MULTIPLICITY FOR SYMMETRIC INDEFINITE FUNCTIONALS: APPLICATION TO HAMILTONIAN AND ELLIPTIC SYSTEMS

by

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[1]Partially supported by Fondecyt Grant 1960-698, Cátedra Presidencial and FONDAP de Matemáticas Aplicadas
[2]Partially supported by a NSF grant and FONDAP de Matemáticas Aplicadas.

0 Introduction

In this article we study existence of critical points for certain superquadratic strongly indefinite even functionals appearing in the study of periodic solutions of Hamiltonian Systems and solutions of certain class of Elliptic Systems.

We first present two abstract critical point theorems for even functionals. These results are well suited for our applications, but that they are interesting by their own. These theorems are then applied to the specific problems we mentioned, when the corresponding Hamiltonians are superquadratic. Critical points theory for even indefinite functionals were studied by Benci in [2], where he developed a general pseudo index theory, a variant of the classical genus theory (c.f. [10]). We want to point out that our approach is totally different from [2]. Taking advantages of the even nature of the functional we shall construct a geometric linking structure which does not require conditions near the origin (see [10] for examples of some standard linking and compare the set up of our linking in Theorem 1.1). Using this linking structure we first establish a critical point theorem for even functionals in a semi-definite setting, using only usual deformation arguments and degree theory for odd operators. Then a Galerkin approximation gives a version of our theorem for strongly indefinite functionals. Since the linking conditions for our functionals are not in a standard form it seems unclear whether one could check the assumptions in [2] in our settings. Furthermore, when applied to specific problems, our theory gives better estimates for the energy level of the obtained critical points.

Next, let us start discussing our results for the case of a Hamiltonian system

$$\begin{aligned} \dot{p} &= -H_q(p,q,t) \\ \dot{q} &= H_p(p,q,t), \end{aligned} \tag{0.1}$$

where $H : R^{2N} \times R$ is a C^1 function, T-periodic in the variable t. We are interested in finding T-periodic solutions in case the Hamiltonian is superquadratic and even in the space-momentum variables.

A usual growth hypothesis on the Hamiltonian, under which results for (0.1) has been obtained, is

(S) There exist $R > 0, \mu > 2$ such that

$$\frac{1}{\mu} H_z(z,t) \cdot z \geq H(z,t) > 0 \quad \forall z \in R^{2n},\ |z| \geq R,\ \forall t \in R.$$

This condition requires H to be superquadratic in all components of the variable z. For example, this condition excludes the case of a Hamiltonian of the form

$$H(p,q,t) = \frac{1}{2}|p|^2 + V(q,t),$$

that corresponds to a second order Hamiltonian system. Usually this case is treated separately. In [5] the first author gave a condition that includes both situation considered above under which the existence of a nontrivial T-periodic solution of (0.1) is proved. The condition given in [5] (see (H2) below) requires only a combined effect of superquadratic nature in p and q with $z = (p,q)$.

On the other hand, for Hamiltonians satisfying the superquadratic condition (S) and being even in space-momentum variables, results showing the existence of infinitely many T-periodic solutions are given by Benci [2].

Thus, an interesting question is whether the new superquadratic condition proposed in [5] with even Hamiltonian is enough to obtain these multiplicity results. In this paper we show that the answer is yes.

Let us consider next our result on Hamiltonian systems in a precise way. We assume that the Hamiltonian satisfies the following hypotheses

(H1) H is of class C^1 and $H(z, t+T) = H(z,t) \quad \forall z \in R^{2N}, \quad \forall t \in R,$

(H2) $\exists R > 0, \alpha > 1, \beta > 1, \frac{1}{\alpha} + \frac{1}{\beta} < 1$, such that

$$\frac{1}{\alpha} H_p(p,q,t) \cdot p + \frac{1}{\beta} H_q(p,q,t) \cdot q \geq H(p,q,t) > 0$$

$$\forall z = (p,q) \in R^{2N}, |z| \geq R,\ \forall t \in R,$$

(H3) $\exists b > 0, c \geq 0$ and $\alpha' > 2$ such that

$$|H_z(z,t)| \leq b|z|^{\alpha'-1} + c \quad \forall z = (p,q) \in R^{2N},\ \forall t \in R.$$

(H4) $H(z,t) = H(-z,t) \quad$ for all $\quad z \in R^{2N}, t \in R.$

We will prove the following

Theorem 0.1 *Assume the hypotheses (H1)-(H4) are satisfied then the Hamiltonian system (0.1) possesses infinitely many solutions* $\{(p_k, q_k)\}$ *such that* $\|(p_k, q_k)\|_\infty \to \infty$ *as* $k \to \infty$. *Moreover,* $\|(p_k, q_k)\|_\infty \geq ck^{1/(\alpha'-2)}$, *for some* $c > 0$.

Remark 0.1 Hypothesis (H3) can be weakened. We can consider the following assumption instead

(H3') $\exists b > 0, c \geq 0$ such that

$$|H_z(z,t)| \leq b(\frac{1}{\alpha}H_p(p,q,t) \cdot p + \frac{1}{\beta}H_q(p,q,t) \cdot q) + c$$

$$\forall z = (p,q) \in R^{2n}, \ \forall t \in R.$$

See remark at the end of Section 2.

Finally, as another application of our abstract theory we present the results obtained in the case of a class of elliptic systems. Precisely we are interested in the existence of solutions for the system

$$-\Delta u = \frac{\partial H}{\partial v}(u,v,x) \quad \text{in} \quad \Omega \tag{0.2}$$

$$-\Delta v = \frac{\partial H}{\partial u}(u,v,x) \quad \text{in} \quad \Omega \tag{0.3}$$

$$u = 0, \ v = 0 \text{ on } \partial\Omega. \tag{0.4}$$

where Ω is a bounded open subset of R^N, with smooth boundary $\partial\Omega$, and the function $H : R^2 \times \bar{\Omega} \to R$, which we call the Hamiltonian, is of class C^1. For easy reference we call this system (ES).

This system has been already studied from the variational point of view by many authors. We mention the work of Hulshof and van der Vorst [8] where assumptions were made on H allowing the existence of positive solutions for (ES). In a paralell work de Figueiredo and Felmer [6] obtained similar results, however here the full superquadratic range was reached. With respect to multiplicity of solutions when the Hamiltonian is even we only know the work by Angenent and van der Vorst [1] covering the case both H_u and H_v are superlinear, but the not whole superquadratic range. In [1] a Morse Index Theory is developed based on Floer Cohomology Theory.

Now we describe precisely our hypotheses on H.

