Existence and nonexistence of $G-$least energy solutions for a nonlinear Neumann problem with critical exponent

Zhi-Qiang Wang *

Department of Mathematics

Utah State University

Logan, UT 84322

1. Introduction

This paper is concerned with the existence, nonexistence and qualitative property of various symmetric minimizers for a family of minimization problems. To demonstrate the problem let us start with a special case. Let B be the unit ball centered at 0 in $\mathbf{R}^N$ ($N \geq 3$) and let G be a closed subgroup of $\mathbf{O(N)}$. Then there is a natural G-action on $H^1(B)$ defined by, for any $g \in G$ and $u \in H^1(B)$,

$$gu(x) := u(g^{-1}x), \quad a.e. \text{ in } B.$$

Define the G-invariant subspace of $H^1(B)$

$$H^1_G(B) = \{u \in H^1(B) \mid gu = u \ \forall g \in G\}.$$

Then we consider the following minimization problem

$$m_{\lambda,G} := \inf_{u \in V_G(B)} E_\lambda(u) \tag{1.1}$$

where

$$E_\lambda(u) = \int_B (|\nabla u|^2 + \lambda u^2)dx, \quad u \in V_G(B), \tag{1.2}$$

and

$$V_G(B) = \{u \in H^1_G(B) | \int_B |u|^{p+1}dx = 1\}. \tag{1.3}$$

Here we assume $\lambda > 0$ is a constant and $p = \frac{N+2}{N-2}$ is the critical Sobolev exponent.

We are interested in the existence and nonexistence of minimizers of (1.1). We shall give conditions on G which guarantee (1.1) to be solvable or nonsolvable. More general G-invariant domains will also be considered in Sections 2 - 4.

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By the symmetric criticality principle ([P]), a minimizer of (1.1), by rescaling, solves the following semilinear elliptic problem

$$\begin{cases} -\Delta u + \lambda u = u^p, u > 0 & \text{in } B \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial B \end{cases} \tag{1.4}$$

where ν is the unit outer normal to ∂B. This problem has been studied extensively in recent years and the existence has been given in [AM] and [Wx] for a solution corresponding to the infimum of E_λ in $V(\Omega) := \{u \in H^1(\Omega) \mid ||u||_{p+1} = 1\}$. (The results in these paper hold for an arbitrary bounded smooth domain Ω). This solution is called least energy solution. Thus it is natural to ask whether there exist least energy solutions in $V_G(B)$, namely, the solutions corresponding to $m_{\lambda,G}(B)$. We shall refer them as *G-least energy solutions* for (1.4), and call $m_{\lambda,G}(B)$ *G-least energy.*

Then the question here is that under what conditions on G, G-least energy problem (1.1) is solvable or nonsolvable. It turns out that the following number plays an important role in classifying the problem. Let us define

$$k_G(B) := \min\{\#G(x) \mid x \in \overline{B} \setminus \{0\}\}, \tag{1.5}$$

where $G(x)$ denotes the orbit of x and $\#G(x)$ is the cardinal number of $G(x)$.

To state our results, we need some preliminary. Let S be the best Sobolev constant, i.e.,

$$S = \min_{u \in \mathcal{D}^{1,2}(\mathbf{R}^N)\setminus\{\theta\}} \frac{\int_{\mathbf{R}^N} |\nabla u|^2 dx}{\{\int_{\mathbf{R}^N} |u|^{p+1} dx\}^{\frac{2}{p+1}}}.$$

Then it is well known ([T]) that S is assumed by the following family of functions: $U_{\epsilon,y}(x) = \epsilon^{-\frac{N-2}{2}} U(\frac{x-y}{\epsilon})$, for $\epsilon > 0$, $y \in \mathbf{R}^N$, with $U(x) = [\frac{N(N-2)}{N(N-2)+|x|^2}]^{\frac{N-2}{2}}$, which are solutions of

$$-\Delta u = u^p \quad \text{in } \mathbf{R}^N.$$

Now we can state our main theorem for (1.1), and more general results will be given in the next sections with Theorem 1.1 as a corollary.

Theorem 1.1.
(i) If $k_G = 1, 2$, then $m_{\lambda,G}(B)$ is always achieved for $\lambda > 0$. Moreover, if $k_G(B) = 1$, we have

$$m_{\lambda,G}(B) < 2^{-\frac{2}{N}} S \text{ for } 0 < \lambda < \infty, \quad \text{and} \quad \lim_{\lambda \to \infty} m_{\lambda,G}(B) = 2^{-\frac{2}{N}} S; \tag{1.6}$$

and if $k_G(B) = 2$, we have

$$m_{\lambda,G}(B) < S \text{ for } 0 < \lambda < \infty, \quad \text{and} \quad \lim_{\lambda \to \infty} m_{\lambda,G}(B) = S. \tag{1.7}$$

(ii) Assume $N \geq 5$. If $k_G \geq 3$, then there exists $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$, $m_{\lambda,G}(B)$ is achieved and for $\lambda_0 < \lambda < \infty$, $m_{\lambda,G}(B)$ is not achieved. Moreover,

$$m_{\lambda,G}(B) < S \ \ \textit{for} \ \ 0 < \lambda < \lambda_0, \ \ \textit{and} \ \ m_{\lambda,G}(B) = S \quad \textit{for} \ \ \lambda_0 \leq \lambda < \infty.$$

Remark 1.1. Note that the work in [AM] and [Wx] can be regarded as a special case of $k_G(B) = 1$ (i.e., G is the trivial group) and that a special case of $k_G(B) = 2$ was treated in [Wz2]. The arguments in these papers can be adapted to analyzing more general cases of $k_G(B) = 1$ and $k_G(B) = 2$. Thus the main goal of this paper is to prove the non-existence result (the assertion (ii) of Theorem 1.1). A more general nonexistence result will be given in Section 3 which holds for any G-invariant domains containing 0 as an interior point. To our knowledges, the question of whether there exist G-least energy solutions for the case $k_G(B) \geq 3$ has not been studied. In this paper we completely settle the question for dimensions $N \geq 5$.

