NONLINEAR ELLIPTIC EQUATIONS ON EXPANDING SYMMETRIC DOMAINS

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Abstract

In this article we study the problem

$$\begin{cases} -\Delta u + u = u^p & \text{in} \quad \Omega_a \\ u > 0, & \\ u = 0, & \text{on} \quad \partial\Omega_a \end{cases}$$

in the case Ω_a is an expanding domain. In particular, for $n \geq 2$ when $\Omega_a = \{x \in \mathbf{R}^n : a < |x| < a+1\}$ is an expanding annulus as $a \to \infty$, we prove the existence of many rotationally non-equivalent solutions obtained as local minimizers of the corresponding energy functional. Moreover, we study the exact symmetry and the shape of these solutions, and under certain conditions we prove the existence of solutions with prescribed symmetry.

1 Introduction

In this paper we discuss the existence of nonradial solutions for the semilinear elliptic equation

$$\begin{cases} -\Delta u + u = u^p & \text{in} \quad \Omega_a \\ u > 0, & \\ u = 0, & \text{on} \quad \partial\Omega_a \end{cases} \tag{1}$$

where

$$\Omega_a \subset \mathbf{R}^n, \qquad n \geq 2$$

is an expanding domain as $a \to \infty$, with smooth boundary having certain symmetries (to be specified later) and

$$1 < p < \frac{n+2}{n-2}.$$

Our study was motivated by [3], [9], [19] (see also [8] and [10]) where many nonradial solutions for (1) are found in the case Ω_a is an annulus. The method used there is to restrict the energy functional to subspaces of functions with symmetries. When further restricted to the Nehari manifold, the absolute minimum of the functional is achieved and gives a solution of (1). Using this method it is due to Coffman [3] for $n = 2$ and to Li [9] for $n \geq 4$ that the number of rotationally nonequivalent solutions tends to infinity as $a \to \infty$. For $n = 3$, the same method is employed in [15] to give a classification of solutions obtained as least energy critical points. It is shown that as $a \to \infty$, only six different types of symmetries allow distinct ground state solutions. The difficulty for $n = 3$ is that the class of symmetries available is not so rich. The question whether a result similar to the case $n = 2$ and $n \geq 4$ holds has not been settled ([9], [15]) until Byeon [2] gives an affirmative answer. We were unaware of [2] until after we submitted this paper and we thank B. Kawohl, Y.Y. Li and the referee for informing us the work of [2] and for their helpful comments.

In this paper we introduce a new approach to the problem and show how to construct new solutions for (1) that are obtained as local minimizers of the energy. We must point out that, in general, solutions given in this paper cannot be obtained as global minimizers of the energy functional. (See remarks in 4.1.) Our method works for any $n \geq 2$ and in particular a consequence of one of our main results (Theorem 2.22) gives the following (see Theorem 1.1) that recovers the result of Byeon. Though the idea of local minimization was used in [2], our approach is different from [2] and stems out of our study for nonlinear Neumann problems ([20] [14]) and for nonlinear Schrödinger equations([21]) where similar phenomena occur.

Theorem 1.1 *For $n = 3$ and $\Omega_a = \{x \in \mathbf{R}^n : a < |x| < a+1\}$, the number of rotationally nonequivalent solutions of (1) tends to infinity as $a \to \infty$.*

Another advantage of our method is that we can get qualitative properties of the solutions constructed such as the shape of solutions and the exact symmetry of solutions. We prove that all solutions obtained are multi-bump solutions with a discrete number of bumps. These qualitative properties in turn enable us to study the exact symmetry of the solutions. Under certain conditions we can construct nonradial solutions with prescribed symmetry. This question was not addressed at all in the previous papers on this problem with the exception [8] where for $n = 2$ the symmetry of global minimizers was examined (See Remark 3.7). In particular, as an example of another main result of this paper (Theorem 3.2) we have the following which also implies Theorem 1.1. Let $D_k \subset \mathbf{O}(2)$ be the group fixing a regular k polygon in $\mathbf{R}^2$.

Theorem 1.2 *Let $n \geq 2$ and $\Omega_a = \{x \in \mathbf{R}^n : a < |x| < a+1\}$. Then for each $k \geq 3$ there is a_k such that for all $a > a_k$ (1) has a solution which has exact symmetry $D_k \times \mathbf{O}(n-2)$.*

Here we define the isotropy subgroup Σ_u for a function $u \in H_0^1(\Omega_a)$ by $\Sigma_u = \{g \in \mathbf{O}(n) \mid u(g^{-1}x) = u(x),\ a.e. \text{ in } \Omega_a\}$ and we say that u has exact symmetry $G \subset \mathbf{O}(n)$ if $\Sigma_u = G$. Elliptic boundary value problems, which are radially invariant (i.e., $\mathbf{O}(n)$-invariant) and which do have nonradial solutions, have been explored in recent years (e.g., [1] [16] [20] [14] and references mentioned above on (1)), and for this type of problems it is natural to ask the general questions that whether one can identify the exact symmetry of a given solution and whether one can find solutions having prescribed symmetry. One of our goals of this paper is to address these questions for problem (1).

Our main results will be given in Sections 2 and 3. See Theorem 2.22 for more details on the qualitative properties of solutions, and Theorem 3.2 and Remark 3.6 for more examples of groups which can be prescribed as the exact symmetry of solutions for (1).

In the following we consider the case Ω_a is an annulus and we shall indicate how the results can be extended to more general domains in Section 4. Thus, let $\Omega_a = \{x \in \mathbf{R}^n : a < |x| < a+1\}$. We write

$$\mathbf{R}^n = \mathbf{R}^l \times \mathbf{R}^{n-l} \text{ and } x = (X_1, X_2)$$

for some integer $1 \leq l \leq n$, $X_1 \in \mathbf{R}^l$, and $X_2 \in \mathbf{R}^{n-l}$. We define

$$P : \mathbf{R}^n \to \mathbf{R}^l, \text{ by } Px = X_1.$$

Denote by $\mathbf{O}(n)$, the orthogonal group of $\mathbf{R}^n$. Let G_1 a subgroup of $\mathbf{O}(l) \subset \mathbf{O}(n)$ and G_2 a subgroup of $\mathbf{O}(n-l) \subset \mathbf{O}(n)$ such that for an integer $k \geq 2$ (which will be fixed throughout) we have:

$$(\mathbf{S_1}) \begin{cases} Fix_{\mathbf{R}^n}(G_1) = \{0\} \times \mathbf{R}^{n-l}, \\ Fix_{\mathbf{R}^n}(G_2) = \mathbf{R}^l \times \{0\}, \\ k = \min_{x \in S^{l-1}} \#G_1 x, \end{cases}$$

where $\#G_1 x$ is the cardinal number of the G_1-orbit of x. We shall establish the existence of k-bump solutions of (1) which are symmetric under

$$G = G_1 \times G_2 = \{(g_1, g_2) : g_i \in G_i, i = 1, 2\} \subset \mathbf{O}(l) \times \mathbf{O}(n-l) \subset \mathbf{O}(n).$$

Example 1.3 *For $n = 3$, take $l = 2$, $G_1 = \mathbf{Z}_k$ rotations of multiples of $2\pi/k$ around the x_3-axis (or $D_k \subset \mathbf{O}(2)$ the group fixing a regular k polygon in $\mathbf{R}^2$), and $G_2 = \mathbf{O}(1)$ symmetry about the x_1x_2-plane.*

We make the following notations,

$$\mathcal{H}_{G,a} = \{u \in H_0^1(\Omega_a) : gu = u, \text{ for any } g \in G\} \tag{2}$$

where by gu we understand the function $(gu)(x) = u(g^{-1}x)$.

$$\mathcal{M}_{G,a} = \{u \in \mathcal{H}_{G,a} : \int_{\Omega_a} |u|^{p+1} = 1\} \tag{3}$$

and

$$\mathcal{K}_{G,a}^{\sigma} = \{u \in \mathcal{M}_{G,a} : \ \gamma_a(u) > \sigma\} \tag{4}$$

where

$$\gamma_a(u) = \frac{1}{a+1} \int_{\Omega_a} |Px||u|^{p+1}(x)\ dx. \tag{5}$$

Here $0 < \sigma < 1$, will be specified later. Note that γ_a is a continuous functional on $H_0^1(\Omega_a)$ and $\mathcal{K}_{G,a}^{\sigma}$ is an open subset of $\mathcal{M}_{G,a}$. We are looking for critical points of the functional

$$E_a(u) = \int_{\Omega_a} |\nabla u|^2 + u^2, \text{ with } u \in \mathcal{K}_{G,a}^{\sigma}. \tag{6}$$

By [17], a critical point in $\mathcal{M}_{G,a}$ of E_a after a rescaling will be a critical point of E_a in $H_0^1(\Omega_a)$ also, and by regularity theory will be a classical solution of (1). Denote

$$m_{a,\sigma} = \inf_{u \in \mathcal{K}_{G,a}^{\sigma}} E_a(u). \tag{7}$$

The requirement that $u \in \mathcal{K}_{G,a}^{\sigma}$ makes possible to avoid the least energy in the minimization process; nevertheless, a complication arises, namely we have to show that $m_{a,\sigma}$ is achieved in the interior of $\mathcal{K}_{G,a}^{\sigma}$.

The outline of the paper is as follows. In Section 2 using a local minimization procedure as described above we shall establish the existence theory of multi-bump solutions. Theorem 1.1 will be a special case of our main result. In Section 3, based on the qualitative properties of solutions given in Section 2 we study the exact symmetry of these solutions, giving Theorem 1.2 as a special case. We finish the paper with some remarks and some possible extensions of the main results.