(E1) $H : R^2 \times \bar{\Omega} \to R$ is of class C^1.

Let us consider real constants

$$p \geq \alpha > p-1 > 0 \text{ and } q \geq \beta > q-1 > 0$$

such that

$$\text{(i)} \qquad \frac{1}{\alpha} + \frac{1}{\beta} < 1$$

$$\text{(ii)} \qquad \{2 - (\frac{1}{p} + \frac{1}{q})\} \max\{\frac{p}{\alpha}, \frac{q}{\beta}\} < 1 + \frac{2}{N}$$

$$\text{(iii)} \qquad \frac{p-1}{p}\frac{q}{\beta} < 1 \text{ and } \frac{q-1}{q}\frac{p}{\alpha} < 1$$

In this paper we will always assume $N \geq 3$. If $N = 2$ or $N = 1$ less restrictive assumptions can be made.

Furthermore, in case $N \geq 5$ we also impose

$$\text{(iv)} \qquad (1 - \frac{1}{p}) \max\{\frac{p}{\alpha}, \frac{q}{\beta}\} < \frac{N+4}{2N} \quad \text{and} \quad (1 - \frac{1}{q}) \max\{\frac{p}{\alpha}, \frac{q}{\beta}\} < \frac{N+4}{2N}.$$

With these constants α, β, p, q satisfying the above conditions (i) - (iv) we now state the other hypotheses on the Hamiltonian H:

(E2) There exists $R > 0$ such that

$$\frac{1}{\alpha}\frac{\partial H}{\partial u}(u, v, x) \cdot u + \frac{1}{\beta}\frac{\partial H}{v}(u, v, x) \cdot v \geq H(u, v, x) > 0$$

for all $(u, v) \in R^2$, $\mid (u, v) \mid \geq R$ and $x \in \bar{\Omega}$.

(E3) There exists $a_2 > 0$ such that

$$\mid \frac{\partial H}{\partial u}(u, v, x) \mid \leq a_2(\mid u \mid^{p-1} + \mid v \mid^{\frac{(p-1)q}{p}} + 1)$$

$$\mid \frac{\partial H}{\partial v}(u, v, x) \mid \leq a_2(\mid u \mid^{q-1} + \mid u \mid^{\frac{(q-1)q}{p}} + 1).$$

(E4) $H(u, v, x) = H(-u, -v, x)$ for all $u, v \in R, x \in R^N$.

Under these general hypotheses on the Hamiltonian H we can prove the following existence result.

Theorem 0.2 *If H satisfies (E1)-(E4) then system (ES) possesses a sequence of strong solutions $\{(u_k, v_k)\}$ such that $\|(u_k, v_k)\| \to \infty$ as $k \to \infty$.*

Here $\| \cdot \|$ represents the norm in the space $W^{2,\frac{p}{p-1}}(\Omega) \cap W_0^{1,\frac{p}{p-1}}(\Omega) \times W^{2,\frac{q}{q-1}}(\Omega) \cap W_0^{1,\frac{q}{q-1}}(\Omega)$.

Remark 0.1 If $H(u,v) = |u|^\alpha + |v|^\beta$ then one could use a fourth order approach and then assumption (iv) would not be necessary; we do not carry the details. See [3]. We do not know if (iv) can be avoided for general Hamiltonians.

This paper is organized in three sections. In Section 1 we present our critical point theorems. Assuming that the functional satisfies some linking condition that does not involve the origin we prove our first theorem on existence of critical points. This requires a semi-finite dimensional splitting. Next the infinite dimensional case is studied using a Galerkin approximation. In Section 2 we study T-periodic solutions of the Hamiltonian system (0.1). We show the existence of infinitely many solutions by applying the general theorems from Section 1. Finally, in Section 3 we analyze the elliptic system (ES) and we show the existence of infinitely many strong solutions.

Acknoledgement. Z.-Q. Wang thanks the hospitality of the faculty members of Departamento de Ingeniería Matemática F.C.F.M., Universidad de Chile, where this work was initiated during his visit.

1 Abstract Critical Points Theorems

In this section we prove two critical point theorems for even functionals. The proofs of these theorems are based on usual deformation arguments and degree theory applied to odd operators. These theorems can be used to find multiple critical points for strongly indefinite functionals as we see in the next two sections.

Let us consider a Hilbert space E with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Assume E has a splitting $E = X \oplus Y$ with $k = \dim X < \infty$.

Then we define the basic sets over which we define later a linking. For $\rho > 0$ we define

$$S_\rho = \{y \in Y \mid \|y\| = \rho\}, \tag{1.1}$$

and for some fixed $y_1 \in Y$ with $\|y_1\| = 1$ and subspaces X_1 and X_2, we consider

$$X \oplus \operatorname{span}\{y_1\} = X_1 \oplus X_2.$$

We assume that $y_1 \in X_2$. Next we define for $M, \sigma > 0$

$$D_{M,\sigma} = \{x_1 + x_2 \in X_1 \oplus X_2 \mid \ \|x_1\| \le M, \|x_2\| \le \sigma\}. \tag{1.2}$$

Now we can state our first abstract theorem.

Theorem 1.1 *Let $I \in C^1(E,R)$ be an even functional satisfying the Palais-Smale condition. Assume there are two linear bounded, invertible operators $L_1, L_2 : E \to E$, and choose $\rho > 0$ and $\sigma > 0$, such that $\sigma\|L_1^{-1}L_2 y_1\| > \rho$.*

Further assume there are numbers $\alpha \leq \beta$ such that

$$\inf_{L_1(S_\rho)} I \geq \alpha, \tag{1.3}$$

$$\sup_{L_2(\partial D_{M,\sigma})} I < \alpha \quad \textit{and} \tag{1.4}$$

$$\sup_{L_2(D_{M,\sigma})} I \leq \beta, \tag{1.5}$$

then I has a critical value $c \in [\alpha, \beta]$.

Proof. Suppose for contradiction that I has no critical values in the interval $[\alpha, \beta]$. Then given $\varepsilon > 0$ so that

$$\sup_{L_2(\partial D_{M,\sigma})} I < \alpha - 2\varepsilon,$$

there is a deformation $\eta : [0,1] \times E \to E$ satisfying:

i) $\eta(t, \cdot)$ is an odd homeomorphism for all $t \in [0,1]$,

ii) $\eta(0, x) = x$ for all $x \in E$,

iii) $\eta(t, x) = x$ for all $x \in I^{\alpha - 2\varepsilon}$ and

iv) $\eta(1, I^{\beta+\varepsilon}) \subset I^{\alpha-\varepsilon}$.