Note also that $m_{\lambda,G}(B)$ is nondecreasing in λ. Then the behavior of $m_{\lambda,G}(B)$ is described in the following figures.

As a consequence, we obtain the existence and nonexistence of G-least energy solutions for equation (1.4). We refer Theorem 2.2 for more qualitative properties of these solutions.

Corollary 1.1.
(i) If $k_G(B) = 1, 2$, then (1.4) has a G-least energy solution for $\lambda > 0$.
(ii) Assume $N \geq 5$. If $k_G \geq 3$, then there exists $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$ (1.4) has a G-least energy solution and for $\lambda_0 < \lambda < \infty$ (1.4) has no G-least energy solutions.

Remark 1.2. As another consequence of our main Theorem, we obtain in the symmetric subspaces sharper estimates on the best embedding constants of some Sobolev type inequalities related to the Cherrier inequality (p. 51, [A]), which states that for any $\epsilon > 0$ there exists $C(\epsilon) > 0$ such that for all $u \in H^1(\Omega)$ (where Ω is a smooth bounded domain)

$$||u||_{p+1}^2 \leq (2^{\frac{2}{N}} S^{-1} + \epsilon)||\nabla u||_2^2 + C(\epsilon)||u||_2^2. \tag{1.9}$$

See Corollary 2.1, 3.2 and 4.2 for sharper versions of this inequality in $H_G^1(\Omega)$.

It turns out that most of the above results hold for more general symmetric domains as we shall show in Section 2 and 3. The paper is organized as follows. In Section 2, besides some preliminaries we shall consider the cases of $k_G(\Omega) = 1$ and $k_G(\Omega) = 2$. In Section 3 we shall give a nonexistence result on G-least energy solutions for $k_G(\Omega) \geq 3$. In Section 4, we continue the discussion for the case of $k_G(\Omega) \geq 3$ giving some existence results that complement the results in Section 3.

2. Preliminaries and the cases of $k_G(\Omega) = 1$ and $k_G(\Omega) = 2$

In this section we first give some preliminaries and then state and prove a result which covers the assertion (i) of Theorem 1.1 as a special case. Let $\Omega \subset \mathbf{R}^N$ ($N \geq 3$) be a smooth bounded domain. Let $G \subset \mathbf{O(N)}$ be a closed subgroup. Assume that Ω is G-invariant and define similarly to (1.5)

$$k_G(\Omega) := \{\#G(x) \ | \ x \in \overline{\Omega} \setminus \{0\} \}. \tag{2.1}$$

We shall consider in this section G-invariant domains that satisfy the following assumption. Let $H(y)$ be the mean curvature of $\partial\Omega$ with respect to the unit outer normal ν.

Assumption (S). There exists $y \in \partial\Omega$ such that $\#G(y) = k_G(\Omega)$ and $H(y) > 0$.

Then we may define the corresponding G-least energy

$$m_{\lambda,G}(\Omega) := \inf_{u \in V_G(\Omega)} E_\lambda(u) \tag{2.2}$$

where $H_G^1(\Omega)$, $E_\lambda(u)$ and $V_G(\Omega)$ can be defined as in (1.2) and (1.3) with B replaced by Ω. Again, minimizers of $m_{\lambda,G}(\Omega)$, by rescaling, solves

$$\begin{cases} -\Delta u + \lambda u = u^p, u > 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

Remark 2.1. Note that 0 may or may not belong to Ω. It turns out that for the cases of $k_G(\Omega) = 1$ or 2 the results do not depend upon whether 0 belongs to Ω. However, for the case of $k_G(\Omega) \geq 3$, as we shall see in Sections 3 and 4, the origin being inside or outside Ω determines the solvability of (2.2).

Theorem 2.1. *Assume Assumption (S) holds.*
(i) If $k_G(\Omega) = 1$, then for $\lambda > 0$, $m_{\lambda,G}(\Omega) < 2^{-\frac{2}{N}} S$, $m_{\lambda,G}(\Omega)$ is achieved and

$$\lim_{\lambda \to \infty} m_{\lambda,G}(\Omega) = 2^{-\frac{2}{N}} S. \tag{2.4}$$

(ii) If $k_G(\Omega) = 2$, then for all $\lambda > 0$, $m_{\lambda,G}(\Omega) < S$, $m_{\lambda,G}(\Omega)$ is achieved and

$$\lim_{\lambda \to \infty} m_{\lambda,G}(\Omega) = S. \tag{2.5}$$

As a consequence we get the following results about the existence of G-least energy solutions.

Theorem 2.2. *Assume Assumption (S) holds.*
(i) If $k_G(\Omega) = 1$, then for $\lambda > 0$, (2.3) has a G-least energy solution u_λ (i.e., $E_\lambda(\frac{u_\lambda}{||u_\lambda||_{p+1}}) = m_{\lambda,G}(\Omega)$) with the property that for λ sufficiently large u_λ is single-peaked on $\partial\Omega$ in the sense that u_λ achieves its maximum over $\overline{\Omega}$ only at one point P_λ such that $P_\lambda \in \partial\Omega$, $\#G(P_\lambda) = 1$, and

$$\lim_{\lambda \to \infty} ||\nabla u_\lambda - \nabla U_{\epsilon_\lambda, P_\lambda}||_2 = 0, \tag{2.6}$$

where $\epsilon_\lambda = [u_\lambda(P_\lambda)]^{-\frac{p-1}{2}}$.
(ii) If $k_G(\Omega) = 2$, then for $\lambda > 0$, (2.3) has a G-least energy solution u_λ satisfying

