ON THE HÉNON EQUATION : ASYMPTOTIC PROFILE OF GROUND STATES

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Abstract. This paper is concerned with the qualitative property of the ground state solutions for the Hénon equation. By studying a limiting equation on the upper half space $\mathbb{R}^N_+$, we investigate the asymptotic energy and the asymptotic profile of the ground states for the Hénon equation. The limiting problem is related to a weighted Sobolev type inequality which we establish in this paper.

Résumé. Nous nous intéresserons, dans cet article, aux propriétés qualitatives des fonctions minimisantes (ou "ground state solutions") de l'équation d'Hénon. L'étude d'une équation limite dans le demi-espace supérieur $\mathbb{R}^N_+$, nous renseignera sur l'énergie et les caractéristiques limites des fonctions minimisantes de l'équation d'Hénon. Notons que le problème limite est en relation avec une inégalité de Sobolev pondérée que nous établirons également.

1. Introduction

In this paper we investigate the Hénon equation

$$\Delta u + |x|^\alpha u^p = 0 \quad u > 0 \quad \text{in} \quad \Omega$$
$$u = 0 \quad \text{on} \quad \partial\Omega, \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$. For $\alpha \geq 0$, $2 < p+1 < 2^* := \frac{2N}{N-2}$ (in the case $N = 1, 2,\ 2^* = +\infty$), it is easy to show that

$$I^{all,\alpha}(\Omega) \equiv \inf_{u \in H^1_0(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{(\int_\Omega |x|^\alpha |u|^{p+1} dx)^{\frac{2}{p}}}$$

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is achieved by a positive function u, which by re-scaling gives a *ground state solution* of (1). $I_\alpha^{all}(\Omega)$ is called the *ground state energy*, or *the least energy*. Numerical computations ([4]) suggest that when Ω is the unit ball $B(0,1)$, for some parameter α the ground state solutions are nonradial. This was confirmed recently in [10], in which it was proved that for each $2 < p+1 < 2^*$ and $N \geq 2$, there exists α^* such that for $\alpha > \alpha^*$ the ground states are nonradial. In fact, they compare $I^{all,\alpha}(B(0,1))$ with another minimization problem

$$I^{rad,\alpha}(B(0,1)) \equiv \inf_{u \in H_0^1(B(0,1))\setminus\{0\}, u(x)=u(|x|)} \frac{\int_{B(0,1)} |\nabla u|^2 dx}{(\int_{B(0,1)} |x|^\alpha |u|^{p+1} dx)^{\frac{2}{p+1}}}.$$

It was shown that, if $p \in (1, (N+2)/(N-2))$ and $N \geq 2$, for sufficiently large $\alpha > 0$,

$$I^{all,\alpha}(B(0,1)) < I^{rad,\alpha}(B(0,1)).$$

More precisely, for $N \geq 2$, they showed that

$$\lim_{\alpha \to \infty} (\frac{N}{\alpha + N})^{\frac{p+3}{p+1}} I^{rad,\alpha}(B(0,1)) \in (0, \infty),$$

and that for some $c > 0$, as $\alpha \to \infty$

$$I^{all,\alpha}(B(0,1)) \leq c\alpha^{2-N+\frac{2N}{p+1}}.$$

Our main interest in this paper is about the asymptotic profiles of both the nonradial ground solutions and the radial ground states (i.e., the shape of these solutions). This is a natural question along the line of the study and has not been addressed at all. In order to study the asymptotic profile of the ground solutions we need to develop finer estimates on the ground state energy than those given above and to derive a limiting equation for the problem, which is essential to locating the asyptotic shape of the ground states. It turns out that the following minimization problem serves as limiting problem for equation (1):

$$J_{N,\beta}(\tilde{\Omega}) \equiv \inf \{ \int_{\tilde{\Omega}} |\nabla u|^2 dx \ \Big| \int_{\tilde{\Omega}} \exp(-\beta t) u^{p+1} dt dy = 1, u \in H(\tilde{\Omega}) \},$$

where $\tilde{\Omega} = (0, \infty) \times \mathbb{R}^{N-1}$, $H(\tilde{\Omega})$ is the completion of $C_0^\infty(\tilde{\Omega})$ with respect to the norm $||u||^2 = \int_{\tilde{\Omega}} |\nabla u|^2 dx$, and $\beta > 0$ and N is the dimension. More precisely, we shall prove that for any $N \geq 1$ and $p > 1$

$$\lim_{\alpha \to \infty} (\frac{N}{\alpha + N})^{\frac{p+3}{p+1}} I^{rad,\alpha}(B(0,1)) = |S^{N-1}|^{(p-1)/(p+1)} J_{1,N},$$

and that for any $N \geq 1$ and $p \in (1, 2^* - 1)$

$$\lim_{\alpha \to \infty} (\frac{N}{\alpha + N})^{\frac{N+2-(N-2)p}{p+1}} I^{all,\alpha}(B(0,1)) = J_{N,N}.$$