See [10]. Here, and in what follows, $I^{\alpha-\varepsilon} = \{x \in E \mid I(x) \leq \alpha - \varepsilon\}$ is the level set, and we drop the subindices from S_ρ and $D_{M,\sigma}$.

Now we first choose $a \geq 1$ such that $ay_1 \in D \setminus \partial D$, $a||L_1^{-1}L_2 y_1|| > \rho$, and $I(L_2(ay_1)) \leq \alpha - 2\epsilon$. By the assumption on ρ and σ and (1.4) this is possible. Then we define a minimax value

$$c = \inf_{\gamma \in \Gamma} \sup_{z \in Q} I(\eta_1(\gamma(L_2 z))), \tag{1.6}$$

where $\eta_1(\cdot) = \eta(1, \cdot)$, $Q = \{z \in D \mid z = x + sy_1, s \geq 0\}$ and

$$\Gamma = \{\gamma \in C(L_2 Q, E) \mid \gamma|_{L_2(\partial Q)} = Id,\ \gamma|_{L_2(Q) \cap I^{\alpha - 2\epsilon}} = Id\}. \tag{1.7}$$

Next we claim that the following intersection property holds:

$$\eta_1(\gamma(L_2 Q)) \cap L_1 S \neq \emptyset \quad \text{for all} \quad \gamma \in \Gamma.$$

Assuming for the moment the claim is true. Then we get $c \geq \alpha$ and, by the definition of η_1, we have for any $\gamma \in \Gamma$,

$$\sup_{\partial Q} I(\eta_1(\gamma(L_2 z))) = \sup_{\partial Q} I(\eta_1(L_2 z)) \leq \alpha - \varepsilon.$$

Then by a usual deformation argument c is a critical value of I, see [10]. Moreover, using the properties of the deformation η_1 it is easy to see that we have

$$c \leq \sup_Q I(\eta_1(L_2 x)) \leq \sup_Q I(L_2 x) \leq \beta.$$

This is a contradiction. Thus it only remains to prove the claim. If the claim is not true, then there exists $\gamma_0 \in \Gamma$, such that $\eta_1(\gamma_0(L_2 Q)) \cap L_1 S = \emptyset$, that is

$$L_1^{-1}\eta_1(\gamma_0(L_2 Q)) \cap S = \emptyset. \tag{1.8}$$

Let us consider a homotopy map for $t \in [0,1]$

$$F_t : B_\rho \times \partial Q \to E$$

defined as

$$F_t(y,z) = y - L_1^{-1}\eta_1\gamma_0 L_2 \psi(t\varphi(z)),$$

where $B_\rho = \{y \in Y \mid \|y\| \leq \rho\}$, $\psi : B^{k+1} \to Q$ and $\varphi : Q \to B^{k+1}$ are homeomorphisms such that $\psi(0) = ay_1$ and $\psi \circ \varphi = Id$, with $B^{k+1} = \{u \in R^{k+1} \mid \|u\| \leq 1\}$. By (1.8) we have that for all $t \in [0,1]$ and $(y,z) \in \partial(B_\rho \times \partial Q) = S \times \partial Q$,

$$F_t(y,z) \neq 0.$$

Therefore the Leray-Schauder degree

$$\deg(F_t, B_\rho \times \partial Q, 0) = c \quad \text{for all } t \in [0,1],$$

for some $c \in Z$. With the choices of ρ, σ and a, we have $\|L_1^{-1}L_2(ay_1)\| \neq \rho$. Thus for $t = 0$ we have $F_0(y,z) = y - L_1^{-1}\eta_1\gamma_0 L_2(ay_1) = y - L_1^{-1}L_2(ay_1) \neq 0$, so that $\deg(F_0, B_\rho \times \partial Q, 0) = 0$. On the other hand, at $t = 1$ we have $F_1(y,z) = y - L_1^{-1}\eta_1\gamma_0 L_2 z = y - L_1^{-1}\eta_1 L_2 z$. Setting $T = \{z \in \partial Q \mid z = x + sy_1, s > 0\}$ we have $\sup\limits_{L_2 T} I = \sup\limits_{L_2(\partial D)} I < \alpha - 2\epsilon$. Thus, for $(y,z) \in B_\rho \times T$,

$$F_1(y,z) = y - L_1^{-1}L_2 z \neq 0.$$

We can then apply the excision property of the Leray-Schauder degree, to obtain

$$\deg(F_1, B_\rho \times \partial Q, 0) = \deg(F_1, B_\rho \times B_1 \times B_2, 0),$$

where $B_1 \times B_2 = \partial Q \backslash T$, with $B_1 = \{x_1 \in X_1 \mid \|x_1\| \leq M\}$ and $B_2 = \{x_2 \in \tilde{X}_2 \mid \|x_2\| \leq \sigma\}$. Here $X_1 \oplus \tilde{X}_2 = X$. Note that $B_\rho \times B_1 \times B_2$ is a symmetric neighborhood of 0 and that $F_1(y,z) = y - L_1^{-1}\eta_1 L_2(z)$ is an odd map and a compact perturbation of identity there. Then by the Borsuk-Ulam Theorem,

$$\deg(F_1, B_\rho \times B_1 \times B_2, 0) \neq 0,$$

giving a contradiction. The proof is now complete.□

In our application we will need an infinite dimensional variant of Theorem 1.1 based on the Galerkin approximation. Let $E = X \oplus Y$ be a splitting of E where both X and Y are infinite dimensional. Assume we have sequences of finite dimensional subspaces $X_n \subset X$, $Y_n \subset Y$ $E_n = X_n \oplus Y_n$ for $n \geq 1$ so that $\overline{\bigcup_{n=1}^{\infty} E_n} = E$. Let L_1, L_2 be as before and $S, D, \partial D$ as defined earlier with ρ, σ, M, and y_1 not depending on n. Instead of the Palais-Smale condition we will require that I satisfies the $(PS)^*$ condition that we recall next: $I \in C^1(E, R)$ satisfies $(PS)^*$ condition on E with respect to E_n, if any sequence $z_\ell \in E_{n_\ell}$ with $n_\ell \to \infty$ as $\ell \to \infty$, satisfying $(I|_{E_n})'(z_n) \to 0$ and $I(z_n) \to c$ has a convergent subsequence in E.