$(ii)_a$. when $N \geq 5$, for λ sufficiently large u_λ is double-peaked on $\partial\Omega$ in the sense that u_λ achieves its maximum over $\overline{\Omega}$ only at two points which lie on $\partial\Omega$ and form a G-orbit $G(P_\lambda)$ of two points such that

$$\lim_{\lambda \to \infty} ||\nabla u_\lambda - \sum_{g \in G} \nabla U_{\epsilon_\lambda, gP_\lambda}||_2 = 0, \tag{2.7}$$

where $\epsilon_\lambda = [u_\lambda(P_\lambda)]^{-\frac{p-1}{2}}$;

$(ii)_b$. when $N = 3, 4$, for λ sufficiently large, either u_λ is double-peaked on $\partial\Omega$ in the sense above, or u_λ is single-peaked at 0 in the sense that u_λ achieves its maximum over $\overline{\Omega}$ only at 0 and

$$\lim_{\lambda \to \infty} ||\nabla u_\lambda - \nabla U_{\epsilon_\lambda, 0}||_2 = 0, \tag{2.8}$$

where $\epsilon_\lambda = [u_\lambda(0)]^{-\frac{p-1}{2}}$.

Remark 2.2. Without consideration of symmetries (i.e., $G = \{Id\}$), it is proved in [AM] and [Wx] that $m_{\lambda,G}(\Omega)$ is always achieved for $\lambda > 0$. The qualitative properties of least energy solutions for semilinear Neumann problems with both subcritical and critical exponents have been studied in [APY], [NPT] and [NT1,2]. For the critical exponent problem (2.3), among other things, it is proved that for λ sufficiently large, the least energy solutions are single-peaked on the boundary ([APY] [NPT]). When $G = \{Id, -Id\}$, a special case for $k_G(\Omega) = 2$, it is proved in [Wz2] that $m_{\lambda,G}(\Omega)$ is always achieved for $\lambda > 0$ and that for $N \geq 5$ the G-least energy solutions are double-peaked on $\partial\Omega$ for λ sufficiently large.

The following estimate was given in [AM] and [Wx].

Lemma 2.1. *There exist positive constants A_N and a_N such that for all $\epsilon > 0, y \in \partial\Omega$ and $\lambda \geq 1$*

$$E_\lambda\left(\frac{U_{\epsilon,y}}{||U_{\epsilon,y}||_{p+1}}\right) = \begin{cases} 2^{-\frac{2}{N}}S - A_N H(y)\epsilon \log\frac{1}{\epsilon} + a_N\lambda\epsilon + O(\epsilon) + o(\lambda\epsilon) & N = 3 \\ 2^{-\frac{2}{N}}S - A_N H(y)\epsilon + a_N\lambda\epsilon^2\log\frac{1}{\epsilon} + O(\epsilon^2\log\frac{1}{\epsilon}) + o(\lambda\epsilon^2\log\frac{1}{\epsilon}) & N = 4 \\ 2^{-\frac{2}{N}}S - A_N H(y)\epsilon + a_N\lambda\epsilon^2 + O(\epsilon^2) + o(\lambda\epsilon^2) & N \geq 5 \end{cases}$$

where A_N and a_N depend only on N, $O(s)$ and $o(s)$ mean, as $s \to 0$, that $\frac{O(s)}{s}$ is bounded and $\frac{o(s)}{s}$ tends to zero, and $H(y)$ is the mean curvature of $\partial\Omega$.

Proof. See [AM], [Wx]. □

As consequences of this lemma, we have the following estimates on $m_{\lambda,G}(\Omega)$.

Lemma 2.2. *Let Assumption (S) be satisfied. Then $m_{\lambda,G}(\Omega) < [k_G(\Omega)]^{\frac{2}{N}}2^{-\frac{2}{N}}S$ for $\lambda > 0$.*

Proof. When $k_G(\Omega) = 1$, there exists $y_0 \in \partial\Omega$ such that $\#G(y_0) = 1$ and $H(y_0) > 0$. Then we have $U_{\epsilon,y_0}(x)$ is $G-$invariant because for all $g \in G$, $U_{\epsilon,y_0}(gx) = U_{\epsilon,g^{-1}y_0}(x) = U_{\epsilon,y_0}(x)$. Then using Lemma 2.1, one gets

$$E_\lambda\left(\frac{U_{\epsilon,y_0}}{||U_{\epsilon,y_0}||_{p+1}}\right) < 2^{-\frac{2}{N}}S,$$

for ϵ small.

When $k := k_G(\Omega) > 1$ is finite, by Assumption (S) there exist $y_0 \in \partial\Omega$ such that $\#G(y_0) = k$ and $H(y_0) > 0$. Let $G(y_0) = \{y_0, y_1, ..., y_{k-1}\}$ and

$$V_{\epsilon,y_0} = \sum_{j=0}^{k-1} U_{\epsilon,y_j}.$$

Then we have $V_{\epsilon,y_0}(x)$ is $G-$invariant. And using Lemma 2.1 a direct computation shows $E_\lambda\left(\frac{V_{\epsilon,y_0}}{||V_{\epsilon,y_0}||_{p+1}}\right) < [k_G(\Omega)]^{\frac{2}{N}}2^{-\frac{2}{N}}S$ for ϵ small. □

The following compactness result was formulated in [Wz4, Lemma 3.4] and it can be proved using the Concentration-Compactness Principle ([L] [S]) and similar arguments to Dirichlet problems. A similar result was also given in [GP].