2 Existence

2.1 Preliminaries

Let D be the strip of width 1, $D = \mathbf{R}^{n-1} \times (0,1) \subset \mathbf{R}^n$. Denote

$$\mathcal{N} = \{u \in H_0^1(D) : \int_D |u|^{p+1} = 1\}. \tag{8}$$

Using the methods in [22] it is not difficult to show that the loss of compactness due to invariance under translations can be overcome, and

$$S = \inf_{u \in \mathcal{N}} \int_D |\nabla u|^2 + u^2 \tag{9}$$

is achieved by a function $v \in \mathcal{N}$. It is known (see [18], [22]) that $\bar{v} = S^{\frac{1}{p-1}} v$ is a classical solution for the problem

$$\begin{cases} -\Delta u + u = u^p & \text{in} \quad D \\ u > 0, & \\ u = 0, & \text{on} \quad \partial D \end{cases} \tag{10}$$

and any solution of (10), eventually after a translation, must satisfy (see [4], [5])

$$v = v(|x'|, x_n) \text{ where } x' = (x_1, ..., x_{n-1}), \tag{11}$$

$$\frac{\partial v(r', x_n)}{\partial r'} < 0 \text{ for } r' = |x'| > 0, \tag{12}$$

$$\frac{\partial v(r', x_n)}{\partial x_n} > 0 \text{ and } v(r', x_n) = v(r', 1 - x_n) \text{ for } 0 < x_n < 1/2. \tag{13}$$

Let $1 < r < a$. For $y \in S_a^{n-1}$ (the $(n-1)$-dimensional sphere centered at the origin and radius a) we construct a map $\varphi = \varphi_{a,r,y}$ from $\Omega_a \cap B_r(y)$ to D. First, suppose $y = (0, 0, ..., 0, a) = N$. For $x \in \Omega_a \cap B_r(y)$, define

$$\psi(x) = (x_1, x_2, ..., x_{n-1}, |x| - a). \tag{14}$$

Note that ψ is a diffeomorphism on its image and

$$\psi^{-1}(x) = (x_1, x_2, ..., x_{n-1}, \sqrt{(a + x_n)^2 - x_1^2 - ... - x_{n-1}^2}). \tag{15}$$

If $y \neq (0, 0, ..., 0, a)$, consider R_y a rotation in $\mathbf{SO}(n)$ that takes y into N. Let

$$R_y = (\alpha_{ij})_{i,j=1...n}.$$

We define $\varphi(x) = \psi(R_y \cdot x)$. Note that there is an ambiguity in the construction of φ in the sense that R_y is not unique. Nevertheless, we shall see that this does not affect the results.

Proposition 2.1 *Let $J_\varphi(x)$ the Jacobian of $\varphi_{a,r,y}$ at x, then for any $y \in S_a^{n-1}$, we have $J_\varphi(x) \to 1$ as $a \to \infty$ and $r/a \to 0$, uniformly in $x \in \Omega_a \cap B_r(y)$.*

Proof: The identity

$$J_\varphi(x) = J_\psi(R_y \cdot x)\det(R_y)$$

shows that in fact it suffices to prove that

$$J_\psi(x) \to 1 \text{ as } a \to \infty \text{ and } r/a \to 0$$

uniformly in $x \in \Omega_a \cap B_r(N)$ where N is the north pole of S_a^{n-1}. We prove even more, namely that

$$\frac{\partial \psi_i}{\partial x_j}(x) \to \delta_{ij}. \tag{16}$$

Indeed the matrix of first order partial derivatives of ψ is the identity except for the last row which is $(\frac{x_1}{|x|}, \frac{x_2}{|x|}, ..., \frac{x_n}{|x|})$. We have $a < |x| < a+1$ and

$$x_1^2 + x_2^2 + ... + x_{n-1}^2 + (x_n - a)^2 < r$$

hence $\left|\frac{x_i}{|x|}\right| < \frac{r}{a}$ for $i = 1, ..., n-1$ and $\left|\frac{x_n - a}{|x|}\right| < \frac{r}{a}$. From $a \to \infty$ and $r/a \to 0$ it follows that $\frac{x_i}{|x|} \to 0$ for $i = 1, ..., n-1$ and $\frac{x_n}{|x|} \to 1$. $\square$

Remark 2.2 *The proof of Proposition 2.1 also implies*

$$\frac{\partial \psi_i^{-1}}{\partial x_j}(x) \to \delta_{ij} \tag{17}$$

as $a \to \infty$ and $r/a \to 0$, uniformly in $x \in \psi(\Omega_a \cap B_r(N))$.

For $2 < 2r < r' < a$, using the mapping $\varphi_{a,r',y}$ and a cut-off function, we now construct operators $T = T_{a,r,r',y}$ and $\bar{T} = \bar{T}_{a,r,r',y}$

$$T : H_0^1(\Omega_a) \to H_0^1(D) \text{ and } \bar{T} : H_0^1(D) \to H_0^1(\Omega_a).$$

First note that if $p \in \Omega_a \cap \partial B_r(N)$ and $p' \in \Omega_a \cap \partial B_{r'}(N)$ we have that

$$\text{dist}(\psi(p), \psi(p')) \geq \frac{1}{2}(r' - 2r).$$

Therefore we can consider a smooth cut-off function $\rho = \rho_{a,r,r'} : D \to [0,1]$, with the properties

$$\rho(x) = \rho(|x'|, x_n) \ (\rho \text{ is axially symmetric}) \tag{18}$$

$$\rho(x) = 1 \text{ for any } x \in \psi(\Omega_a \cap B_r(N)) \subset D \tag{19}$$

$$\rho(x) = 0 \text{ for any } x \in D \setminus \psi(\Omega_a \cap B_{r'}(N)) \tag{20}$$

and

$$|\nabla \rho(x)| \leq \frac{4}{r' - 2r} \text{ for any } x \in D. \tag{21}$$

For $u \in H_0^1(\Omega_a)$ we now define

$$Tu(x) = \begin{cases} \rho(x) \cdot u(\varphi_{a,r',y}^{-1}(x)) & \text{if} \quad x \in \varphi_{a,r',y}(\Omega_a \cap B_{r'}(y)) \\ 0 & \text{if} \quad x \in D \setminus \varphi_{a,r',y}(\Omega_a \cap B_{r'}(y)) \end{cases} \tag{22}$$

For $u \in H_0^1(D)$ we define

$$\bar{T}u(x) = \begin{cases} \rho(\varphi_{a,r',y}(x)) \cdot u(\varphi_{a,r',y}(x)) & \text{if} \quad x \in \Omega_a \cap B_{r'}(y) \\ 0 & \text{if} \quad x \in \Omega_a \setminus B_{r'}(y) \end{cases} \tag{23}$$

2.2 Concentration-Compactness under Symmetry

Let $a_m \to \infty$ and $u_m \in \mathcal{M}_{G,a_m}$ any sequence such that $(E_{a_m}(u_m))$ is bounded. Consider the sequence $(u_m) \subset H^1(\mathbf{R}^n)$ by prolongation with zero outside Ω_{a_m}. We need the Concentration - Compactness Lemma due to P.L. Lions ([12]). The following is a consequence of a more detailed version of the Concentration - Compactness Lemma ([13]), as reformulated and proved in [21].

Lemma 2.3 *Let (u_m) be bounded in $H^1(\mathbf{R}^n)$ with $\int |u_m|^{p+1} = 1$. Then there is a subsequence (denoted still by (u_m)), a nonnegative nonincreasing sequence (α_i) satisfying $\lim_{s\to\infty}\sum_{i=1}^s \alpha_i = 1$, and sequences $(y_{m,i}) \subset \mathbf{R}^n$ associated with each $\alpha_i > 0$ satisfying*

$$\liminf_{m\to\infty} |y_{m,i} - y_{m,j}| \to \infty, \text{ as } m \to \infty, \forall i \neq j, \tag{24}$$

such that the following property holds: If $\alpha_s > 0$ for some $s \geq 1$, then for any $\epsilon > 0$ there exist $R > 0$ such that for all $r \geq R$ and all $r' \geq R$

$$\limsup_{m\to\infty} \sum_{i=1}^{s} \left| \alpha_i - \int_{B_r(y_{m,i})} |u_m|^{p+1} \right| + \left| (1 - \sum_{i=1}^{s} \alpha_i) - \int_{\mathbf{R}^n \setminus \cup_{i=1}^s B_{r'}(y_{m,i})} |u_m|^{p+1} \right| < \epsilon. \tag{25}$$

Remark 2.4 *We shall need the definition of (α_i) later. As in the proof given in [21], we may define (α_i) as follows. For $s \in \mathbf{N}$ and $r > 0$ we define a family of concentration functions:*

$$Q_{m,s}(r) := \sup_{y_i \in R^N} \int_{\bigcup_{i=1}^s B_r(y_i)} |u_m|^{p+1} dx.$$

Then there exist a subsequence of (u_m) (still denoted by (u_m)) such that $\lim_{m\to\infty} Q_{m,s}(r)$ exists for all $s \in \mathbf{N}$ and $r \in \mathbf{N}$, and $\{\alpha_i\}_1^\infty$ is defined by

$$\alpha_1 = \lim_{r\to\infty} \lim_{m\to\infty} Q_{m,1}(r),$$

and for $s \geq 2$

$$\alpha_s = \lim_{r\to\infty} \lim_{m\to\infty} Q_{m,s}(r) - \sum_{i=1}^{s-1} \alpha_i.$$

Now, with (u_m) being in $\mathcal{M}_{G,a_m}$ we shall give more information about the sequences $(y_{m,i})$ and conclude by passing to another subsequence that we have $(y_{m,i}) \subset S_{a_m}^{n-1}$ and that for each $s \geq 1$ with $\alpha_s > \alpha_{s+1}$ we have $Gy_{m,i} \subset \{y_{m,1}, ..., y_{m,s}\}$ for any $1 \leq i \leq s$. Namely, we have

Lemma 2.5 *Let $a_m \to \infty$ and $u_m \in \mathcal{M}_{G,a_m}$ with $(E_{a_m}(u_m))$ bounded. Then there is a subsequence (denoted still by (u_m)), a nonnegative nonincreasing sequence (α_i) satisfying $\lim_{s\to\infty}\sum_{i=1}^s \alpha_i = 1$, and sequences $(y_{m,i}) \subset S_{a_m}^{n-1}$ associated with each $\alpha_i > 0$ satisfying*

$$\liminf_{m\to\infty} |y_{m,i} - y_{m,j}| \to \infty, \text{ as } m \to \infty, \forall i \neq j, \tag{26}$$

such that the following property holds: If $\alpha_s > \alpha_{s+1}$ for some $s \geq 1$, then

$$Gy_{m,i} \subset \{y_{m,1}, ..., y_{m,s}\} \text{ for any } 1 \leq i \leq s; \tag{27}$$

and if $\alpha_s > 0$, then for any $\epsilon > 0$ there exist $R > 0$ such that for all $r \geq R$ and all $r' \geq R$

$$\limsup_{m\to\infty} \sum_{i=1}^{s} \left| \alpha_i - \int_{B_r(y_{m,i})} |u_m|^{p+1} \right| + \left| (1 - \sum_{i=1}^{s} \alpha_i) - \int_{\mathbf{R}^n \setminus \cup_{i=1}^s B_{r'}(y_{m,i})} |u_m|^{p+1} \right| < \epsilon. \tag{28}$$