Theorem 1.2 *Let $I \in C^1(E, R)$ be an even functional satisfying the $(PS)^*$ condition on E with respect to E_n. Assume $L_i E_n = E_n$, for $i = 1, 2$ and n large and let $\rho > 0, \sigma > 0$ such that $\sigma \|L_1^{-1} L_2 y_1\| > \rho$. Assume there are constants $\alpha \leq \beta$ such that for all n large*

$$\inf_{L_1(S \cap E_n)} I \geq \alpha, \tag{1.9}$$

$$\sup_{L_2((\partial D) \cap E_n)} I < \alpha \quad \textit{and} \tag{1.10}$$

$$\sup_{L_2(D \cap E_n)} I \leq \beta, \tag{1.11}$$

then I has a critical value $c \in [\alpha, \beta]$.

Proof. Note first that since I satisfies (PS)* condition on E with respect to E_n, I satisfies (PS) condition on E_n for n large. Applying Theorem 1.1 on each subspace E_n, for n large, we get a sequence of points $z_n \in E_n$ satisfying

$$I(z_n) \in [\alpha, \beta] \quad \text{and} \quad (I|_{E_n})'(z_n) = 0.$$

By the $(PS)^*$ condition, z_n has a convergent subsequence which converges to a critical point of I in the range $[\alpha, \beta]$.□

Remark 1.1 *Looking at the proof of Theorem 1.1, we see that it suffices to assume L_1, L_2 are odd homeomorphisms of E. Similar idea of the proof was used in [9] and [11].*

2 Superquadratic Hamiltonian Systems

This section is devoted to give a proof of Theorem 0.1, that is we prove the existence of infinitely many solutions of the Hamiltonian System (0.1). For this purpose we use Theorem 1.2. Let us start introducing some basic notation.

Let $E = W^{\frac{1}{2},2}(S^1, R^{2N})$ be the Sobolev space of functions $z \in L^2(S^1, R^{2N})$ satisfying

$$\pi \sum_{j \in Z} |j| \; |a_j|^2 < \infty,$$

where $z(t) = \sum_{j \in Z} a_j e^{ijt}$, with $a_{-j} = \bar{a}_j \in C^{2N}$. For $z, \eta \in E$ we consider the inner product

$$\langle z, \eta \rangle = \pi \sum_{j \in Z - \{0\}} |j| a_j \cdot \bar{b}_j + 2\pi a_0 \cdot \bar{b}_0, \tag{2.1}$$

where $\eta(t) = \sum_{j \in Z} b_j e^{ijt}$ with $b_{-j} = \bar{b}_j \in C^{2N}$. We denote by $\| \cdot \|$ the norm associated to the inner product $\langle \cdot, \cdot \rangle$.

For given smooth $z = (p, q)$ and $\eta = (\phi, \psi)$ in E we define the bilinear and quadratic form B and Q as

$$B(z, \eta) = \int_0^{2\pi} (p \cdot \dot{\psi} + \phi \cdot \dot{q}) dt \quad \text{and} \quad Q(z) = \frac{1}{2} B(z, z). \tag{2.2}$$

Both Q and B can be extended continuously to the whole space E, and the bilinear form B induces a linear, bounded, selfadjoint operator $L : E \to E$ defined by

$$B(z, \eta) = \langle Lz, \eta \rangle, \quad \text{for all} \quad z, \eta \in E. \tag{2.3}$$

We consider following subspaces of E

$$E_+^j = \text{span}\{\sin(jt)e_k - \cos(jt)e_{k+N}, \cos(jt)e_k + \sin(jt)e_{k+N}, \quad k = 1, ..., N\}, \tag{2.4}$$

$$E_-^j = \text{span}\{\sin(jt)e_k + \cos(jt)e_{k+N}, \cos(jt)e_k - \sin(jt)e_{k+N}, \quad k = 1, ..., N\}, \tag{2.5}$$

for $j \in N \setminus \{0\}$, and

$$E_0 = \text{span}\{e_1, ..., e_{2N}\}. \tag{2.6}$$

Here $\{e_1, ..., e_{2N}\}$ is the canonical basis in R^{2N}. Defining the subspaces E_+ and E_- as

$$E_+ = \bigoplus_{j \geq 1} E_+^j \quad \text{and} \quad E_- = \bigoplus_{j \geq 1} E_-^j, \tag{2.7}$$

we have the splitting $E = E_+ \oplus E_- \oplus E_0$. We observe that E_+, E_-, and E_0 are the positive, negative, and null eigenspace of the linear operator L, respectively. Consequently Q is positive on E_+, negative on E_-, and it vanishes on E_0. We see that on the space E the norm can be written as

$$\|z\|^2 = Q(z_+) - Q(z_-) + |z_0|^2 \tag{2.8}$$

where $z = z_+ + z_- + z_0$ with $z_+ \in E_+, z_- \in E_-$ and $z_0 \in E_0$.

In order to find 2π-periodic solutions of (0.1) we consider the functional $I : E \to R$ defined as

$$I(z) = Q(z) - \int_0^{2\pi} H(z,t)dt, \tag{2.9}$$

which is of class C^1 in E thanks to hypotheses (H1) and (H3), see [10]. It is well known that the 2π-periodic solutions of (0.1) are the critical points of the functional I.

Next we show that I satisfies the necessary compactness assumptions. We consider the subspaces $E_n = \bigoplus_{1\le j\le n} (E_+^j \oplus E_-^j) \oplus E_0$, for $n \ge 1$, and we see that $\overline{\bigcup_{n=1}^{\infty} E_n} = E$. We study the $(PS)^*$ condition for I on E with respect to this sequence of subspaces.

Lemma 2.1 *The functional I satisfies $(PS)^*$ condition on E with respect to the family of subspaces $\{E_n\}$.*

Proof. The proof is similar to Lemma 1.2 in [5], where (PS) condition is proved for a related functional. Thus we will be sketchy here. Let $z_\ell \in E_{n_\ell}$ be such that $I(z_\ell) \to c$ and $(I|_{E_{n_\ell}})'(z_\ell) \to 0$ as $\ell \to \infty$. We write z_ℓ as $z = (p,q)$ for simplicity. For ℓ large we have

$$\begin{aligned} c + o(1) + o(1)\|p\| &= I(z) - (I|_{E_{n_\ell}})'(z) \cdot p \\ &= -\int_0^{2\pi} H(z,t) - H_p(z,t) \cdot p, \end{aligned}$$

and similarly,

$$c + o(1) + o(1)\|q\| = -\int_0^{2\pi} H(z,t) - H_q(z,t) \cdot q.$$

Here $o(1) \to 0$ as $\ell \to \infty$. Then using $(H2)$ we get

$$\frac{1}{\alpha}\|p\| + \frac{1}{\beta}\|q\| \ge (1 - (\frac{1}{\alpha} + \frac{1}{\beta})) \int_0^{2\pi} H(z,t)dt - C, \tag{2.10}$$

where C denotes here and in what follows an appropriate constant.