Furthermore, through more delicate analysis, we find asymptotic profiles of the minimizers of $I^{all,\alpha}(B(0,1))$ and $I^{rad,\alpha}(B(0,1))$ as $\alpha \to \infty$. Roughly speaking, under suitable transformations the minimizers of $I^{all,\alpha}(B(0,1))$ converge to the minimizers of $J_{N,N}(\tilde{\Omega})$ and the minimizers of $I^{rad,\alpha}(B(0,1))$ converge to the minimizers of $J_{1,N}(\tilde{\Omega})$. The precise statements will be given in Section 3 and Section 4. As a byproduct of our delicate analysis, we show that the symmetry breaking also occurs for $N = 1$. In order to study the limiting problems we need to first establish some weighted Sobolev type inequalities in the half space $\mathbb{R}^N_+$. These inequalities should have independent interest of their own.

In recent years, extensive work have been done for analyzing the limiting profile of least energy solutions of singularly perturbed elliptic problems including elliptic Dirichlet, Neumann boundary value problems and nonlinear Schrödinger equations in $\mathbb{R}^N$. Symmetry breaking of ground state solutions has been observed for some of these problems when the problems are radially invariant. Most of these problems have an associated limiting problem which are usually of the following form

$$-\Delta u + u = f(u), \quad \text{in } \mathbb{R}^N.$$

The existence and uniqueness of the ground state solutions for the limiting problems are used to get information for the ground state solutions of the singularly perturbed problems. For Hénon equation (1) we shall see that the appropriate limiting problem is much more complicated. The analysis of the limiting problem will be done in Section 2. After getting information for the limiting problem we shall study the asymptotic property of the ground state solutions of the Hénon equation (1).

2. Limiting equations and a Weighted Sobolev type inequality

As we mentioned, by a suitable transformation of the ground states, we obtain, in Section 3, the following limiting problem for equation (1).

$$\Delta u + \exp(-\beta t)u^p = 0, u > 0 \text{ in } \tilde{\Omega}, \qquad (2)$$
$$u = 0 \text{ on } \partial\tilde{\Omega},$$

where $\beta > 0$ and $\tilde{\Omega} = (0, \infty) \times \mathbb{R}^{N-1}$. We shall first study this equation in this section. In fact, we shall work on more general situations and consider more general $\tilde{\Omega}$. Let $(t, y) \in (-\infty, \infty) \times \mathbb{R}^{N-1}$. Let $\tilde{\Omega}$ be a domain in $(-\infty, \infty) \times \mathbb{R}^{N-1}$. Throughout this section, we assume that there exists $L > 0$ such that

$$\tilde{\Omega} \subset (-L, \infty) \times \mathbb{R}^{N-1}.$$

It is well known that there exists no solution for

$$\Delta u + u^p = 0, \qquad u > 0 \text{ in } \tilde{\Omega},$$
$$u = 0 \text{ on } \partial\tilde{\Omega}.$$

Moreover, for $p \in [1, (N+2)/(N-2))$,

$$\sup_{\varphi \in C_0^\infty((0,\infty)\times\mathrm{R}^{N-1})\setminus\{0\}} \frac{\left(\int_{(0,\infty)\times\mathrm{R}^{N-1}} \varphi^{p+1} dydt\right)^{2/(p+1)}}{\int_{(0,\infty)\times\mathrm{R}^{N-1}} |\nabla\varphi|^2 dydt} = \infty.$$

On the other hand, we have the following weighted Sobolev inequality.