Lemma 2.3. *Let $\lambda > 0$ be fixed. Let $u_n \geq 0$ be a $(PS)_c$ sequence for E_λ in $V_G(\Omega)$ (i.e., as $n \to \infty$, $E_\lambda(u_n) \to c$ and $E'_\lambda(u_n) \to 0$). Then there exist integers $l_1 \geq 0$, $l_2 \geq 0$, a solution $u_0 \geq 0$ of*

$$\begin{cases} -\Delta u + \lambda u = cu^p, & in\ \Omega \\ \frac{\partial u}{\partial \nu} = 0 & on\ \partial\Omega, \end{cases} \tag{2.9}$$

and $x_{n,j} \in \Omega, \epsilon_{n,j} \to 0$, as $n \to \infty$, for $j = 1, 2, ..., l_1 + l_2$, such that

$$\left\| u_n - u_0 - c^{\frac{1}{1-p}} \sum_{j=1}^{l_1+l_2} U_{\epsilon_{n,j}, x_{n,j}} \right\|_{W^{1,2}(\Omega)} \to 0, \quad as\ n \to \infty, \tag{2.10}$$

and

$$c = \int_\Omega (|\nabla u_0|^2 + \lambda u_0^2) dx + l_1 c^{\frac{2}{1-p}} \int_{\mathbf{R}^N} |\nabla U|^2 dx + l_2 c^{\frac{2}{1-p}} \int_{\mathbf{R}^N_+} |\nabla U|^2 dx. \tag{2.11}$$

Recall that E_λ satisfies $(PS)_c$ condition in $V_G(\Omega)$ if any sequence $u_n \in V_G(\Omega)$ satisfying $E_\lambda(u_n) \to c$ and $E'_\lambda(u_n) \to 0$ has a convergent subsequence.

Lemma 2.4. *Assume $k_G(\Omega) = 1$, then E_λ satisfies $(PS)_c$ condition in $V_G(\Omega)$ for all $c < 2^{-\frac{2}{N}} S$. Assume $k_G(\Omega) \geq 2$, then E_λ satisfies $(PS)_c$ condition in $V_G(\Omega)$ for all $c < S$.*

Proof. The case of $k_G(\Omega) = 1$ is trivial with the application of Lemma 2.3. Assume $k_G(\Omega) \geq 2$. Applying Lemma 2.3, we get $l_1 \geq 0, l_2 \geq 0$ such that (2.11) holds. If all $x_{n,j}$ as $n \to \infty$ are away from the origin then we get at least $k_G(\Omega)$ of them. Since $k_G(\Omega) \geq 2$, by (2.11) we get $c \geq S$. If there exists at least one $\{x_{n,j}\}$ converging to the origin as $n \to \infty$ we get $l_1 \geq 1$ and therefore $c \geq S$ again. In both cases we get a contradiction. □

Now it is easy to see that the existence of G-minimizers in Theorem 1.1 follows from Lemma 2.2 and Lemma 2.4. (2.4) follows from the arguments in [APY] [Wz3] and (2.5) follows from the arguments in [Wz2] where special G actions with $k_G(\Omega) = 1$ and 2 were considered. The details are left to readers.

For Theorem 2.2, the existence of least energy solutions follows from Theorem 2.1. About the qualitative properties, the assertion (i) of Theorem 2.2 follows from similar arguments as in [APY] and [NPT] and the assertion (ii) follows from a slightly modified argument in [Wz2] where a special case of $k_G(\Omega) = 2$ was considered ($G = \{Id, -Id\}$). To shorten the paper we leave the details to readers.

Corollary 2.1. *Let Assumption (S) be satisfied with $k_G \geq 2$. Then for any $\epsilon > 0$ there exists $C(\epsilon) > 0$ such that for all $u \in H^1_G(\Omega)$*

$$||u||^2_{p+1} \leq (S^{-1} + \epsilon)||\nabla u||^2_2 + C(\epsilon)||u||^2_2. \tag{1.10}$$

3. A nonexistence result for the case of $k_G(\Omega) \geq 3$

In this section we shall prove a nonexistence result which covers the assertion (ii) in Theorem 1.1. Let $\Omega \subset \mathbf{R}^N$ be a smooth bounded domain satisfying

$$0 \in \Omega. \tag{3.1}$$

Assume that Ω is G-invariant and satisfies $k_G(\Omega) \geq 3$.

Theorem 3.1. *Let $N \geq 5$. Let Ω be a bounded smooth G-invariant domain satisfying $k_G(\Omega) \geq 3$ and $0 \in \Omega$. Then there exists $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$, $m_{\lambda,G}(\Omega)$ is achieved and for $\lambda > \lambda_0$, $m_{\lambda,G}(\Omega)$ is not achieved.*

Remark 3.1. We do not know whether $m_{\lambda_0,G}(\Omega)$ is achieved. Also we do not know whether λ_0 depends upon G (even among subgroups of G having the same $k_G(\Omega)$). However an estimate of λ_0 in terms of the domain will be given at the end of this Section.

We need a few technical results (the proof of Lemma 3.1 will be given later).

Lemma 3.1. *Let Ω be a bounded smooth G-invariant domain satisfying $k_G(\Omega) \geq 3$ and $0 \in \Omega$. If for a sequence $\lambda_n \to \infty$ as $n \to \infty$ $m_{\lambda_n,G}$ is achieved at $u_n \in V_G(\Omega)$, then*

(i) u_n achieves its maximum over $\overline{\Omega}$ only at 0.