On the other hand, it is proved in [5] that (H2) implies

$$H(p,q,t) \geq C(|p|^\alpha + |q|^\beta) - C, \;\; \forall (p,q) \in R^{2N}, \;\; t \in R. \tag{2.11}$$

Thus, from (2.10) and (2.11), we find

$$\|p\| + \|q\| \geq C(\|p\|_\alpha^\alpha + \|q\|_\beta^\beta) - C.$$

Here $\|\cdot\|_p$ denotes the usual norm in $L^p(S^1)$. Next, using that $|(I|_{E_{n_\ell}})'(z)z_+| \leq \|z_+\|$, where $z = z_+ + z_- + z_0$ with $z_+ \in E_+, z_- \in E_-, z_0 \in E_0$ we find that

$$\|z_+\| \leq C(1 + \|p\|_\alpha^{\alpha-1} + \|q\|_\beta^{\beta-1}).$$

Similarly we find

$$\|z_-\| \leq C(1 + \|p\|_\alpha^{\alpha-1} + \|q\|_\beta^{\beta-1}).$$

Putting all this together, we obtain that z_ℓ is bounded in E. Note that

$$(I|_{E_{n_\ell}})'(z) = Lz - P_{n_\ell} g'(z), \tag{2.12}$$

where P_{n_ℓ} is the orthogonal projection from E to E_{n_ℓ} and

$$g(z) = -\int_0^{2\pi} H(z,t)dt$$

satisfies that $g'(z)$ is compact. Since z_ℓ is bounded we have that $(z_\ell)_0$ has a convergent subsequence in E_0 then from (2.12) and the fact that g' is compact we get $z_\ell - (z_\ell)_0$ has a convergent subsequence in E. □

To use Theorem 1.2 we need to check the linking conditions (1.9), (1.10) and (1.11) in Theorem 1.2. To that end, let k be a fixed integer and let us consider for $n \geq k$ in E_n the subspaces

$$X_n = \bigoplus_{1\leq j\leq n} E_-^j \oplus E_0 \oplus \bigoplus_{1\leq j\leq k-1} E_+^j \quad \text{and} \tag{2.13}$$

$$Y_n = \bigoplus_{k\leq j\leq n} E_+^j. \tag{2.14}$$

We observe that $E_n = X_n \oplus Y_n$. We start proving (1.9) and for this we consider the operator L_1 as the identity.

Lemma 2.2 *There exist $\alpha_k > 0$ and $\rho_k > 0$ both independent of $n \geq k$ such that for all $n \geq k$,*

$$\inf_{z \in L_1(S_{\rho_k} \cap Y_n)} I(z) \geq \alpha_k \tag{2.15}$$

where $S_{\rho_k} = \{y \in E_+ \mid \|y\| = \rho_k\}$. Furthermore, we have $\alpha_k \to \infty$ as $k \to \infty$.

Proof. We first recall that E is embedded in $L^\gamma(S^1, R^{2N})$ for any $\gamma \in [1, \infty)$, so that there is $a(\gamma) > 0$ such that

$$\|z\|_\gamma \leq a(\gamma)\|z\| \quad \text{for all} \quad z \in E, \tag{2.16}$$

and also that for all $z \in \bigoplus_{k \leq j} E_+^j$, we have

$$\|z\| \geq \sqrt{k}\|z\|_2. \tag{2.17}$$

We see that from (H3) we have for all $(p,q) \in R^{2N}, t \in R$

$$|H(p,q,t)| \leq C(|z|^{\alpha'} + 1). \tag{2.18}$$

Now, consider $z = (p,q) \in Y_n$ then for a constant $a > 0$ depending only on α' we have

$$\|z\|_{\alpha'}^{\alpha'} \leq \|z\|_2 \|z\|_{2(\alpha'-1)}^{\alpha'-1} \leq \frac{a}{\sqrt{k}}\|z\|\|z\|^{\alpha'-1}.$$

Then for $z = L_1(p,q)$ we have

$$\begin{aligned} I(z) &\geq \|z\|^2 - C(\|z\|_{\alpha'}^{\alpha'} + 1) \\ &\geq \|z\|^2 - C(\frac{a}{\sqrt{k}}\|z\|^{\alpha'} + 1). \end{aligned}$$

Then we choose ρ_k as

$$\rho_k = (\frac{\sqrt{k}}{2Ca})^{1/(\alpha'-2)}. \tag{2.19}$$

Then, for $z \in S_{\rho_k} \cap Y_n$ we find that

$$I(z) \geq \frac{1}{2}(\frac{\sqrt{k}}{2Ca})^{2/(\alpha'-2)} - C. \tag{2.20}$$

Defining α_k as the right hand side of (2.20) and noting that both ρ_k and α_k are independent of $n \geq k$, we complete the proof of the lemma.□

Our next goal is to prove (1.10) and (1.11). For that purpose let us define the operator $L_\sigma : E \to E$ as

$$L_\sigma(z) = (\sigma^{u-1}p, \sigma^{v-1}q), \tag{2.21}$$

for $z = (p,q) \in E$, and given numbers $\sigma > 0, u > 1, v > 1$. It is easy to see that the operator L_σ is linear, bounded and invertible. Using Fourier series it is also easy to prove the following lemma.

Lemma 2.3 *$L_\sigma(E_n) \subset E_n$ for any $n \geq 1$.*

Now we choose the numbers $u > 1, v > 1$ satisfying

$$\frac{1}{\alpha} < \frac{u}{u+1} \quad \text{and} \quad \frac{1}{\beta} < \frac{v}{v+1},$$

where α and β are given in (H2). We have

Lemma 2.4 *There exist positive numbers β_k, σ_k and M_k satisfying $\sigma_k > \rho_k$ and all independent of $n \geq k$, such that for all $n \geq k$*

$$\sup_{L_{\sigma_k}(\partial D \cap E_n)} I \leq 0 \quad \textit{and} \quad \sup_{L_{\sigma_k}(D \cap E_n)} I \leq \beta_k, \tag{2.22}$$

where

$$D = \{z \in E_- \oplus E_0 \oplus \bigoplus_{1 \leq j \leq k} E_+^j \ \mid \ \|z_- + z_0\| \leq M_k, \|z_+\| \leq \sigma_k\}. \tag{2.23}$$