Proposition 2.1. *Let* $p \in [1, (N+2)/(N-2)]$ *for* $N \geq 3$, $p \in [1, \infty)$ *for* $N = 1, 2$ *and* $\beta > 0$. *Then, there exists a constant* $C > 0$, *depending only on* β, p, N *and* L, *such that for any* $\varphi \in C_0^\infty(\tilde{\Omega})$,

$$\left(\int_{\tilde{\Omega}} \exp(-\beta t)\varphi^{p+1} dydt\right)^{\frac{2}{p+1}} \leq C \int_{\tilde{\Omega}} |\nabla\varphi|^2 dydt.$$

Proof. Since $\exp(-\beta t) \leq \exp(\beta L)$ in $\tilde{\Omega}$, the case $p = (N+2)/(N-2)$ comes from Sobolev inequality.

Let $\varphi \in C_0^\infty(\tilde{\Omega})$. We see from integration by parts and Cauchy's inequality that

$$\int_{-\infty}^{\infty} \exp(-\beta t)(\varphi(y,t))^2 dt = \frac{2}{\beta}\int_{-\infty}^{\infty} \exp(-\beta t)\varphi(y,t)\varphi_t(y,t)dt$$
$$\leq \frac{1}{2}\int_{-\infty}^{\infty} \exp(-\beta t)(\varphi(y,t))^2 dt + \frac{8}{\beta^2}\int_{-\infty}^{\infty} \exp(-\beta t)(\varphi_t(y,t))^2 dt.$$

Then, it follows that

$$\int_{-\infty}^{\infty} \exp(-\beta t)(\varphi(y,t))^2 dt \leq \frac{16}{\beta^2}\int_{-\infty}^{\infty} \exp(-\beta t)(\varphi_t(y,t))^2 dt. \tag{3}$$

Thus, by integrating both sides over $\mathbb{R}^{N-1}$ in above inequality, we deduce that for any $\varphi \in C_0^\infty(\tilde{\Omega})$,

$$\begin{aligned}\int_{\tilde{\Omega}} \exp(-\beta t)(\varphi(y,t))^2 dydt &\leq \frac{16}{\beta^2}\int_{\tilde{\Omega}} \exp(-\beta t)(\varphi_t(y,t))^2 dydt \\ &\leq \frac{16}{\beta^2}\exp(\beta L)\int_{\tilde{\Omega}} |\nabla\varphi(y,t)|^2 dydt. \end{aligned}\tag{4}$$

This proves the case $p = 1$. The case $p = 2^* - 1$ with $N \geq 3$ comes from the Sovolev inequalities.

From now on, we assume that $p \in (1, 2^* - 1)$.

Let $N \geq 3$. Since $p + 1 \in (2, 2N/(N-2))$, there exists $s \in (0,1)$ such that $p + 1 = 2s + (1-s)2N/(N-2)$. Then, from Hölder's inequality, we see that

$$\int_{\tilde{\Omega}} \exp(-\beta t)(\varphi(y,t))^{p+1} dydt$$
$$\leq \Big(\int_{\tilde{\Omega}} \exp(-\beta t)\phi^2 dydt\Big)^s \Big(\int_{\tilde{\Omega}} \exp(-\beta t)\phi^{2N/(N-2)} dydt\Big)^{1-s}$$
$$\leq \exp(\beta L)\Big(\int_{\tilde{\Omega}} \exp(-\beta t)\varphi^2 dydt\Big)^s \Big(\int_{\tilde{\Omega}} \varphi^{2N/(N-2)} dydt\Big)^{1-s}.$$

Then, by Sobolev inequalities, there exists a constant $C > 0$ that $\varphi \in C_0^\infty(\tilde{\Omega})$,

$$\int_{\tilde{\Omega}} \exp(-\beta t)(\varphi(y,t))^{p+1} dydt \leq C\Big(\int_{\tilde{\Omega}} |\nabla\varphi(y,t)|^2 dydt\Big)^{(p+1)/2}.$$

Thus, the case of $N \geq 3$ is finished.