(ii) there exist $\delta_n > 0$ $\delta_n \to 0$ as $n \to \infty$, such that

$$\lim_{n\to\infty} ||\nabla(u_n - S^{\frac{-1}{p-1}} U_{\delta_n,0})||_2 = 0. \tag{3.2}$$

Lemma 3.2. ([Wz2]) *Let $\lambda_n > 0, \delta_n > 0, P_n \in \Omega, 0 \leq v_n \in H^1(\Omega)$ be such that as $n \to \infty$, $\lambda_n \to \infty, \delta_n \to 0, P_n \to P_0, v_n \rightharpoonup 0$ in $W^{1,2}(\Omega)$ and*

$$\lim_{n\to\infty} ||\nabla v_n - \nabla U_{\delta_n,P_n}||_2 = 0. \tag{3.3}$$

Assume that P_0 is an interior point of Ω. Then if $N \geq 5$ and $E_{\lambda_n}(\frac{v_n}{||v_n||_{p+1}}) < S$, there exist $\epsilon_n > 0, y_n \in \Omega$ such that for a subsequence(still denoted by v_n), as $n \to \infty$, $\frac{\epsilon_n}{\delta_n} \to 1, y_n \to P_0$ and

$$E_{\lambda_n}(\frac{v_n}{||v_n||_{p+1}}) \geq E_{\lambda_n}(\frac{U_{\epsilon_n,y_n}}{||U_{\epsilon_n,y_n}||_{p+1}}) + O(\epsilon_n^2) + o(\lambda_n \epsilon_n^2), \tag{3.4}$$

$$\lambda_n \epsilon_n^2 = O(\epsilon_n). \tag{3.5}$$

Proof. See Lemma 4.7 of [Wz2]. □

Lemma 3.3. ([BN]) *Let y be an interior point of Ω. Then for $N \geq 5$*

$$E_\lambda\left(\frac{U_{\epsilon,y}}{||U_{\epsilon,y}||_{p+1}}\right) = S + b_N \lambda\epsilon^2 + O(\epsilon^2) + o(\lambda\epsilon^2) \tag{3.6}$$

where $b_N > 0$ is a constant depending only on N, and $O(\cdot)$ and $o(\cdot)$ are uniform in λ and y as $\epsilon \to 0$ for $\lambda \geq 1$ and for y in a compact subset of Ω.

Proof. For a proof of this lemma, see p.444, formulas (1.11) - (1.13) in [BN]. □

Proof of Lemma 3.1. First, since $\frac{U_{\epsilon,0}}{||U_{\epsilon,0}||_{p+1}} \in V_G(\Omega)$, Lemma 3.3 implies that $m_{\lambda,G}(\Omega) \leq S$ for all $\lambda > 0$. Let $u_n \in V_G(\Omega)$ be such that $E_{\lambda_n}(u_n) = m_{\lambda_n,G}(\Omega)$. Now because we assume $m_{\lambda_n,G}(\Omega)$ are achieved with $\lambda_n \to \infty$, we get $m_{\lambda_n,G}(\Omega) < S$ for all n. We write $m_n := m_{\lambda_n,G}(\Omega)$, then

$$\begin{cases} -\Delta u_n + \lambda_n u_n = m_n u_n^p, u_n > 0 & \text{in } \Omega \\ \frac{\partial u_n}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Let $P_n \in \overline{\Omega}$ be such that $u_n(P_n) = \max_{x\in\overline{\Omega}} u_n(x)$. Then we have

$$M_n := u_n(P_n) \geq \left(\frac{\lambda_n}{m_n}\right)^{\frac{1}{p-1}}.$$

Let $\epsilon_n = M_n^{-\frac{p-1}{2}}$, then

$$\lambda_n \epsilon_n^2 = \frac{\lambda_n}{M_n^{p-1}} \leq m_n \leq S.$$

Assume $P_n \to P_0 \in \overline{\Omega}$, $\lambda_n \epsilon_n^2 \to a \geq 0$. First we claim

$$\lim_{n\to\infty} \frac{|P_n|}{\epsilon_n} = \alpha < +\infty. \tag{3.7}$$

Proof of the Claim. If this is not true, then $\theta_n := \frac{|P_n|}{\epsilon_n} \to \infty$ as $n \to \infty$. Define $A_n := \Omega \cap B_{\sqrt{\theta_n}\epsilon_n}(P_n)$, and $v_n(x) = u_n|_{A_n}$. Then we assert that there exist $g_0 = id, g_1, g_2 \in G$ such that for n large,

$$|g_i P_n - g_j P_n| \geq 2\sqrt{\theta_n}\epsilon_n, \quad i \neq j, \ i,j = 0,1,2.$$

To see this, let us write $\tilde{P}_n = \frac{P_n}{|P_n|} \in S^{N-1}$. Then up to a subsequence there exists $\tilde{P}_0 \in S^{N-1}$, such that $\tilde{P}_n \to \tilde{P}_0$ as $n \to \infty$. Since $k_G(\Omega) \geq 3$, there exist $g_1, g_2 \in G$ and $\sigma_0 > 0$ such that (with $g_0 = id$)

$$|g_i\tilde{P}_0 - g_j\tilde{P}_0| \geq \sigma_0, \quad i \neq j, \; i,j = 0,1,2.$$

Now for n large

$$\begin{aligned} |g_iP_n - g_jP_n| &= |g_i\tilde{P}_n - g_j\tilde{P}_n||P_n| \\ &= |g_i\tilde{P}_n - g_j\tilde{P}_n|\theta_n\epsilon_n \\ &\geq \frac{1}{2}\sigma_0\sqrt{\theta_n}\sqrt{\theta_n}\epsilon_n \\ &\geq 2\sqrt{\theta_n}\epsilon_n \end{aligned}$$

where we have used the fact that $\sqrt{\theta_n} \to \infty$.