Proof. From (H2) and (2.11) we see that for all $z = (p,q) \in E$ we have

$$\int_0^{2\pi} H(z,t)dt \geq C(\|p\|_\alpha^\alpha + \|q\|_\beta^\beta) - 2\pi C.$$

Let $z = L_\sigma(p,q)$ with $(p,q) \in D$ with M_k and $\sigma = \sigma_k$ to be determined in a moment. Then $z = (\sigma^{u-1}p_+, \sigma^{v-1}q_+) + (\sigma^{u-1}p_-, \sigma^{v-1}q_-) + (\sigma^{u-1}p_0, \sigma^{v-1}q_0)$, and

$$\begin{aligned} I(z) \quad \leq \quad & \sigma^{u+v-2}(\|(p_+,q_+)\|^2 - \|(p_-,q_-)\|^2) - \\ & C(\sigma^{(u-1)\alpha}\|p_+ + p_- + p_0\|_\alpha^\alpha + \sigma^{(v-1)\beta}\|q_+ + q_- + q_0\|_\beta^\beta) + 2\pi C. \end{aligned} \tag{2.24}$$

By continuity of the corresponding projections given $(p,q) = w = w_0 + w_+ + w_-$ with $w_0 \in E_0, w_+ \in E_+, w_- \in E_-$ we have

$$\|p_0\|_\alpha \leq C_\alpha \|p\|_\alpha, \qquad \|q_0\|_\beta \leq C_\beta \|q\|_\beta$$

and

$$\|p_+ + p_-\|_\alpha \leq C_\alpha \|p\|_\alpha, \qquad \|q_+ + q_-\|_\beta \leq C_\beta \|q\|_\beta.$$

Thus, from (2.24) it follows that

$$\begin{aligned} I(z) \quad \leq \quad & \sigma^{u+v-2}(\|(p_+,q_+)\|^2 - \|(p_-,q_-)\|^2) - \\ & C\{\sigma^{(u-1)\alpha}(\|p_+ + p_-\|_\alpha^\alpha + |p_0|^\alpha) + \sigma^{(v-1)\beta}(\|q_+ + q_-\|_\beta^\beta + |q_0|^\beta)\} + 2\pi C. \end{aligned}$$

Assume, without lose of generality, that $\alpha \leq \beta$. Then we have by Hölder inequality

$$\|p_+\|_\alpha \leq \frac{1}{2}\|p_+ + p_-\|_\alpha + C\|p_+ - p_-\|_\beta.$$

And then, using a theorem of M. Riesz we find that

$$\|p_+\|_\alpha \leq \frac{1}{2}\|p_+ + p_-\|_\alpha + C\|q_+ + q_-\|_\beta. \tag{2.25}$$

See [4] and note that $\bar{p}_+ = q_+$ and $\bar{p}_- = -q_-$, conjugate in the sense given in [4]. Since $p_+ \in \bigoplus_{1\leq j\leq k}(E_+^j + E_-^j)$, a finite dimensional subspace, there exists $C(k) > 0$ such that

$$\|p_+\| \leq C(k)\|p_+\|_\alpha. \tag{2.26}$$

Then we get from (2.25) and (2.26) that

$$\|p_+\| \leq C(\|p_+ + p_-\|_\alpha + \|q_+ + q_-\|_\beta),$$

where C depends on k. This implies the existence of a constant $C > 0$ such that either

$$\|q_+ + q_-\|_\beta \geq C\|p_+\|, \tag{2.27}$$

or

$$\|p_+ + p_-\|_\alpha \geq C\|p_+\|. \tag{2.28}$$

Setting $\sigma = \|(p_+, q_+)\|$, in case (2.27) holds we get

$$I(z) \leq \sigma^{u+v} - C(\sigma^{u\alpha} - 1),$$

or in case (2.28) holds

$$I(z) \leq \sigma^{u+v} - C(\sigma^{v\beta} - 1).$$

In both cases we can choose $\sigma_k = \sigma$ such that if $\|(p_+, q_+)\| = \sigma_k$ then

$$I(z) \leq 0.$$

We may assume $\sigma_k > \rho_k$(given in Lemma 2.2). For this fixed $\sigma_k > 0$, we have

$$I(z) \leq \sigma_k^{u+v} - \sigma_k^{u+v-2}\|(p_-, q_-)\|^2 - C(\sigma_k^{(u-1)\alpha}|p_0|^\alpha + \sigma_k^{(v-1)\beta}|q_0|^\beta).$$

Thus we may choose M_k large enough so that when

$$\|(p_-, q_-)\|^2 + |p_0|^2 + |q_0|^2 = M_k^2$$

$I(z) \leq 0$ also holds. Thus we have proved the first inequality in (2.22).

Finally for fixed $M_k > 0, \sigma_k > 0$ as above take β_k simply as the maximum value of I over $L_{\sigma_k}(D \cap E_n)$. Note that all these arguments are independent of $n \geq k$. This finishes the proof.□

Now we are in a position to prove Theorem 0.1.

Proof of Theorem 0.1 For a given $k \geq 1$, Lemmas 2.1-2.4 allow us to use Theorem 1.2. First we choose $y_1 \in E_+^k$ with $||y_1|| = 1$. Then it is easy to check that with $L_1 = Id$ and L_2 is given in (2.21) with $\sigma = \sigma_k$ we have

$$\sigma_k ||L_1^{-1} L_2 y_1|| = (\sigma_k)^{\frac{u+v}{2}} \geq \sigma_k > \rho_k.$$

Then the functional I possesses a critical value $c_k \in [\alpha_k, \beta_k]$. Since $\alpha_k \to \infty$ as $k \to 0$, we get infinitely many critical values of I. Therefore infinitely solutions of (0.1).□

Remark 2.1 As mentioned in Remark 0.1, hypothesis (H3) can be replaced by (H3'). In order to prove this theorem we need to redefine the Hamiltonian outside a given ball in R^{2N}, to obtain a polynomial growth at infinity. Then using (H3'), one can show that the obtained critical point is inside the given ball. We do not carry out the details. See [5].

3 Superquadratic Elliptic Systems

In this section we give the proof of Theorem 0.2. We follow the pattern used in Section 2, with some modifications, that is we set up a functional analytic framework where strong solutions of (ES) correspond to critical points of a functional. Then we show that this functional satisfies the compactness assumption of Theorem 1.2 and also the geometric conditions (1.9), (1.10) and (1.11).