First, the sequences $(y_{m,i})$ in Lemma 2.3 are by no means unique. We have the following

Observation 2.6 *Let $s \geq 1$ fixed. If there are $\bar{R} > 0$ and $(\hat{y}_{m,i})$, $i = 1, ..., s$ such that $|\hat{y}_{m,i} - y_{m,i}| \leq \bar{R}$ for any m, and $i = 1, ..., s$ then Lemma 2.3 holds with $(y_{m,i})$ substituted by $(\hat{y}_{m,i})$.*

Proof: Note that $\liminf |\hat{y}_{m,i} - \hat{y}_{m,j}|$ finite for some $i \neq j$ as $m \to \infty$ implies $\liminf |y_{m,i} - y_{m,j}|$ is finite which contradicts (24). For $R > 0$, let $\hat{R} = R + \bar{R}$ so that $B_R(y_{m,i}) \subset B_{\hat{R}}(\hat{y}_{m,i})$ for any m and i. For any $\hat{r}, \hat{r}' \geq \hat{R}$, let $r, r' \geq R$ such that

$$B_R(y_{m,i}) \subset B_{\hat{r}}(\hat{y}_{m,i}) \subset B_r(y_{m,i}), \tag{29}$$

and

$$\mathbf{R}^n \setminus \cup_{i=1}^s B_{r'}(y_{m,i}) \subset \mathbf{R}^n \setminus \cup_{i=1}^s B_{\hat{r}'}(\hat{y}_{m,i}) \subset \mathbf{R}^n \setminus \cup_{i=1}^s B_R(y_{m,i}). \tag{30}$$

Therefore

$$\int_{B_R(y_{m,i})} |u_m|^{p+1} \leq \int_{B_{\hat{r}}(\hat{y}_{m,i})} |u_m|^{p+1} \leq \int_{B_r(y_{m,i})} |u_m|^{p+1},$$

and

$$\int_{\mathbf{R}^n \setminus \cup_{i=1}^s B_{r'}(y_{m,i})} |u_m|^{p+1} \leq \int_{\mathbf{R}^n \setminus \cup_{i=1}^s B_{\hat{r}'}(\hat{y}_{m,i})} |u_m|^{p+1} \leq \int_{\mathbf{R}^n \setminus \cup_{i=1}^s B_R(y_{m,i})} |u_m|^{p+1}.$$

This implies Lemma 2.3 is true with $y_{m,i}$, substituted by $\hat{y}_{m,i}$. $\Box$

Observation 2.7 *The points $(y_{m,i})$ can be assumed to be on $S^{n-1}_{a_m}$.*

Proof: Take $0 < \epsilon < \alpha_i$. From (25), there is $R > 0$ such that for m large we have

$$\int_{B_r(y_{m,i})} |u_m|^{p+1} > \alpha_i - \epsilon > 0 \text{ for all } r \geq R. \tag{31}$$

Therefore,

$$\operatorname{dist}(y_{m,i}, S^{n-1}_{a_m}) \leq R + 1$$

otherwise $\operatorname{supp}(u_m) \subset \Omega_{a_m}$ and $B_R(y_{m,i})$ would not intersect. Let $\hat{y}_{m,i}$ the points where the directions of $y_{m,i}$ intersect $S^{n-1}_{a_m}$. Then $|\hat{y}_{m,i} - y_{m,i}| \leq \bar{R} = R + 1$, therefore for m large we can apply Observation 2.6 replacing $y_{m,i}$ by $\hat{y}_{m,i}$. $\Box$

From now on, we consider the points $y_{m,i}$ to be on $S^{n-1}_{a_m}$. We need some preliminaries.

Proposition 2.8 *Let $(y_m) \subset \mathbf{R}^n$, $|y_m| \to \infty$. Then there are $\bar{R} > 0$, $1 \leq \lambda \leq n$ and $\xi_1, ..., \xi_\lambda$ orthonormal vectors in $\mathbf{R}^n$ such that for a subsequence (denoted by (y_m))*

$$|y_m - <y_m, \xi_1> \xi_1 - ... - <y_m, \xi_\lambda> \xi_\lambda| \leq \bar{R},$$

$$<y_m, \xi_i> \to \infty, \quad 1 \leq i \leq \lambda, \quad <y_m, \xi_{i+1}> / <y_m, \xi_i> \to 0, \quad 1 \leq i \leq \lambda - 1 \quad \textit{as } m \to \infty.$$

Proof: We construct the vectors ξ_i as follows: let $a_{m,1} = |y_m| \to \infty$ and $\eta_{m,1} = y_m / a_{m,1}$. For a subsequence, $\eta_{m,1} \to \xi_1$ as $m \to \infty$. Note that

$$y_m = a_{m,1} \eta_{m,1} \text{ implies } 1 = < \frac{y_m}{a_{m,1}}, \eta_{m,1} >, \text{ i.e. } \frac{<y_m, \xi_1>}{a_{m,1}} \to 1. \tag{32}$$

Denote $a_{m,2} = |y_m - <y_m, \xi_1> \xi_1|$. If a subsequnce of $(a_{m,2})$ is bounded, the conclusion follows with $\lambda = 1$. Therefore assume $a_{m,2} \to \infty$. Since

$$\frac{y_m - a_{m,1}\xi_1}{a_{m,1}} \to 0 \text{ and } |y_m - a_{m,1}\xi_1| \geq |y_m - <y_m, \xi_1> \xi_1|, \tag{33}$$

we have

$$a_{m,2}/a_{m,1} \to 0. \tag{34}$$

Let $\eta_{m,2} = (y_m - < y_m, \xi_1 > \xi_1)/a_{m,2}$. For a subsequence, $\eta_{m,2} \to \xi_2$ as $m \to \infty$. Since $\eta_{m,2} \perp \xi_1$ for any m, it follows $\xi_2 \perp \xi_1$. We have

$$y_m = < y_m, \xi_1 > \xi_1 + a_{m,2}\eta_{m,2}, \text{ hence } \frac{< y_m, \xi_2 >}{a_{m,2}} \to 1. \tag{35}$$

From (32), (34), (35) we have

$$\frac{< y_m, \xi_2 >}{< y_m, \xi_1 >} \to 0.$$

We go on in this manner. Assume we constructed $\xi_1, ..., \xi_i$ such that

$$< y_m, \xi_i > \to \infty, \quad < y_m, \xi_i > / < y_m, \xi_{i-1} > \to 0$$

and

$$\frac{< y_m, \xi_i >}{a_{m,i}} \to 1. \tag{36}$$

Let $a_{m,i+1} = |y_m - < y_m, \xi_1 > \xi_1 - ... - < y_m, \xi_i > \xi_i|$. If $(a_{m,i+1})$ has a bounded subsequence, again the conclusion follows. So, assume $a_{m,i+1} \to \infty$. Let

$$\eta_{m,i+1} = (y_m - < y_m, \xi_1 > \xi_1 - ... - < y_m, \xi_i > \xi_i)/a_{m,i+1}.$$

For a subsequence, $\eta_{m,i+1} \to \xi_{i+1}$ as $m \to \infty$. Since $\eta_{m,i+1} \perp \xi_1, ..., \xi_i$ for any m, it follows $\xi_{i+1} \perp \xi_1, ..., \xi_i$. Since

$$\frac{y_m - < y_m, \xi_1 > \xi_1 - ... - < y_m, \xi_{i-1} > \xi_{i-1} - a_{m,i}\xi_i}{a_{m,i}} \to 0 \quad \text{and}$$

$$|y_m - < y_m, \xi_1 > \xi_1 - ... - < y_m, \xi_{i-1} > \xi_{i-1} - a_{m,i}\xi_i| \geq |y_m - < y_m, \xi_1 > \xi_1 - ... - < y_m, \xi_i > \xi_i|,$$

$$\text{we get } a_{m,i+1}/a_{m,i} \to 0. \tag{37}$$

Also

$$y_m = < y_m, \xi_1 > \xi_1 + ... + < y_m, \xi_i > \xi_i + a_{m,i+1}\eta_{m,i+1}, \text{ hence } \frac{< y_m, \xi_{i+1} >}{a_{m,i+1}} \to 1. \tag{38}$$

From (36), (37), (38) we have

$$\frac{< y_m, \xi_{i+1} >}{< y_m, \xi_i >} \to 0.$$

If the process does not stop for some $\lambda < n$(i.e., if $(a_{m,\lambda+1})$ has no bounded subsequence), then we take $\lambda = n$ and we have

$$y_m = < y_m, \xi_1 > \xi_1 + ... + < y_m, \xi_n > \xi_n,$$

and the conclusion of the proposition holds. □

From Observation 2.6 and Proposition 2.8 we conclude that by passing to a subsequence in Lemma 2.3 we can take the sequences $(y_{m,i})$ to be of the particular form:

$$y_{m,i} = b^i_{m,1}\xi^i_1 + ... + b^i_{m,\lambda_i}\xi^i_{\lambda_i}, \tag{39}$$

where $1 \leq \lambda_i \leq n$ and as $m \to \infty$, $b^i_{m,j} = < y_{m,i}, \xi^i_j > \to \infty$ for $1 \leq j \leq \lambda_i$ and $b^i_{m,j+1}/b^i_{m,j} \to 0$ for $1 \leq j \leq \lambda_i - 1$.

