*
Assume

$$a < b < a+1 \quad \text{and } 0 < \lambda$$

or

$$0 < a = b \quad \text{and } 0 < \lambda << 1.$$

Then there exists a sequence $(t_n) \subset]0, \infty[$ *such that* $(u_n^{t_n})$ *contains a convergent subsequence. In particular, there exists a minimizer for* $S(a, b, \lambda)$.

The proof is similar to the proof of theorem 6.

Remark. For $\lambda > 0$, $S(0, 0, \lambda)$ is equal to S and is never achieved.

Remark. For both problems $S(a, b)$ and $S(a, b, \lambda)$ the ground state solutions are positive everywhere and smooth everywhere but the origin. This follows from the elliptic regularity theory in a standard way.

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