We shall consider the spaces E^s, which are obtained as the domains of fractional powers of the operator

$$-\triangle : H^2(\Omega) \cap H_0^1(\Omega) \subset L^2(\Omega) \to L^2(\Omega)$$

where $\triangle$ denotes the Laplacian and $H^2(\Omega)$, $H_0^1(\Omega)$ are the usual Sobolev spaces. Namely $E^s = D((-\triangle)^{\frac{s}{2}})$ for $0 \leq s \leq 2$, and the corresponding operator is denoted by A^s

$$A^s : E^s \to L^2(\Omega).$$

The spaces E^s are Hilbert spaces with inner product

$$(u, v)_{E^s} = \int_\Omega A^s u A^s v \, dx. \tag{3.1}$$

Its associated norm is denoted by $\| u \|_{E^s}^2$. In E^s we find the Poincaré's inequality for the operator A^s

$$\| A^s u \|_{L^2(\Omega)} \geq \lambda_1^{\frac{s}{2}} \| u \|_{L^2(\Omega)} \quad \forall u \in E^s \tag{3.2}$$

where λ_1 is the first eigenvalue of $-\triangle$.

Next we define the spaces on which we set up the problem. For numbers $s > 0$ and $t > 0$ with $s + t = 2$ we define the Hilbert space $E = E^s \times E^t$ and the bilinear form $B : E \times E \to R$ by the formula

$$B((u,v),(\phi,\psi)) = \int_\Omega A^s u A^t \psi + A^s \phi A^t v \, dx. \tag{3.3}$$

The form B is continuous and symmetric and there exists a selfadjoint bounded linear operator $L : E \to E$ so that

$$B(z,\eta) = (Lz,\eta)_E \tag{3.4}$$

for all $z, \eta \in E$. Here $(\cdot,\cdot)_E$ denotes the natural inner product in E induced by E^s and E^t. We can also define the quadratic form $Q : E \to R$ associated to B and L as

$$Q(z) = \frac{1}{2}(Lz,z)_E = \int_\Omega A^s u A^t v \, dx \tag{3.5}$$

for all $z = (u,v) \in E$. We can define the subspaces

$$E^+ = \{(u, A^{-t}A^s u) \mid u \in E^s\}$$

and

$$E^- = \{(u, -A^{-t}A^s u) \mid u \in E^s\}$$

which give a natural splitting $E = E^+ \oplus E^-$. The spaces E^+ and E^- are the positive and negative eigenspaces of L, they are consequently orthogonal with respect to the bilinear form B, that is we also find that

$$\frac{1}{2} \| z \|_E^2 = Q(z^+) - Q(z^-) \tag{3.6}$$

where $z = z^+ + z^-$, $z_\pm \in E^\pm$.

Next we define the functional associated to the Hamiltonian. Now we will choose the numbers s and t defining the orders of the Sobolev spaces involved. From inequality (ii) in the Introduction we see the existence of $s, t \in R$, $s + t = 2$ such that

$$(1 - \frac{1}{p}) \max\{\frac{p}{\alpha}, \frac{q}{\beta}\} < \frac{1}{2} + \frac{s}{N} \tag{3.7}$$

and

$$(1 - \frac{1}{q}) \max\{\frac{p}{\alpha}, \frac{q}{\beta}\} < \frac{1}{2} + \frac{t}{N}. \tag{3.8}$$

Using (iii) and (iv) if $N \geq 5$, we can choose $s > 0$ and $t > 0$. From the fact that $\frac{p}{\alpha} \geq 1$ and $\frac{q}{\beta} \geq 1$ we find, following from (3.7) and (3.8) that

$$\frac{1}{p} > \frac{1}{2} - \frac{s}{N} \qquad \text{and} \qquad \frac{1}{q} > \frac{1}{2} - \frac{t}{N}. \tag{3.9}$$

These last inequalities and Sobolev Embedding Theorem give the compact inclusions

$$E^s \to L^p(\Omega), \qquad E^t \to L^q(\Omega).$$

Now we can define a functional $\Phi : E \to R$ as

$$I(z) = Q(z) - g(z) = \int_\Omega A^s u A^t v \, dx - \int_\Omega H(u(x), v(x), x) \, dx. \tag{3.10}$$

for $z = (u, v) \in E$. The functional I is of class C^1 and the functional g has a compact gradient. We refer the reader for details and proof of the aspects discussed so far to [6]. In particular see in [6] that critical points of I correspond to the strong solutions of (ES).

Now we define a Galerkin scheme in order to apply Theorem 1.2. We consider a basis of $L^2(\Omega)$ constituted by eigenfunctions $\{\phi_j\}$ of

$$-\triangle\phi = \lambda\phi \qquad \text{in } \Omega$$

$$\phi = 0 \qquad \text{on } \partial\Omega$$

with associated eigenvalues $\{\lambda_j\}$. If $u \in L^2(\Omega)$ we write

$$u = \sum_{j=1}^{\infty} a_j \phi_j$$

its Fourier series with respect to the basis $\{\phi_j\}$. Then we define for $n \geq 1$

$$E_n^s = \{u \in L^2(\Omega) \mid \sum_{j=1}^{n} \lambda_j^s \mid a_j \mid^2 < \infty\} \quad \text{and} \quad E_n^t = \{u \in L^2(\Omega) \mid \sum_{j=1}^{n} \lambda_j^t \mid a_j \mid^2 < \infty\}. \tag{3.11}$$

Then we consider $E_n = E_n^s \times E_n^t$, and we see that $\overline{\bigcup_{n=1}^{\infty} E_n} = E$. The following lemma gives the compactness needed for I.

Lemma 3.1 *The functional I satisfies $(PS)^*$ condition on E with respect to the family of subspaces $\{E_n\}$.*

Proof. The proof of this lemma is easily obtained with minor modifications of Proposition 2.1 in [6]. We omit the details. □

Next we fix a $k \in N$ and define a splitting of space E_n for $n \geq k$. Let

$$X_n = \bigoplus_{1\leq j\leq n} E_-^j \oplus \bigoplus_{1\leq j\leq k-1} E_+^j \quad \text{and} \quad Y_n = \bigoplus_{k\leq j\leq n} E_+^j, \tag{3.12}$$

where $E_+^j = \{(u, A^{-t}A^s u) \,|\, u = a_j\phi_j\}$ and $E_-^j = \{(u, -A^{-t}A^s u) \,|\, u = a_j\phi_j\}$. We observe that $E_n = X_n \oplus Y_n$. As in Section 2, we consider the operator L_1 as the identity. We have

Lemma 3.2 *There exist $\alpha_k > 0$ and $\rho_k > 0$ independent of $n \geq k$ such that for all $n \geq k$,*

$$\inf_{z\in L_1(S_{\rho_k}\cap Y_n)} I(z) \geq \alpha_k \tag{3.13}$$

where $S_{\rho_k} = \{y \in E_+ \mid \|y\| = \rho_k\}$. Furthermore, $\alpha_k \to \infty$ as $k \to \infty$.