Observation 2.9 *Let (y_m) have form (39). If for $g \in \mathbf{O}(n)$ the orthonormal frames $(\xi_1, ..., \xi_\lambda)$ and $(g\xi_1, ..., g\xi_\lambda)$ are distinct, then $|gy_m - y_m| \to \infty$.*

Proof: We have

$$|gy_m - y_m| = b_{m,l}|(g\xi_l - \xi_l) + \frac{b_{m,l+1}}{b_{m,l}}(g\xi_{l+1} - \xi_{l+1}) + ... + \frac{b_{m,\lambda}}{b_{m,l}}(g\xi_\lambda - \xi_\lambda)| \to \infty \text{ as } m \to \infty,$$

where l is the first index such that $g\xi_l \neq \xi_l$. □

Observation 2.10 *Let $(y_{m,i})$ have form (39), then there exists $t_i \in \mathbf{N}$ such that $\#Gy_{m,i} = t_i$ and $dist(gy_{m,i}, hy_{m,i}) \to \infty$ as $m \to \infty$ for any $g, h \in G$ with $g(\xi_1^i, ..., \xi_{\lambda_i}^i) \neq h(\xi_1^i, ..., \xi_{\lambda_i}^i)$.*

Proof: By construction,

$$gy_{m,i} = b_{m,1}^i g\xi_1^i + ... + b_{m,\lambda_i}^i g\xi_{\lambda_i}^i$$

for all $g \in G$. If $\{g(\xi_1^i, ..., \xi_{\lambda_i}^i) \mid g \in G\}$ contains infinitely many elements, by Observation 2.9 we have a contradiction with (u_m) being bounded in $L^{p+1}(\Omega_{a_m})$ for

$$\int_{\Omega_m} |u_m|^{p+1} \geq \int_{GB_R(y_{m,i})} |u_m|^{p+1} \to \infty.$$

Thus $\#Gy_{m,i} = t_i$ for some $t_i \in \mathbf{N}$ and there exist $g_j \in G$ with $j = 1, ..., t_i$ such that $Gy_{m,i} = \{g_1 y_{m,i}, ..., g_{t_i} y_{m,i}\}$ satisfying that $g_j(\xi_1^i, ..., \xi_{\lambda_i}^i)$, $j = 1, ..., t_i$, are all different. Then Observation 2.9 gives the last assertion. □

Proposition 2.11 *For two sequences $(y_{m,i})$, $(y_{m,j})$ in Lemma 2.3 corresponding to $\alpha_i \neq \alpha_j$ we have*

$$dist(Gy_{m,i}, Gy_{m,j}) \to \infty \text{ as } m \to \infty.$$

Proof: Assume $\alpha_i > \alpha_j$ and take $0 < 2\epsilon < (\alpha_i - \alpha_j)$. Then there is $R > 0$ such that (25) holds. If for a subsequence $\text{dist}(Gy_{m,i}, Gy_{m,j}) \leq \bar{R}$, then there are $g_m, \bar{g}_m \in G$ such that $\text{dist}(g_m y_{m,i}, \bar{g}_m y_{m,j}) \leq \bar{R}$. Let $r = R + \bar{R}$ such that $B_R(g_m y_{m,i}) \subset B_r(\bar{g}_m y_{m,j})$. Then we have a contradiction

$$\alpha_i - \epsilon < \int_{B_R(g_m y_{m,i})} |u_m|^{p+1} \leq \int_{B_r(\bar{g}_m y_{m,j})} |u_m|^{p+1} < \alpha_j + \epsilon.$$

□

Proposition 2.12 *If in Lemma 2.3 we have $dist(Gy_{m,i}, Gy_{m,j}) \leq \bar{R}$ for some $\bar{R} > 0$ for any m, then $Gy_{m,j}$ is included in the $\bar{R}$ neighborhood of $Gy_{m,i}$.*

Proof: We have that there are $(\bar{g}_m), (\hat{g}_m) \subset G$ such that

$$|\bar{g}_m y_{m,j} - \hat{g}_m y_{m,i}| \leq \bar{R}.$$

Then for any $g \in G$ we have

$$|gy_{m,j} - g\bar{g}_m^{-1}\hat{g}_m y_{m,i}| = |\bar{g}_m y_{m,j} - \hat{g}_m y_{m,i}| \leq \bar{R}.$$

This implies that $Gy_{m,j}$ is contained in the $\bar{R}$ neighborhood of $Gy_{m,i}$. □

We close up the proof of Lemma 2.5.

Proof of Lemma 2.5: Let $s_j \in \mathbf{N}$ be the sequence where a strict decrease occurs for α_i, i.e., $\alpha_{s_j} > \alpha_{s_j+1}$. We do an induction on s_j. For $j = 1$, we have $\alpha_1 = ... = \alpha_{s_1}$ and we show first $Gy_{m,i} \subset \{y_{m,1}, ..., y_{m,s_1}\}$ for any $1 \leq i \leq s_1$. Consider $y_{m,1}$ first. If there is $g \in G$ such that $gy_{m,1}$ is not among $\{y_{m,1}, ..., y_{m,s_1}\}$ we conclude that for some $j_0 \leq s_1$

$$\limsup_{m\to\infty} \mathrm{dist}(gy_{m,1}, y_{m,j_0}) \leq \bar{R}.$$

If this is not true then along a subsequence

$$\lim_{m\to\infty} \mathrm{dist}(gy_{m,1}, y_{m,j}) = \infty$$

for all $j = 2, ..., s_1$. Take $0 < 2\epsilon < \alpha_{s_1} - \alpha_{s_1+1}$. By Lemma 2.3, there is $R > 0$ such that (25) holds. Then by definition,

$$\alpha_{s_1+1} \geq \lim_{m\to\infty} \int_{\bigcup_{j=1}^{s_1} B_R(y_{m,j}) \bigcup B_R(gy_{m,1})} |u_m|^{p+1} dx - \sum_{j=1}^{s_1} \alpha_j \geq \alpha_1 - 2\epsilon,$$

a contradiction. By Observation 2.6 we replace y_{m,j_0} by $gy_{m,1}$. Let $\#Gy_{m,1} = t$ and consider $y_{m,t+1}$ if $t + 1 \leq s_1$ (otherwise we are done with this step). If for some $g \in G$, $(gy_{m,t+1})$ is not among the sequences, we claim there is $t < j_0 \leq s_1$ and $\bar{R} > 0$ such that

$$\limsup_{m\to\infty} \mathrm{dist}(gy_{m,t+1}, y_{m,j_0}) \leq \bar{R}.$$

Otherwise, we have for a subsequence

$$\lim_{m\to\infty} \mathrm{dist}(gy_{m,t+1}, y_{m,j}) = \infty$$

for all $j = t + 1, ..., s_1$. Also by Observation 2.10, Proposition 2.12 and (24),

$$\lim_{m\to\infty} \mathrm{dist}(gy_{m,t+1}, y_{m,j}) = \infty$$

for all $j = 1, ..., t$. Now, similar argument as above shows a contradiction with the definition of α_{s_1+1}. Next, assume the conclusion is true for s_l and consider s_{l+1}. If for some $g \in G$, gy_{m,s_l+1} is not among $\{y_{m,1}, ..., y_{m,s_{l+1}}\}$, we claim again there is $s_l < j_0 \leq s_{l+1}$ such that

$$\limsup_{m\to\infty} \mathrm{dist}(gy_{m,s_l+1}, y_{m,j_0}) \leq \bar{R}.$$

Otherwise by Observation 2.10 and Proposition 2.11, for a subsequence,

$$\lim_{m\to\infty} \mathrm{dist}(gy_{m,s_l+1}, y_{m,j}) = \infty$$

for all $j = 1, ..., s_{l+1}$. Using definition of $\alpha_{s_{l+1}+1}$ and the argument above we get a contradiction. Thus we may replace y_{m,j_0} by gy_{m,s_l+1}. Let $\#Gy_{m,s_l+1} = t$ and go on to consider y_{m,s_l+t+1} the same way and we then finish the induction. □

Let $a_m \to \infty$ and $u_m \in \mathcal{M}_{G,a_m}$ such that $(E_{a_m}(u_m))$ is a bounded sequence. Applying Lemma 2.5 we get that there is a subsequence (still denoted by (u_m)) and the sequence (α_i). Let $s \geq 1$ be such that $\alpha_s > \alpha_{s+1}$. Define $M_m = \{y_{m,1}, ..., y_{m,s}\}$. By Lemma 2.5 we make the following conventions, $\bar{M}_m = M_m/G = \{\bar{y}_{m,1}, ..., \bar{y}_{m,\bar{s}}\}$, $\bar{\alpha}_\iota = \alpha_i$ for any $y_{m,i}$ in the class $\bar{y}_{m,\iota}$ and denote by k_ι the number of points in this class. Observe that by $(\mathbf{S_1})$, $k_\iota \geq 2$. Let $\bar{\Lambda}$ the set of all $\bar{\alpha}_\iota$, $\iota = 1, ..., \bar{s}$. For a sequence as above we have

Proposition 2.13 *Let $a_m \to \infty$ and $u_m \in \mathcal{M}_{G,a_m}$ such that $(E_{a_m}(u_m))$ is a bounded sequence. Then there is a subsequence (still denoted by (u_m)) such that*

$$\liminf_{m\to\infty} E_{a_m}(u_m) \geq S \sum_{\iota=1}^{\bar{s}} k_\iota \bar{\alpha}_\iota^{\frac{2}{p+1}}. \tag{40}$$

Proof: Let $\epsilon > 0$, free for the moment. For m large we pick $r_m > R$ and $r'_m > R$ with the following properties:

$$r_m \to \infty, \tag{41}$$

$$r'_m > 3r_m, \tag{42}$$

$$\frac{r'_m}{a_m} \to 0 \tag{43}$$

so that the balls $B_{r'_m}(y_{m,i})$ are disjoint for $i = 1, ..., s$. From (21) and (42) we obtain

$$|\nabla \rho_m(x)| \leq 4/r_m, \tag{44}$$

for any $x \in D$. For $i = 1, ..., s$ fixed, let $v_{m,i} = T_{a_m,r_m,r'_m,y_{m,i}} u_m$. From (25) we have

$$\int_{B_{r_m}(y_{m,i})} |u_m|^{p+1} > \alpha_i - \epsilon. \tag{45}$$

Denote $\varphi_m = \varphi_{a_m,r'_m,y_{m,i}}$, $\psi_m = \psi_{a_m,r'_m}$ and $\rho_m = \rho_{a_m,r_m,r'_m}$. The change of variables formula gives

$$\int_{\varphi_m(B_{r'_m}(y_{m,i})\cap\Omega_{a_m})} |v_{m,i}|^{p+1}(x)\ dx = \int_{B_{r'_m}(y_{m,i})\cap\Omega_{a_m}} \rho^{p+1}(\varphi_m(x))|u_m|^{p+1}(x) J_{\varphi_m}(x)\ dx \tag{46}$$

By Proposition 2.1 and (45) we conclude

$$\liminf_{m\to\infty} \int_{\varphi_m(B_{r'_m}(y_{m,i})\cap\Omega_{a_m})} |v_{m,i}|^{p+1}(x)\ dx \geq \alpha_i - \epsilon. \tag{47}$$

From (9) we get

$$\liminf_{m\to\infty} \int_{\varphi_m(B_{r'_m}(y_{m,i})\cap\Omega_{a_m})} |\nabla v_{m,i}|^2(x) + v_{m,i}^2(x)\ dx \geq S(\alpha_i - \epsilon)^{\frac{2}{p+1}}. \tag{48}$$