Now, we define

$$\tilde{v}_n(x) = \epsilon_n^{\frac{N-2}{2}} u_n(\epsilon_n x + P_n), \quad \text{for } \; x \in B_n := \frac{A_n - \{P_n\}}{\epsilon_n}.$$

Then calculating directly, we have

$$\begin{cases} -\Delta\tilde{v}_n + \lambda_n\epsilon_n^2\tilde{v}_n = m_n\tilde{v}_n^p, & \text{in } B_n \\ \frac{\partial\tilde{v}_n}{\partial\nu} = 0 & \text{on } \partial(\frac{\Omega_n - \{P_n\}}{\epsilon_n}) \cap \partial B_n \\ 0 \leq \tilde{v}_n \leq 1, \; \tilde{v}_n(0) = 1. \end{cases}$$

By elliptic theory ([GT]), there exists $\tilde{v}(x) \in C^2(B_\infty)$ such that $\tilde{v}_n \to \tilde{v}$ in $C^2_{loc}(B_\infty)$ and

$$\begin{cases} -\Delta\tilde{v} + a\tilde{v} = S\tilde{v}^p, & \text{in } B_\infty \\ \frac{\partial\tilde{v}}{\partial\nu} = 0 & \text{on } \partial\Omega_\infty \cap \partial B_\infty \\ 0 \leq \tilde{v} \leq 1, \; \tilde{v}(0) = 1. \end{cases}$$

If

$$\lim_{n\to\infty} \frac{d(P_n, \partial\Omega)}{\epsilon_n} = \infty,$$

then $B_\infty = \mathbf{R}^N$. By Pohozaev's identity, $a = 0$, and $\tilde{v} = S^{\frac{1}{p-1}}U(x)$. Thus,

$$\begin{aligned} S = \int_{\mathbf{R}^N} |\nabla\tilde{v}|^2 dx &\leq \liminf_{n\to\infty} \int_{B_n} |\nabla\tilde{v}_n|^2 dx \\ &\leq \limsup_{n\to\infty} \int_{A_n} |\nabla v_n|^2 dx \\ &\leq \frac{1}{3}\lim_{n\to\infty} E_{\lambda_n}(u_n) \leq \frac{1}{3}S, \end{aligned}$$

a contradiction.

If

$$\lim_{n\to\infty} \frac{d(P_n, \partial\Omega)}{\epsilon_n} = \beta < \infty,$$

then $P_n \to P_0 \in \partial\Omega$ and $B_\infty = \mathbf{R}^N_+$. We then get $\tilde{v}(x) = S^{\frac{1}{p-1}} U(x)$ again. Then

$$\begin{aligned} \frac{S}{2} = \int_{\mathbf{R}^N_+} |\nabla \tilde{v}|^2 dx &\le \liminf_{n\to\infty} \int_{B_n} |\nabla \tilde{v}_n|^2 dx \\ &\le \limsup_{n\to\infty} \int_{A_n} |\nabla v_n|^2 dx \\ &\le \frac{1}{3} \lim_{n\to\infty} E_{\lambda_n}(u_n) \le \frac{1}{3} S, \end{aligned}$$

a contradiction again. Thus we have proved the claim (3.7).

It follows that $P_n \to 0$ as $n \to \infty$.

Next we repeat the rescaling again and define for $x \in \Omega_n := \frac{\Omega - \{P_n\}}{\epsilon_n}$

$$\tilde{u}_n(x) = \epsilon_n^{\frac{N-2}{2}} u_n(\epsilon_n x + P_n).$$

Then

$$\begin{cases} -\Delta \tilde{u}_n + \lambda_n \epsilon_n^2 \tilde{u}_n = m_n \tilde{u}_n^p, & \text{in } \Omega_n \\ \frac{\partial \tilde{u}_n}{\partial \nu} = 0 & \text{on } \partial\Omega_n \\ 0 \le \tilde{u}_n \le 1, \ \tilde{u}_n(0) = 1. \end{cases}$$

By elliptic theory ([GT]) again, $\tilde{u}_n \to \tilde{u}$ in $C^2_{loc}(\Omega_\infty)$, and $0 \le \tilde{u}(x) \le 1, \ \tilde{u}(0) = 1$. Also

$$\int_{\Omega_\infty} |\nabla \tilde{u}|^2 dx \le \liminf_{n\to\infty} \int_{\Omega_n} |\nabla \tilde{u}_n|^2 dx \le \limsup_{n\to\infty} \int_{\Omega} |\nabla u_n|^2 dx \le S. \tag{3.8}$$

$$\begin{cases} -\Delta \tilde{u} + a\tilde{u} = S\tilde{u}^p, & \text{in } \Omega_\infty \\ \frac{\partial \tilde{u}}{\partial \nu} = 0 & \text{on } \partial\Omega_\infty \\ 0 \le \tilde{u} \le 1, \ \tilde{u}(0) = 1. \end{cases} \tag{3.9}$$

By (3.7), $\Omega_\infty = \mathbf{R}^N$. And thus $a = 0$, $\tilde{u} = U(x)$ by Pohozaev's identity.

Next, we show $P_n = 0$. If $P_n \ne 0$ then again we argue that there exists $\tilde{P}_0 \in S^{N-1}$, such that $\tilde{P}_n = \frac{P_n}{|P_n|} \to \tilde{P}_0$ as $n \to \infty$ and that there exists $g \in G$ such that $g\tilde{P}_0 \ne \tilde{P}_0$. Therefore for n large $gP_n \ne P_n$. By (3.7) $y_n := \frac{gP_n - P_n}{\epsilon_n}$ are bounded. Because $\tilde{u}_n(x)$ also achieves its maximum at y_n and because $\tilde{u}_n(x) \to U(x)$ in $C^2_{loc}(\mathbf{R}^N)$ we conclude $y_n \to 0$ as $n \to \infty$. Since $\tilde{u}_n$ achieves a minimum on the line segment between 0 and y_n (which are two maxima) there exist b_n which lie on the line segment between 0 and y_n such that

$$\frac{\partial \tilde{u}_n}{\partial e_n}(b_n) = 0, \ \ \frac{\partial^2 \tilde{u}_n}{\partial (e_n)^2}(b_n) \ge 0,$$

where $e_n = \frac{gP_n - P_n}{||gP_n - P_n||} \in S^{N-1}$. But since $b_n \to 0$, $e_n \to e_0 \in S^N$, we get

$$0 = \lim_{n\to\infty} \frac{\partial \tilde{u}_n}{\partial e_n}(b_n) = \frac{\partial U}{\partial e_0}(0),$$

and

$$0 \le \lim_{n\to\infty} \frac{\partial^2 \tilde{u}_n}{\partial (e_n)^2}(b_n) = \frac{\partial^2 U}{\partial (e_0)^2}(0),$$

which is a contradiction for $\frac{\partial^2 U}{\partial (e_0)^2}(0) < 0$. Thus $P_n = 0$. Then it follows from (3.8) that

$$\lim_{n\to\infty} ||\nabla(u_n - S^{\frac{-1}{p-1}} U_{\epsilon_n,0})||_2 = 0.$$