Proof. The proof is obtained by minor modifications of Lemma 2.2. We note that we need to replace (2.17) by the following: given $z \in \bigoplus_{k\leq j\leq n} E_+^j$ then

$$\|z\|_E \geq \max\{\lambda_k^s, \lambda_k^t\}\|z\|_{L^2}.$$

We note that the sequence of eigenvalues λ_k diverges to ∞ as $k \to \infty$.□

Next we define the operator $L_\sigma : E \to E$ as

$$L_\sigma(z) = (\sigma^{\mu-1}u, \sigma^{\nu-1}v), \tag{3.14}$$

for $z = (u, v) \in E$, and where σ will be given later and μ and ν are chosen so that

$$\frac{1}{\alpha} < \frac{\mu}{\mu+1} \quad \text{and} \quad \frac{1}{\beta} < \frac{\nu}{\nu+1},$$

where α and β are given in (H2). It is easy to see that $L_\sigma(E_n) \subset E_n$ for any $n \geq 1$. We have

Lemma 3.3 *There exist positive numbers β_k, σ_k and M_k, which are all independent of $n \geq k$ and satisfy $\sigma_k > \rho_k$, such that*

$$\sup_{L_{\sigma_k}(\partial D\cap E_n)} I \leq 0 \quad \textit{and} \quad \sup_{L_{\sigma_k}(D\cap E_n)} I \leq \beta_k, \tag{3.15}$$

where

$$D = \{z \in E_- \oplus \bigoplus_{1\leq j\leq k} E_+^j \mid \|z_-\|_E \leq M_k, \|z_+\|_E \leq \sigma_k\}. \tag{3.16}$$

Proof. Let us consider $z = L_\sigma(u,v)$ with $(u,v) \in D$. Then we can write $z = (\sigma^{\mu-1}u_+, \sigma^{\nu-1}v_+) + (\sigma^{\mu-1}u_-, \sigma^{\nu-1}v_-)$. Then, using the definition of Q and the spaces E_+ and E_- we have

$$Q(z) = \sigma^{\mu+\nu-2}(\|z_+\|_E^2 - \|z_-\|_E^2) \tag{3.17}$$

On the other hand, from (E2) we have

$$\int_\Omega H(z,x)\,dx \geq C\{\int_\Omega (\sigma^{\alpha(\mu-1)} \mid u_+ + u_- \mid^\alpha + \sigma^{\beta(\nu-1)} \mid v_+ + v_- \mid^\beta)\,dx - \mid \Omega \mid\}. \tag{3.18}$$

The functions u_+ and u_- can be written as

$$u_+ = \sum_{i=1}^k \alpha_i \phi_i \quad \text{and} \quad u_- = \sum_{i=1}^k \gamma_i \phi_i + \hat{u}_-,$$

where $\hat{u}_-$ is orthogonal to ϕ_i, $i = 1, ..., k$ in the L^2 sense. Then using Holder inequality we obtain

$$\sum_{i=1}^k \lambda_i^{s-t}(\alpha_i^2 + \alpha_i\gamma_i) = \int_\Omega (u_+ + u_-)A^{s-t}u_+ \leq \|u_+ + u_-\|_{L^\alpha(\Omega)} \|A^{s-t}u_+\|_{L^{\alpha'}(\Omega)}. \tag{3.19}$$

Then, there exists a constant C_k such that

$$\sum_{i=1}^k \lambda_i^{s-t}(\alpha_i^2 + \alpha_i\gamma_i) \leq C_k \|u_+ + u_-\|_{L^\alpha(\Omega)} \|u_+\|_{L^2(\Omega)}. \tag{3.20}$$

Similarly, since $v_+ = A^{s-t}u_+$ and $v_- = -A^{s-t}u_-$ we have

$$\sum_{i=1}^k \lambda_i^{s-t}(\alpha_i^2 - \alpha_i\gamma_i) = \int_\Omega (v_+ + v_-)u_+ \leq \|v_+ + v_-\|_{L^\beta(\Omega)} \|u_+\|_{L^{\beta'}(\Omega)}. \tag{3.21}$$

Then there is a constant C_k so that

$$\sum_{i=1}^k \lambda_i^{s-t}(\alpha_i^2 - \alpha_i\gamma_i) \leq C_k \|v_+ + v_-\|_{L^\beta(\Omega)} \|u_+\|_{L^2(\Omega)}. \tag{3.22}$$

Depending on the sign of $\sum_{i=1}^k \lambda_i^{s-t}\alpha_i\gamma_i$ we use (3.20) or (3.22) to conclude that

$$\|u_+\|_{L^2(\Omega)} \leq C_k \|u_+ + u_-\|_{L^\alpha(\Omega)} \quad \text{or} \quad \|u_+\|_{L^2(\Omega)} \leq C_k \|v_+ + v_-\|_{L^\beta(\Omega)},$$

and then

$$I(z) \leq \sigma^{\mu+\nu-2}(\|z_+\|_E^2 - C_k \sigma^{\alpha(\mu-1)} \|u_+\|_{L^2(\Omega)}^\alpha) + C|\Omega|$$

or

$$I(z) \leq \sigma^{\mu+\nu-2}(\|z_+\|_E^2 - C_k \sigma^{\beta(\nu-1)} \|u_+\|_{L^2(\Omega)}^\beta) + C|\Omega|.$$

Thus, we may choose $\|z_+\|_E = \sigma_k$ large enough so that $\sigma_k > \rho_k$ and $I(z) \leq 0$. Then taking $\|z_+\|_E \leq \sigma_k$

$$I(z) \leq \|z_+\|_E^{\mu+\nu} - \|z_+\|_E^{\mu+\nu-2}\|(u_-, v_-)\|_E^2 + C|\Omega|$$

and then chosing $\|(u_-, v_-)\|_E^2 = M_k$ large enough we find that

$$I(z) \leq 0.$$

In this way we finished with the proof of the first part of Lemma 3.3. Next we choose β_k so the second inequality holds.□

Proof of Theorem 0.2 For a given $k \geq 1$, Lemmas 3.1-3.3 allow us to use Theorem 1.2. Then the functional I possesses a critical value $c_k \in [\alpha_k, \beta_k]$. Since $\alpha_k \to \infty$ as $k \to \infty$, we get infinitely many critical values of I, therefore infinitely solutions of (ES).□

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