Then we have

$$\begin{aligned}
&S(\alpha_i - \epsilon)^{\frac{2}{p+1}} \leq \liminf_{m\to\infty} \int_{\varphi_m(B_{r'_m}(y_{m,i})\cap\Omega_{a_m})} |\nabla v_{m,i}|^2(x) + v_{m,i}^2(x)\ dx \\
&= \liminf_{m\to\infty} \int_{\varphi_m(B_{r'_m}(y_{m,i})\cap\Omega_{a_m})} \rho_m^2(x)|\nabla(u_m(\varphi_m^{-1}(x)))|^2 + \rho_m^2(x) u_m^2(\varphi_m^{-1}(x))\ dx \\
&\leq \liminf_{m\to\infty} \int_{\varphi_m(B_{r'_m}(y_{m,i})\cap\Omega_{a_m})} |\nabla(u_m(\varphi_m^{-1}(x)))|^2 + u_m^2(\varphi_m^{-1}(x))\ dx \\
&= \liminf_{m\to\infty} \int_{B_{r'_m}(y_{m,i})\cap\Omega_{a_m}} |\nabla u_m|^2(x) J_{\varphi_m}(x) + u_m^2(x) J_{\varphi_m}(x)\ dx \\
&= \liminf_{m\to\infty} \int_{B_{r'_m}(y_{m,i})\cap\Omega_{a_m}} |\nabla u_m|^2(x) + u_m^2(x)\ dx
\end{aligned} \tag{49}$$

where for the last equality we used Proposition 2.1, i.e., $\max |J_{\varphi_m}(x) - 1| \to 0$ uniformly as $m \to \infty$. Since the balls $B_{r'_m}(y_{m,i})$ are disjoint we get

$$\liminf_{m\to\infty} E_{a_m}(u_m) \geq S \sum_{\iota}^{\bar{s}} k_\iota (\bar{\alpha}_\iota - \epsilon)^{\frac{2}{p+1}}. \tag{50}$$

Letting $\epsilon \to 0$, the conclusion follows. □

2.3 Existence Theorem

Studying the function

$$f(x) = a_1 x_1^\alpha + a_2 x_2^\alpha + \dots + a_s x_s^\alpha$$

where $s \geq 2$, $a_i > 0$, $x_i \geq 0$, $\sum_{i=1}^s x_i = \beta$ and $0 < \alpha < 1$, we make the following

Observation 2.14 *We note that the minimum is achieved for $x_j = \beta$ where j is such that*

$$a_j = \min_{i=1,\dots,s} \{a_i\} \text{ and } \min f(x) = a_j \beta^\alpha.$$

Observation 2.15 *Suppose $s = 2$ and $a_1 = 2^{\frac{p-1}{p+1}}$, $a_2 = k^{\frac{p-1}{p+1}}$ and $\alpha = \frac{2}{p+1}$. Let $\sigma = \frac{2k+1}{2k+4}$. If $x_1 + x_2 = \beta \in (3/4, 1]$ then if $0 < x_1 \leq 1 - \sigma = \frac{3}{2k+4}$ it follows $f(x) > k^{\frac{p-1}{p+1}} \beta^{\frac{2}{p+1}}$.*

We prove the following

Proposition 2.16 *(a) For $\sigma = \frac{2k+1}{2k+4}$ we have $m_{a,\sigma} \to k^{\frac{p-1}{p+1}} S$ as $a \to \infty$, and*

(b) for any sequence $a_m \to \infty$ and $u_m \in \mathcal{K}_{G,a_m}^\sigma$ with $E_{a_m}(u_m) \to k^{\frac{p-1}{p+1}} S$ we have $\gamma_{a_m}(u_m) \to 1$.

Proof: We prove part (a) of the proposition in two steps,
Step 1: we show

$$\limsup_{a \to \infty} m_{a,\sigma} \leq k^{\frac{p-1}{p+1}} S.$$

For this we construct test functions $u_a \in \mathcal{K}_{G,a}^\sigma$ with $E_a(u_a) \to k^{\frac{p-1}{p+1}} S$ as $a \to \infty$. Let $\bar{v}$ a solution of the minimizing problem (9). We have that

$$\int_D |\nabla \bar{v}|^2 + \bar{v}^2 = S, \text{ and } \int_D \bar{v}^{p+1} = 1. \tag{51}$$

Denote

$$v = \frac{\bar{v}}{k^{\frac{1}{p+1}}}. \tag{52}$$

Choose $r' > 3r$ and let $a > r'$ sufficiently large so that the balls $B_{r'}(y_i)$ are disjoint, where y_i, $i = 1, \dots, k$, are the orbit under G of y_1, a point in $\mathbf{R}^l \times \{0\}$ with fixed direction. This is possible because of $(\mathbf{S_1})$. Then for $i = 1, \dots, k$, let

$$u_{a,i} = \bar{T}_{a,r,r',y_i} v \tag{53}$$

and denote

$$u_a = \sum_{i=1}^k u_{a,i}. \tag{54}$$

Because of (11), any two rotations used in the construction on $\bar{T}_{a,r,r',y_i}$ will produce the same u_a. This shows that u_a is G-invariant. For $r \to \infty$, $\frac{r'}{a} \to 0$ we get

$$\int_{\Omega_a} u_a^{p+1} \to 1, \tag{55}$$

$$\gamma(u_a) \to 1. \tag{56}$$

Therefore for a sufficiently large, $\frac{u_a}{\|u_a\|_{p+1}} \in \mathcal{K}^{\sigma}_{G,a}$. The energy of u_a as $a \to \infty$ is

$$E_a(\frac{u_a}{\| u_a \|_{p+1}}) = \frac{\sum_{i=1}^{k} \int_{\Omega_a} |\nabla u_{a,i}|^2 + u_{a,i}^2}{\| u_a \|_{p+1}^2} = k \int_D |\nabla v|^2 + v^2 + o(1) = k \frac{S}{k^{\frac{2}{p+1}}} + o(1) = k^{\frac{p-1}{p+1}} S + o(1). \tag{57}$$

Step 2: suppose

$$\liminf_{a \to \infty} m_{a,\sigma} < k^{\frac{p-1}{p+1}} S.$$

Then there is a sequence $a_m \to \infty$ and $u_m \in \mathcal{K}^{\sigma}_{G,a_m}$ such that $E_{a_m}(u_m) \to d < k^{\frac{p-1}{p+1}} S$ as $m \to \infty$. With the conventions preceding Proposition 2.13, we get the sequence (α_i) and we let $\Lambda = \{\alpha_i : \alpha_i > 0\}$. Suppose Λ contains infinitely many elements (in case Λ is finite take s the number of elements in Λ). Fix $1/4 > \epsilon > 0$ and $s \geq 2$ such that $\alpha_s > \alpha_{s+1}$ and $\beta = \sum_{i=1}^{s} \alpha_i > 1 - \epsilon > \frac{3}{4}$, so $\beta = \sum_{\iota=1}^{\bar{s}} k_\iota \bar{\alpha}_\iota$. By Proposition 2.13 we have

$$d = \lim_{m \to \infty} E_{a_m}(u_m) \geq S \sum_{\iota=1}^{\bar{s}} k_\iota \bar{\alpha}_\iota^{\frac{2}{p+1}} = S \sum_{\iota=1}^{\bar{s}} k_\iota^{\frac{p-1}{p+1}} (k_\iota \bar{\alpha}_\iota)^{\frac{2}{p+1}}. \tag{58}$$

Some remarks are in order here. Note that if $x \in \{0\} \times \mathbf{R}^{n-l}$ then it is fixed under G_1, its orbit is contained in $\{0\} \times \mathbf{R}^{n-l}$ and has at least two elements. If $x \in \mathbf{R}^l \times \{0\}$, then its orbit contains at least k points, and if x is neither in $\{0\} \times \mathbf{R}^{n-l}$ nor in $\mathbf{R}^l \times \{0\}$ (i.e. both components of x are nonzero), then its orbit contains at least $2k$ elements. For any m, a number of $y_{m,i}$'s in the set $\{y_{m,1}, ..., y_{m,s}\}$ corresponding to $\{\alpha_1, ..., \alpha_s\}$ may be contained in $\{0\} \times \mathbf{R}^{n-l} \cap S^{n-1}_{a_m}$. Eventually passing to a subsequence we can assume this number is fixed and denote it by s_1. Without loss of generality we can assume $y_{m,1}, ..., y_{m,s_1}$ are the only $y_{m,i}$'s in $\{0\} \times \mathbf{R}^{n-l}$ and belong to classes $\bar{y}_{m,1}, ..., \bar{y}_{m,\bar{s}_1}$. Then from (58)

$$d \geq S 2^{\frac{p-1}{p+1}} \sum_{\iota=1}^{\bar{s}_1} (k_\iota \bar{\alpha}_\iota)^{\frac{2}{p+1}} + S k^{\frac{p-1}{p+1}} \sum_{\iota=\bar{s}_1+1}^{\bar{s}} (k_\iota \bar{\alpha}_\iota)^{\frac{2}{p+1}}, \tag{59}$$

and by Observation 2.14 we have,

$$\begin{aligned} d &\geq S 2^{\frac{p-1}{p+1}} \left(\sum_{\iota=1}^{\bar{s}_1} k_\iota \bar{\alpha}_\iota \right)^{\frac{2}{p+1}} + S k^{\frac{p-1}{p+1}} \left(\sum_{\iota=\bar{s}_1+1}^{\bar{s}} k_\iota \bar{\alpha}_\iota \right)^{\frac{2}{p+1}} \\ &= S 2^{\frac{p-1}{p+1}} \left(\sum_{\iota=1}^{\bar{s}_1} k_\iota \bar{\alpha}_\iota \right)^{\frac{2}{p+1}} + S k^{\frac{p-1}{p+1}} \left(\beta - \sum_{\iota=1}^{\bar{s}_1} k_\iota \bar{\alpha}_\iota \right)^{\frac{2}{p+1}}. \end{aligned} \tag{60}$$

As before, let $r_m \to \infty$, such that $r_m / a_m \to 0$ and $B_{r_m}(y_{m,i}) \cap B_{r_m}(y_{m,j}) = \emptyset$ for $i \neq j$. Then $|Px|$ is bounded by r_m, uniformly in $x \in \cup_{i=1}^{s_1} B_{r_m}(y_{m,i})$. Therefore,

$$\frac{1}{a_m + 1} \int_{\left(\cup_{i=1}^{s_1} B_{r_m}(y_{m,i})\right) \cap \Omega_{a_m}} |Px| |u_m|^{p+1}(x) \; dx \to 0. \tag{61}$$