The proof of Lemma 3.1 is complete now. □

Proof of Theorem 3.1. First we shall show there exists $0 < \lambda_0 < \infty$ such that $m_{\lambda,G}(\Omega)$ is not achieved for $\lambda_0 < \lambda$. If this is not the case, we get a sequence $\lambda_n \to \infty$ and $u_n \in V_G(\Omega)$ such that $m_{\lambda_n,G}(\Omega) = E_{\lambda_n}(u_n)$. By Lemma 3.1, there exist $\delta_n \to 0$ such that (3.2) holds for u_n. Then $v_n := S^{\frac{1}{p-1}} u_n$ satisfies (3.3) with $P_n = 0$. Applying Lemma 3.2 we have $\epsilon_n > 0$, $y_n \in \Omega$ such that $\frac{\epsilon_n}{\delta_n} \to 1, y_n \to 0$ and

$$E_{\lambda_n}\left(\frac{v_n}{||v_n||_{p+1}}\right) \ge E_{\lambda_n}\left(\frac{U_{\epsilon_n,y_n}}{||U_{\epsilon_n,y_n}||_{p+1}}\right) + O(\epsilon_n^2) + o(\lambda_n \epsilon_n^2), \tag{3.10}$$

$$\lambda_n \epsilon_n^2 = O(\epsilon_n). \tag{3.11}$$

On the other hand, noting that $||v_n||_{p+1} = S^{\frac{1}{p-1}}$ we have

$$E_{\lambda_n}(u_n) = S^{\frac{-2}{p-1}} \int_\Omega |\nabla v_n|^2 + \lambda_n v_n^2 = E_{\lambda_n}\left(\frac{v_n}{||v_n||_{p+1}}\right). \tag{3.12}$$

Noting that $y_n \to 0 \in \Omega$, by applying Lemma 3.3, we get for n large

$$E_{\lambda_n}(u_n) > S,$$

which is a contradiction. Therefore we conclude that there exists $0 < \lambda_0 < \infty$ such that for $0 < \lambda < \lambda_0$, $m_{\lambda,G}(\Omega) < S$, and for $\lambda_0 < \lambda$, $m_{\lambda,G}(\Omega) = S$ and $m_{\lambda,G}(\Omega)$ is not achieved. Next by Lemma 2.4, we get for $\lambda < \lambda_0$, $m_{\lambda,G}$ is achieved. □

We finish this section by giving an estimate of λ_0 in terms of the domain.

Corollary 3.1. *Under the assumption of Theorem 3.1, $\lambda_0 \ge |\Omega|^{-\frac{2}{N}} S$, where $|\Omega|$ is the volume of Ω.*

Proof. Note that $v = |\Omega|^{-\frac{1}{p+1}} \in V_G(\Omega)$. We get $m_{\lambda,G}(\Omega) \le E_\lambda(v) = \lambda|\Omega|^{\frac{2}{N}}$. Since for $\lambda \ge \lambda_0$ we have from Theorem 3.1 that $m_{\lambda,G}(\Omega) = S$ we get $\lambda_0 \ge |\Omega|^{-\frac{2}{N}} S$. □

As a corollary of Theorem 3.1 we have the following inequality.

Corollary 3.2. *Let Ω be a G-invariant domain satisfying $k_G(\Omega) \ge 3$ and $0 \in \Omega$. Then there exists $C_0 > 0$ such that for all $u \in H^1_G(\Omega)$*

$$||u||^2_{p+1} \le S^{-1}||\nabla u||^2_2 + C_0||u||^2_2.$$

Remark 3.2. Our result here implies that if $k_G(\Omega) \ge 3$ the global minimization problem (2.2) can not give solutions to (2.3) when λ is larger. We have initiated in [Wz4] and [MSW] a local minimization approach to construct multi-peaked solutions in this case.

4. More on the case of $k_G(\Omega) \ge 3$

In this section we shall consider a situation that complements the one in Section 3 in the sense that we assume that $\Omega \subset \mathbf{R}^N$ is a bounded smooth G-invariant domain satisfying $k_G(\Omega) \ge 3$ and

$$0 \notin \overline{\Omega}. \tag{4.1}$$

We also assume in this section Assumption (S).

Theorem 4.1. *Let Ω be a bounded smooth G-invariant domain with $k_G(\Omega) \ge 3$ and satisfy Assumption (S) and $0 \notin \overline{\Omega}$. Then*
(i) $m_{\lambda,G}(\Omega) < [k_G(\Omega)]^{\frac{2}{N}} 2^{-\frac{2}{N}} S$ for all $\lambda > 0$.
(ii) $m_{\lambda,G}(\Omega)$ is achieved for all $\lambda > 0$.
(iii) $\lim_{\lambda\to\infty} m_{\lambda,G}(\Omega) = [k_G(\Omega)]^{\frac{2}{N}} 2^{-\frac{2}{N}} S$.