We also have,

$$\begin{aligned} &\limsup_{m \to \infty} \frac{1}{a_m + 1} \int_{\Omega_{a_m} \setminus \left(\cup_{i=1}^{s_1} B_{r_m}(y_{m,i})\right)} |Px| |u_m|^{p+1}(x) \; dx \\ &\leq \limsup_{m \to \infty} \int_{\Omega_{a_m} \setminus \left(\cup_{i=1}^{s_1} B_{r_m}(y_{m,i})\right)} |u_m|^{p+1}(x) \; dx \to 1 - \sum_{\iota=1}^{\bar{s}_1} k_\iota \bar{\alpha}_\iota. \end{aligned} \tag{62}$$

From (61) and (62) it follows

$$\sigma < \gamma_{a_m}(u_m) \leq 1 - \sum_{\iota=1}^{\bar{s}_1} k_\iota \bar{\alpha}_\iota, \tag{63}$$

hence $\sum_{\iota=1}^{\bar{s}_1} k_\iota \bar{\alpha}_\iota \leq 1 - \sigma = \frac{3}{2k+4}$. By Observation 2.15 and (60) we get

$$S2^{\frac{p-1}{p+1}} \left(\sum_{\iota=1}^{\bar{s}_1} k_\iota \bar{\alpha}_\iota \right)^{\frac{2}{p+1}} + Sk^{\frac{p-1}{p+1}} \left(\beta - \sum_{\iota=1}^{\bar{s}_1} k_\iota \bar{\alpha}_\iota \right)^{\frac{2}{p+1}} \geq Sk^{\frac{p-1}{p+1}} \beta^{\frac{2}{p+1}}, \tag{64}$$

with equality if and only if $\bar{\alpha}_\iota = 0$, for $\iota = 1, ..., \bar{s}_1$. Hence $d \geq k^{\frac{p-1}{p+1}} S \beta^{\frac{2}{p+1}}$. If ϵ is such that $\beta^{\frac{2}{p+1}} > (1-\epsilon)^{\frac{2}{p+1}} > dk^{-\frac{p-1}{p+1}} S^{-1}$, we obtain the desired contradiction.

For part (b), let $a_m \to \infty$ and $u_m \in \mathcal{K}^\sigma_{G,a_m}$ with $E_m(u_m) \to k^{\frac{p-1}{p+1}} S$. The argument above shows that $\alpha_i = 0$ for any $y_{m,i} \notin \mathbf{R}^l \times \{0\}$. Therefore all points $y_{m,i}$ with $\alpha_i > 0$ are in $\mathbf{R}^l \times \{0\}$, hence $\gamma_{a_m}(u_m) \to 1$. □

Remark 2.17 *There are only k $y_{m,i}$'s for which $\alpha_i > 0$, and they are the orbit of one point only.*

Proof: We apply Observation 2.14 to show that every $y_{m,i}$ has orbit with k elements only, and then the inequality

$$\sum_{\iota=1}^{\bar{s}} \bar{\alpha}_\iota^{\frac{2}{p+1}} \geq \left(\sum_{\iota=1}^{\bar{s}} \bar{\alpha}_\iota \right)^{\frac{2}{p+1}} \tag{65}$$

shows $\bar{\alpha}_\iota = 0$ except for one ι. □

Proposition 2.18 *Let $\sigma > \frac{2k+1}{2k+4}$ be fixed. Then there is $a_0 > 0$ such that*

$$m_{a,\sigma} = \inf_{u \in \mathcal{K}^\sigma_{G,a}} E_a(u)$$

is achieved in the interior of $\mathcal{K}^\sigma_{G,a}$ for any $a \geq a_0$.

Proof: Consider $\bar{\mathcal{K}}^\sigma_{G,a}$, the closure in $H^1_0(\mathbf{R}^n)$ of $\mathcal{K}^\sigma_{G,a}$. For a fixed, any minimizing sequence in $\bar{\mathcal{K}}^\sigma_{G,a}$ has a convergent subsequence in the weak topology and the limit is still in $\bar{\mathcal{K}}^\sigma_{G,a}$. Since E_a is weakly lower semicontinuous it follows that

$$\inf_{u \in \bar{\mathcal{K}}^\sigma_{G,a}} E_a(u)$$

is achieved. Suppose the proposition is false. By Proposition 2.16 there is a sequence $a_m \to \infty$ and $u_m \in \partial \mathcal{K}^\sigma_{G,a_m}$ (the boundary of $\mathcal{K}^\sigma_{G,a_m}$ in $H^1_0(\mathbf{R}^n)$) such that $E_m(u_m) \to k^{\frac{p-1}{p+1}} S$. Since γ_a is continuous in $H^1_0(\mathbf{R}^n)$, it follows that

$$\gamma_{a_m}(u_m) = \sigma < 1, \tag{66}$$

contradicting Proposition 2.16 (b). □

Corollary 2.19 *For $a \geq a_0$, E_a has a critical point in $\mathcal{K}^\sigma_{G,a}$.*

If u is a critical point of E_a in $\mathcal{K}^\sigma_{G,a}$, then $\bar{u} = (E_a(u))^{\frac{1}{p-1}}u$ is a weak solution of (1). By regularity theory, $\bar{u}$ is a classical solution and by Maximum Principle it follows $\bar{u}(x) > 0$ for any $x \in \Omega_a$.

We now turn to study more about the properties of these solutions.

Remark 2.20 *For any $\epsilon > 0$ and any $a_m \to \infty$, there are $R > 0$ and $y_m \in S^{n-1}_{a_m}$ with the property $Gy_m = \{y_{m,1}, ..., y_{m,k}\}$ such that $x \in \Omega_{a_m} \setminus \cup_{i=1}^k B_R(y_{m,i})$ implies $\bar{u}_{a_m}(x) < \epsilon$.*

Proof: Let
$$u_m(x) = \frac{\bar{u}_m(x)}{\| \bar{u}_m \|_{L^{p+1}}}.$$

Then $u_m \in \mathcal{K}^\sigma_{G,a_m}$ and $E_{a_m}(u_m) \to k^{\frac{p-1}{p+1}}S$. Let $\epsilon' > 0$. By Remark 2.17 and Lemma 2.5 we have that there are $R > 0$, $y_m \in S^{n-1}_{a_m}$ such that $Gy_m = \{y_{m,1}, ..., y_{m,k}\}$ (has k elements) and
$$\int_{\Omega_{a_m} \setminus \cup_{i=1}^k B_R(y_{m,i})} u_m^{p+1} < \epsilon'. \tag{67}$$

Since $\bar{u}_m$ are solutions of (1) we can use a boot strap argument to conclude from (67) that
$$\bar{u}_m(x) = O(\epsilon') \text{ for } x \in \Omega_{a_m} \setminus \cup_{i=1}^k B_R(y_{m,i}). \tag{68}$$

For sufficiently small ϵ', the conclusion follows. $\square$

For a G-invariant solution $\bar{u}_a$ obtained as before, denote
$$Q_a = \{x \in \Omega_a : \bar{u}_a(x) = \max_{y \in \Omega_a} \bar{u}_a(y)\}.$$

Proposition 2.21 *For a sequence $a_m \to \infty$, there are $r_m \to 0$ and $y_m \in S^{n-1}_{a_m}$ such that $Gy_m = \{y_{m,1}, ..., y_{m,k}\}$ and $Q_{a_m} \subset \cup_{i=1}^k B_{r_m}(y_{m,i})$.*

Proof: For $\epsilon > 0$ let y_m and R given by Remark 2.20. Let $R'_m \geq 3R_m \geq 3R$ such that $R_m \to \infty$, $R'_m/a_m \to 0$ and $B_{R'_m}(y_{m,i}) \cap B_{R'_m}(y_{m,j}) = \emptyset$ for any $i \neq j$. Denote $\bar{v}_m = T_{a_m,R_m,R'_m,y_m}\bar{u}_m$, with $\bar{u}_m = \bar{u}_{a_m}$. Then on the domain given by the image of φ_{a_m,R_m,y_m}, denoted by $D_m \subset D$, $\bar{v}_m$ satisfies an elliptic equation whose coefficients, on any compact subset of D tend uniformly to the coefficients of $-\Delta v + v = v^p$. Moreover, $v_m|_{\partial D_m \cap \partial D} = 0$ and by Remark 2.20, $\bar{v}_m(x) < \epsilon$ for any $x \in \partial D_m$. Also
$$\frac{\int_D |\nabla \bar{v}_m|^2 + \bar{v}_m^2 \, dx}{\left(\int_D \bar{v}_m^{p+1} \, dx\right)^{\frac{2}{p+1}}} \to S.$$

By elliptic theory, $\bar{v}_m$ converges in $C^2_{loc}(D)$ to a solution $\bar{v}$ of (10). Such a solution must satisfy (11), (12) and (13). Let $1/2 > r > 0$, then there is a $\delta > 0$ such that $|\nabla \bar{v}|(x) > \delta$ for any $x \in (B_R(C) \setminus B_r(C)) \cap D$, where $C = (0, ..., 0, 1/2)$. Since as $m \to \infty$, we have $|\nabla \bar{v}_m|(x) \to |\nabla \bar{v}|(x)$ for any $x \in (B_R(C) \setminus B_r(C)) \cap D$ it follows that for m sufficiently large, all maximum points of $\bar{v}_m$ are in $B_r(C)$. Since r is arbitrary, the conclusion follows. $\square$

Collecting the results we have the following

Theorem 2.22 *Given G satisfying $(\mathbf{S_1})$, there is $a_0 > 0$ such that equation (1) has a G-invariant solution $\bar{u}_a$ for any $a \geq a_0$. Moreover, this solution satisfies*

(i). $E_a(\bar{u}_a) \to kS^{\frac{p+1}{p-1}}$, $\| \bar{u}_a \|^2_{L^{p+1}} \to k^{\frac{2}{p+1}} S^{\frac{2}{p-1}}$ *as* $a \to \infty$.

(ii). There exists $P_a \in \mathbf{R}^l \times \{0\} \cap \Omega_a$ such that $\#GP_a = k$ and the set $Q_a = \{x \in \Omega_a : \bar{u}_a(x) = \max_{y\in\Omega_a} \bar{u}_a(y)\}$ is contained in balls with radius tending to zero as $a \to \infty$, centered at the points GP_a.

(iii). $\bar{u}_a$ is concentrated around a G-orbit which contains k points in the following sense that for any sequence $a_m \to \infty$ there is a subsequence (denoted still (a_m)) and v a least energy solution of (10) such that

$$\lim_{a_m\to\infty} \int_{\Omega_{a_m}} |\nabla(\bar{u}_{a_m} - \sum_{y\in GP_{a_m}} v(\cdot - y))|^2 + |\bar{u}_{a_m} - \sum_{y\in GP_{a_m}} v(\cdot - y)|^2 \; dx = 0.$$

The assertion (i) follows from Propositions 2.16 and 2.18, (ii) and (iii) from the proof of Proposition 2.21. Note that Theorem 1.1 is a consequence of (i) when we take $G_1 = Z_k \subset \mathbf{O}(2)$ and $G_2 = \mathbf{O}(1)$ and solutions are distinquished by their energy.

3 Exact Symmetry

3.1 Main Results

In order to discuss the exact symmetry of these solutions we are going to strengthen the assumptions on G, besides $(\mathbf{S_1})$. We shall assume that G is maximal with property $(\mathbf{S_1})$ in the sense that

$$(\mathbf{S_2}) \quad \begin{array}{c} \text{for any group } \mathbf{O}(n) \geq H \geq G \text{ with } H \neq G \text{ we have} \\ \#H\xi > k \text{ for any } \xi \text{ such that } \#G\xi = k. \end{array}$$

For $a > 0$ large, let $\bar{u}_a$ be the solution obtained in the previous section and let us define the isotropy subgroup

$$\Sigma_{\bar{u}_a} = \{g \in \mathbf{O}(n) : g\bar{u}_a = \bar{u}_a\}. \tag{69}$$

Remark 3.1 *$\Sigma_{\bar{u}_a}$ is a closed subgroup of $\mathbf{O}(n)$.*

Proof: Let $(g_i)_i \subset \Sigma_{\bar{u}_a}$ a sequence convergent to $g \in \mathbf{O}(n)$. Since $g_i\bar{u}_a = \bar{u}_a$ it follows $g\bar{u}_a = \bar{u}_a$, i.e. $g \in \Sigma_{\bar{u}_a}$. □

Theorem 3.2 *If G satisfies $(\mathbf{S_1})$ and $(\mathbf{S_2})$ then for sufficiently large a, we have $\Sigma_{\bar{u}_a} = G$.*

First we need the following

Lemma 3.3 *For G satisfying $(\mathbf{S_1})$ and $(\mathbf{S_2})$, there is $\delta > 0$ such that for any closed subgroup $\mathbf{O}(n) \geq H \geq G$ and $H \neq G$ we have $H\xi \setminus N_\delta(G\xi) \neq \emptyset$, for any $\xi \in S^{n-1}$ with $\#G\xi = k$.*

Here $N_\delta(G\xi)$ denotes the δ neighborhood of $G\xi$ in $\mathbf{R}^n$.

Proof: Let ξ_1, ..., ξ_k a G-orbit of ξ_1, and let

$$\delta = \min\{\frac{1}{8}d, \ \frac{1}{2\sqrt{n}}\}, \text{ where } d = \min_{\alpha\neq\beta} \operatorname{dist}(\xi_\alpha, \xi_\beta). \tag{70}$$

Suppose by contradiction that

$$H\xi_1 \subset \cup_{\alpha=1,\ldots,k} N_\delta(\xi_\alpha). \tag{71}$$

Observation 3.4 *Let $\hat{h} \in H$ such that $\hat{h}\xi_\alpha \in N_\delta(\xi_\beta)$. Then for any $h \in H$, $h\xi_\alpha \in N_\delta(\xi_\alpha)$ if and only if $\hat{h}h\xi_\alpha \in N_\delta(\xi_\beta)$.*

Indeed, if $h\xi_\alpha \in N_\delta(\xi_\alpha)$ then because H is a group of isometries we have

$$|\hat{h}h\xi_\alpha - \xi_\beta| \le |\hat{h}h\xi_\alpha - \hat{h}\xi_\alpha| + |\hat{h}\xi_\alpha - \xi_\beta| = |h\xi_\alpha - \xi_\alpha| + |\hat{h}\xi_\alpha - \xi_\beta| < 2\delta. \tag{72}$$

Since $\hat{h}h\xi_\alpha \in N_\delta(G\xi_1)$, it follows $\hat{h}h\xi_\alpha \in N_\delta(\xi_\beta)$. Conversely, the same argument and the inequalities

$$|h\xi_\alpha - \xi_\alpha| = |\hat{h}h\xi_\alpha - \hat{h}\xi_\alpha| \le |\hat{h}h\xi_\alpha - \xi_\beta| + |\hat{h}\xi_\alpha - \xi_\beta| < 2\delta \tag{73}$$

complete the proof of the observation.

We construct functions $f^i_\alpha : H \to [-1, 1]$, $i = 1, ..., n$, $\alpha = 1, ..., k$ given by

$$f^i_\alpha(h) = \begin{cases} x_i(h\xi_\alpha) & \text{if} \quad h\xi_\alpha \in N_\delta(\xi_\alpha) \\ 0 & \text{otherwise} \end{cases} \tag{74}$$

where x_i are the coordinate functions in $\mathbf{R}^n$. It is not difficult to see that because of (71), the functions f^i_α are continuous. Also we have

$$\sum_{i=1}^{n} (f^i_\alpha(h))^2 = 1 \text{ or } 0 \tag{75}$$

according to $h\xi_\alpha$ being in $N_\delta(\xi_\alpha)$ or not.

Let $\hat{h} \in H$ such that $\hat{h}\xi_\alpha \in N_\delta(\xi_\beta)$. Then for any $h \in H$ we have

$$x_i(\hat{h}h\xi_\alpha) = \sum_{j=1}^{n} \hat{h}^i_j x_j(h\xi_\alpha). \tag{76}$$

Let $g \in G$ such that $\xi_\alpha = g\xi_\beta$. By (74) and Observation 3.4 we conclude

$$f^i_\beta(\hat{h}hg) = \sum_{j=1}^{n} \hat{h}^i_j f^j_\alpha(h), \text{ for any } h \in H \text{ i.e. } \left(f^i_\beta\right)^g_{\hat{h}} = \sum_{j=1}^{n} \hat{h}^i_j f^j_\alpha. \tag{77}$$

Since H is closed subgroup of $\mathbf{O}(n)$ it is compact. This guarantees the existence of a bi-invariant measure on H (the Haar measure, see [7]). That is, there is a linear functional $I : C(H) \to \mathbf{R}$ with the properties of the integral (the Haar integral) such that if $f \in C(H)$, $h \in H$, $f_h(x) = f(hx)$ and $f^h(x) = f(xh)$ for any $x \in H$ then $I(f) = I(f_h) = I(f^h)$. With $\hat{h} \in H$ such that $\hat{h}\xi_\alpha \in N_\delta(\xi_\beta)$, from (77) we now have

$$I\left(f^i_\beta\right) = I\left(\left(f^i_\beta\right)^g_{\hat{h}}\right) = I\left(\sum_{j=1}^{n} \hat{h}^i_j f^j_\alpha\right) = \sum_{j=1}^{n} \hat{h}^i_j I\left(f^j_\alpha\right). \tag{78}$$

We claim that the vectors $I(f_\alpha) \in \mathbf{R}^n$ with components $I(f^i_\alpha)$, $i = 1, ..., n$ are nonzero. Since $\sum_{i=1}^{n}(x_i(\xi_\alpha))^2 = 1$ we have that

$$|x_j(\xi_\alpha)| \ge \frac{1}{\sqrt{n}} \tag{79}$$

for at least one j. If $h\xi_\alpha \in N_\delta(\xi_\alpha)$ then $|f^j_\alpha(h) - x_j(\xi_\alpha)| < \delta$ and from (70) and (79) we get

$$|f^j_\alpha(h)| \ge \frac{1}{2\sqrt{n}} \text{ for any } h \in H \text{ such that } h\xi_\alpha \in N_\delta(\xi_\alpha). \tag{80}$$

We now show that the set $H_\alpha = \{h \in H : h\xi_\alpha \in N_\delta(\xi_\alpha)\}$ has nonzero measure which proves the claim that $I(f_\alpha)$ are nonzero (at least one component is the integral of a constant sign function on a nonzero measure set). If $\rho : H \times \mathbf{R}^n \to \mathbf{R}^n$ is the action of H, let

$$\rho_\alpha : H \to \mathbf{R}^n \text{ to be } \rho_\alpha(h) = \rho(h, \xi_\alpha).$$

Because ρ_α are continuous, it follows H_α are open sets. Hence $V = \cap_{\alpha=1,\ldots,k} H_\alpha$ is open in H. Since H is compact any open neighborhood V of e has nonzero measure. Otherwise one can extract a finite open cover from translates of V which implies the Haar measure of H is 0. Therefore we can consider the vectors $\eta_\alpha \in S^{n-1}$ obtained by normalizing to unit length the vectors $I(f_\alpha)$. From (78) it follows that $H\eta_1 = \{\eta_1, \ldots, \eta_k\}$. We have $\#G\eta_1 \geq k$ and because $G \subset H$, it follows $\#G\eta_1 = k$. This contradicts $(\mathbf{S_2})$. The proof of the lemma is complete. □

Corollary 3.5 *Under the hypotheses of Lemma 3.3, there is $\delta' > 0$ such that $H\eta \setminus N_{\delta'}(G\xi) \neq \emptyset$, for any $\eta, \xi \in S^{n-1}$ with $\#G\xi = k$.*

Proof: Take $\delta' = \delta/2$. Suppose $H\eta \in N_{\delta'}(G\xi)$. Since $\#G\xi = k$, by Lemma 3.3, there is $h \in H$ such that $\mathrm{dist}(h\xi, G\xi) > \delta$. Therefore

$$\delta' > \mathrm{dist}(h\eta, G\xi) \geq \mathrm{dist}(h\xi, G\xi) - \mathrm{dist}(h\eta, h\xi) > \delta - \mathrm{dist}(\eta, \xi) > \delta',$$

contradiction. □

Proof of Theorem 3.2: Assume $\Sigma_{\bar{u}_{a_m}} \neq G$ for a sequence $a_m \to \infty$. Since $\bar{u}_m = \bar{u}_{a_m}$ are solutions of (1), by maximum principle we have

$$\bar{u}_m(x_0) > 1 \text{ for any } x_0 \text{ local maximum point of } \bar{u}_m. \tag{81}$$

Let $x_m \in \Omega_{a_m}$ be any local maximum of u_m. For $\epsilon > 0$ sufficiently small and R as in the Remark 2.20, estimate (68), implies

$$x_m \in \cup_{i=1}^k B_R(y_{m,i}), \text{ i.e. } \mathrm{dist}(x_m, Gy_{m,1}) < R. \tag{82}$$

The vectors $\eta_m = \frac{x_m}{|x_m|}$ and $\xi_{m,1} = \frac{y_{m,1}}{|y_{m,1}|}$ are on the unit sphere. By Remark 3.1 and Corollary 3.5, it follows that there is $h_m \in \Sigma_{\bar{u}_{a_m}}$ such that

$$\mathrm{dist}(h_m\eta_m, G\xi_{m,1}) > \delta'.$$

Therefore, $\mathrm{dist}(h_m x_m, Gy_{m,1}) > a_m\delta'$. By symmetry $h_m x_m$ is also a local maximum of u_m. For $a_m \geq R/\delta'$, this contradicts (82). □

3.2 Examples

We consider the condition

$$(\mathbf{S_2'}) \begin{cases} G_1 \text{ acts irreducible on } \mathbf{R}^l \text{ and for any } G_1 \leq H_1 \leq \mathbf{O}(l), \\ H_1 \neq G_1 \text{ we have } \#H_1\xi > k \text{ for any } \xi \text{ with } \#G\xi = k. \end{cases}$$

It is not difficult to see that if $G = G_1 \times \mathbf{O}(n-l)$ and G_1 satisfies $(\mathbf{S_2'})$, then G satisfies $(\mathbf{S_2})$.