Lemma 4.1. *Assume $k_G(\Omega) \ge 3$. If $0 \notin \overline{\Omega}$, then E_λ satisfies $(PS)_c$ condition for all $c < [k_G(\Omega)]^{\frac{2}{N}} 2^{-\frac{2}{N}} S$. This especially implies that when $k_G(\Omega) = \infty$, E_λ satisfies $(PS)_c$ condition for all c.*

Proof. Let $u_n \subset V_G(\Omega)$ be a $(PS)_c$ sequence for some $c < [k_G(\Omega)]^{\frac{2}{N}} 2^{-\frac{2}{N}} S$. We may assume $u_n \ge 0$ for all $n \ge 0$. By Lemma 2.4 there exist integers $l_1 \ge 0$, $l_2 \ge 0$, a solution $u_0 \ge 0$ of

$$\begin{cases} -\Delta u + \lambda u = cu^p, & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

and $x_{n,j} \in \Omega$, $\epsilon_{n,j} \to 0$, as $n \to \infty$, for $j = 1, 2, ..., l_1 + l_2$, such that

$$\left\| u_n - u_0 - c^{\frac{1}{1-p}} \sum_{j=1}^{l_1+l_2} U_{\epsilon_{n,j}, x_{n,j}} \right\|_{W^{1,2}(\Omega)} \to 0, \quad \text{as } n \to \infty, \tag{4.2}$$

and

$$c = \int_\Omega (|\nabla u_0|^2 + \lambda u_0^2)dx + l_1 c^{\frac{2}{1-p}} \int_{\mathbf{R}^N} |\nabla U|^2 dx + l_2 c^{\frac{2}{1-p}} \int_{\mathbf{R}^N_+} |\nabla U|^2 dx. \tag{4.3}$$

Since for all $x \in \Omega$, $\#G(x) \geq k_G(\Omega)$, and since $\{x_{n,j}\}$ appear in orbits, we have $l_1 \geq k_G(\Omega)$ or $l_2 \geq k_G(\Omega)$ if they are not zero. However, if $l_1 \geq k_G(\Omega)$ or $l_2 \geq k_G(\Omega)$ using (4.3) we get

$$c \geq [k_G(\Omega)]^{\frac{2}{N}} 2^{-\frac{2}{N}} S$$

which is a contradiction. □

Proof of Theorem 4.1. (i). This follows from Lemma 2.1 and similar arguments as in the proof of Lemma 2.2 and 2.3.

(ii). This follows from (i) and Lemma 4.1.

(iii). The proof follows from the following lemma whose proof is similar to that of Lemma 3.1 and is omitted here. □

Lemma 4.2. *Let Ω be a bounded smooth domain and be G-invariant with $k_G(\Omega) \geq 3$. Assume that Assumption (S) is satisfied and that $0 \notin \overline{\Omega}$. If for a sequence λ_n as $n \to \infty$ $m_{\lambda_n,G}(\Omega)$ is achieved at $u_n \in V_G(\Omega)$, then there exists $P_n \in \partial\Omega$ with $\#G(P_n) = k_G(\Omega)$, $\delta_n \to 0$, such that*

(i) u_n achieves its maximum over $\overline{\Omega}$ only at gP_n for $g \in G$.

(ii)

$$\lim_{n\to\infty} ||\nabla(u_n - S^{\frac{-1}{p-1}} \sum_{g\in G} U_{\delta_n, gP_n})||_2 = 0. \tag{4.4}$$

Corollary 4.1. *Under the assumptions of Theorem 4.1, equation (2.3) always has a G-least energy solution for $\lambda > 0$. Moreover, for λ sufficiently large the G-least energy solutions are $k_G(\Omega)$-peaked on $\partial\Omega$.*

Remark 4.1. The above corollary generalizes a result in [Wz1] where Z_k-invariant domains were considered. The $k-$peakedness of these solutions can be proved using the arguments in [Wz1].

Corollary 4.2. *Let Ω be a G-invariant domain satisfying Assumption (S) with $k_G(\Omega) \geq 3$ and $0 \notin \overline{\Omega}$. Then for any $\epsilon > 0$ there exists $C(\epsilon) > 0$ such that for all $u \in H^1_G(\Omega)$*

$$||u||^2_{p+1} \leq \left([k_G(\Omega)]^{-\frac{2}{N}} 2^{\frac{2}{N}} S^{-1} + \epsilon\right) ||\nabla u||_2^2 + C(\epsilon)||u||_2^2, \tag{4.5}$$

where $[k_G(\Omega)]^{-\frac{2}{N}}$ is understood to be zero when $k_G(\Omega) = \infty$.

Corollary 4.3. *Let Ω be a G-invariant domain satisfying Assumption (S) and $0 \notin \overline{\Omega}$. Assume $k_G(\Omega) = \infty$. Then the embedding from $H^1_G(\Omega)$ into $L^{p+1}(\Omega)$ is compact.*

Proof. Let $u_n \in H^1_G(\Omega)$ be a bounded sequence and assume u_n weakly converges to $u \in H^1(\Omega)$. Then by the Sobolev embedding there exists a convergent subsequence (still denoted by $\{u_n\}$) in $L^2(\Omega)$, say, $u_n \to u$ in $L^2(\Omega)$. Now we show that u_n converges in $L^{p+1}(\Omega)$ too. Let $\delta > 0$. We may first find an $\epsilon > 0$ such that $\epsilon||\nabla(u_n - u)||_2^2 < \frac{\delta}{2}$. By Corollary 4.2, there is $C(\epsilon) > 0$ such that (4.5) holds. For this $C(\epsilon) > 0$ there exists $M > 0$ such that for $n > M$, $C(\epsilon)||u_n - u||_2^2 < \frac{\delta}{2}$. Thus for $n > M$, $||u_n - u||_{p+1}^2 < \delta$. □

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