In the following we shall consider $G = G_1 \times \mathbf{O}(n-l)$, where G_1 acts irreducible on $\mathbf{R}^l$. Our examples of groups that act irreducible on $\mathbf{R}^l$ come from [6]. We shall take G_1 to be a Coxeter group (finite group generated by reflections about hyperplanes in $\mathbf{R}^l$). The classification of all irreducible such groups is given in [6] in terms of Coxeter graphs.

We have the following

Remark 3.6 *For sufficiently large a, problem (1) has solutions with exact symmetry $G = G_1 \times \mathbf{O}(n-l)$ where:*

- *for $l = 2$ we have the dihedral groups $G_1 = D_k$ for $k \geq 3$ (corresponding to the graphs H_2^k).*
- *for $l \geq 3$, there are at least two possibilities for G_1: the group that leaves invariant the regular simplex (tetrahedron) in which case $k = l + 1$ (corresponds to the graph A_l), and the group that leaves invariant the cube, for which $k = 2l$ (corresponds to the graph B_l).*
- *for $l = 3$, besides the two above there is another choice for G_1: the group that leaves invariant the dodecahedron (icosahedron) for which $k = 12$ (corresponds to the graph I_3).*
- *for $l = 4$, G_1 is denoted F_4, is a finite group larger than the group of the cube.*

In all cases above, G_1 satisfies $(\mathbf{S_2'})$. In the case G_1 is the symmetry group of the cube or F_4 see the arguments in [14]. For G_1 the symmetry group of the regular simplex, it is not difficult to see that there are only two orbits having $l+1$ points each, and if $\xi \in S^{l-1}$ is a point with $l + 1 = \#G_1\xi$ then $-\xi$ generates the other orbit with $l+1$ points. Therefore, if H_1 is a subgroup of $\mathbf{O}(n)$ containing strictly G_1, then

$$\min_{\xi \in S^{l-1}} \#H_1\xi > l + 1.$$

Proof of Theorem 1.2: Consider $G = D_k \times \mathbf{O}(n-2)$. By Theorem 2.22 we have solutions for each $k \geq 2$ distinguished by their energy level. Moreover, by Theorem 3.2 the isotropy of any such solution is G. □

For the existence part (Theorem 2.22), condition $(\mathbf{S_1})$ is sufficient. In this case there are numerous possibilities of splitting $\mathbf{R}^l$ in a direct sum of orthogonal subspaces, invariant under the action of G_1. The number k (implicitly the energy level) distinguishes between solutions corresponding to different groups.

Remark 3.7 *We give an example in which $(\mathbf{S_2})$ is not satisfied and the exact symmetry result (Theorem 3.2) fails to hold. This partially indicates certain necessity of our condition $(\mathbf{S_2})$ for G being prescribed symmetry. Consider $n = 2$ and $G_1 = G = \mathbf{Z}_k$. Then it is easy to see $(\mathbf{S_2})$ is not satisfied. By the result of Coffman [3] (or applying Theorem 2.22 in this paper) we still get many rotationally non-equivalent solutions u_k for a large. Then Kawohl in [8] examined the exact symmetry of these solutions and showed that $\Sigma_{u_k} = D_k$, which is larger than $\mathbf{Z}_k$, i.e., $\Sigma_{u_k} \neq \mathbf{Z}_k$.*

We believe that only under $(\mathbf{S_1})$ and $(\mathbf{S_2'})$, the solutions obtained in Section 2 must have the isotropy group exactly $G_1 \times \mathbf{O}(n-l)$.

4 Concluding Remarks

4.1 Least Energy Solutions

In the case $l = n$,

$$P : \mathbf{R}^n \to \mathbf{R}^n$$

is just the identity map, so $a \leq |Px| \leq a + 1$ for any $x \in \Omega_a$. This implies as $a \to \infty$, $\gamma_a(u) \to 1$ for any $u \in \mathcal{M}_{G,a}$. For a sufficiently large, $\mathcal{M}_{G,a} = \mathcal{K}^\sigma_{G,a}$, i.e. the minimizers of the energy in $\mathcal{K}^\sigma_{G,a}$ are global minimizers in $\mathcal{M}_{G,a}$. So by a different method we recover the known results; namely we have the following

Theorem 4.1 *For $n = 2$ or $n \geq 4$, as $a \to \infty$ the number of rotationally nonequivalent solutions of problem (1) tends to infinity.*

Moreover, suppose $G = G_1 \times G_2$, and $k_1 = \min_{x \in S^{l-1}} \#G_1 x$ and $k_2 = \min_{x \in S^{n-l-1}} \#G_2 x$. Theorem 2.22 is applicable with $k = \min\{k_1, k_2\}$, and gives precise information on the energy level and the concentration of these solutions.

For $n = 3$ the global minimizers of the energy are discussed in [15]. As $a \to \infty$ solutions can concentrate:

- at one point under symmetry $G = \{e\} \times \mathbf{O}(2)$, (axially symmetric solutions),
- at two points under symmetry $G = \mathbf{Z}_2 \times \mathbf{O}(2)$, (axially symmetric solutions),
- at four points with G the group leaving a regular tetrahedron invariant,
- at six points with G the group leaving a cube invariant (solutions concentrate at the vertices of an octahedron-dual to the cube),
- at twelve points with G the group leaving invariant an icosahedron (or the dual dodecahedron),
- there is a rotationally invariant solution for $G = \mathbf{O}(3)$ whose energy tends to infinity as $a \to \infty$.

By arguments similar to those in Section 3, now we can show that for $n = 3$ there are G-invariant least energy solutions having exact symmetry G, when G is one of the following: $\{e\} \times \mathbf{O}(2)$, $\mathbf{Z}_2 \times \mathbf{O}(2)$, the tetrahedral subgroup, the cube subgroup, the icosahedral subgroup, and $\mathbf{O}(3)$. However, the solutions having exact symmetry $D_k \times \mathbf{O}(1)$ ($k \geq 3$) from Theorem 1.2 cannot be global minimizers.

4.2 Extensions

With almost no changes of our proof, we may consider the following

$$\left\{ \begin{array}{rcll} -\Delta u + \lambda u & = & u^p & \text{in} \quad \Omega_a \\ u & > & 0, & \\ u & = & 0, & \text{on} \quad \partial\Omega_a \end{array} \right. \tag{83}$$

where $\lambda > -\pi^2$ is a constant. Then all conclusions for (1) hold for (83) for a large. We only point out that for a large $\int_{\Omega_a} (|\nabla u|^2 + \lambda u^2) dx$ is a norm for $H_0^1(\Omega_a)$. This is due to the fact $\lambda_1(-\Delta)$, the first eigenvalue with respect to Ω_a converges to π^2, the first eigenvalue on D (e.g., [11]). One may also consider more general elliptic operators than the Laplacian. For a more general nonlinear term $f(u)$ instead of u^p, one may use the Nehari manifold approach (e.g., [15] [22]).

Finally we want to point out that with minor changes, our method applies to more general expanding domains with symmetries. Consider C being a smooth hypersurface in $\mathbf{R}^n$ diffeomorphic to S^{n-1}. Let G satisfying $(\mathbf{S_1})$ and suppose C is G invariant. For $a > 0$ denote

$$C_a = \{x \in \mathbf{R}^n : x/a \in C\}.$$

We take Ω_a to be

$$\Omega_a = \{x + \lambda\nu(x) \in \mathbf{R}^n : x \in C_a, \ \lambda \in (0,1)\},$$

where $\nu(x)$ is the exterior unit normal at $x \in C_a$.

We have the following

Remark 4.2 *For Problem (1) where the domain Ω_a is as above, Theorem 2.22 still holds.*

Without loss of generality we can assume

$$\operatorname{dist}(0, C) = 1, \text{ where } 0 \text{ is the origin in } \mathbf{R}^n.$$

We keep the same notations for $\mathcal{H}_{G,a}$, $\mathcal{M}_{G,a}$ and $\mathcal{K}^{\sigma}_{G,a}$. The only major change is in the construction of the operators

$$T : H^1_0(\Omega_a) \to H^1_0(D) \text{ and } \bar{T} : H^1_0(D) \to H^1_0(\Omega_a).$$

This can be done using diffeomorphisms $\Phi : C \to S^{n-1}$; around any point $x \in C$ for a sufficiently small neighborhood there is a diffeomorphism Φ which restricted to this neighborhood is close to a rigid transformation in $\mathbf{R}^n$. Proposition 2.16 holds with the same value for σ, and so Theorem 2.22 holds. Roughly speaking, for a minimizing sequence in $\mathcal{K}^{\sigma}_{G,a_m}$ the concentration in $\{0\} \times \mathbf{R}^{n-l} \cap \Omega_{a_m}$ is prohibited by the requirement $\gamma_{a_m}(u_m) > \sigma$, and the concentration at points that are neither in $\{0\} \times \mathbf{R}^{n-l}$ nor in $\mathbf{R}^l \times \{0\}$ cannot happen because the G orbit of such a point contains more than k elements.

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