For the case $N=1$, we see that

$$\exp(-\frac{\beta t}{2(p-1)})\varphi(t)=\int_{-\infty}^{t}-\frac{\beta}{2(p-1)}\exp(-\frac{\beta s}{2(p-1)})\varphi(s)+\exp(-\frac{\beta s}{2(p-1)})\varphi'(s)ds.$$

Thus, it follows from Cauchy's inequality and (3) that for some $C=C(p,\beta,L)>0$,

$$\begin{aligned}&\sup_{t\in(-\infty,\infty)}|\exp(-\frac{\beta t}{2(p-1)})\varphi(t)|^2\\&\le(\frac{\beta}{2(p-1)})^2\int_{-L}^{\infty}\exp(-\frac{\beta s}{2(p-1)})ds\int_{-\infty}^{\infty}\exp(-\frac{\beta s}{2(p-1)})(\varphi(s))^2ds\\&\quad+\int_{-L}^{\infty}\exp(-\frac{\beta s}{2(p-1)})ds\int_{-\infty}^{\infty}\exp(-\frac{\beta s}{2(p-1)})(\varphi'(s))^2ds\\&\le C\int_{-\infty}^{\infty}(\varphi'(s))^2ds.\end{aligned}$$

Then, for any $p\ge 1$, we deduce from (3) that for some $C=C(\beta,p,L)>0$,

$$\begin{aligned}&\int_{-\infty}^{\infty}\exp(-\beta t)(\varphi(t))^{p+1}dt\\&\le\sup_{t\in(-\infty,\infty)}|\exp(-\frac{\beta t}{2(p-1)})\varphi(t)|^{p-1}\int_{-\infty}^{\infty}\exp(-\frac{\beta t}{2})(\varphi(t))^2dt\\&\le C\Big(\int_{-\infty}^{\infty}(\varphi')^2dt\Big)^{(p+1)/2}.\end{aligned}$$

For the case $N=2$, we see that

$$\begin{aligned}\exp(-\frac{\beta}{2}t)|\varphi(y,t)|&\le\int_{-\infty}^{y}\exp(-\frac{\beta}{2}t)|\frac{\partial\varphi(y,t)}{\partial y}|dy,\\\exp(-\frac{\beta}{2}t)|\varphi(y,t)|&\le\int_{-\infty}^{t}\exp(-\frac{\beta}{2}t)|\frac{\partial\varphi(y,t)}{\partial t}|+\frac{\beta}{2}\exp(-\frac{\beta}{2}t)|\varphi(y,t)|dt.\end{aligned}$$

Then, multiplying each sides and integrating over $\tilde{\Omega}$, we see that

$$\begin{aligned}\int_{\tilde{\Omega}}\exp(-\beta t)(\varphi(y,t))^2dtdy&\le\Big(\int_{\tilde{\Omega}}\exp(-\frac{\beta}{2}t)|\nabla\varphi(y,t)|dydt\Big)^2\\&\quad+\frac{\beta}{2}\int_{\tilde{\Omega}}\exp(-\frac{\beta}{2}t)|\nabla\varphi(y,t)|dydt\int_{\tilde{\Omega}}\exp(-\frac{\beta}{2}t)|\varphi(y,t)|dtdy.\end{aligned}$$

Replacing φ by φ^m in above inequality, we deduce from Cauchy's inequality that

$$\begin{aligned}
&\int_{\tilde{\Omega}} \exp(-\beta t)\varphi^{2m} \\
&\leq \Big(\int_{\tilde{\Omega}} \exp(-\frac{\beta}{2}t) m|\varphi|^{m-1}|\nabla\varphi| \Big)^2 \\
&\quad + \frac{\beta}{2} \int_{\tilde{\Omega}} \exp(-\frac{\beta}{2}t) m|\varphi|^{m-1}|\nabla\varphi| \int_{\tilde{\Omega}} \exp(-\frac{\beta}{2}t)|\varphi|^m \\
&\leq m^2 \int_{\tilde{\Omega}} \exp(-\frac{\beta}{2}t)\varphi^{2(m-1)} \int_{\tilde{\Omega}} \exp(-\frac{\beta}{2}t)|\nabla\varphi|^2 \\
&\quad + \frac{m\beta}{2}\Big(\int_{\tilde{\Omega}} \exp(-\frac{\beta}{2}t)\varphi^{2(m-1)} \int_{\tilde{\Omega}} \exp(-\frac{\beta}{2}t)|\nabla\varphi|^2 \Big)^{\frac{1}{2}} \int_{\tilde{\Omega}} \exp(-\frac{\beta}{2}t)|\varphi|^m.
\end{aligned}$$

Thus, there exists $C = C(m, \beta, L) > 0$ such that for any $\varphi \in C_0^\infty(\tilde{\Omega})$,

$$\begin{aligned}
&\int_{\tilde{\Omega}} \exp(-\beta t)\varphi^{2m} \\
&\leq C \int_{\tilde{\Omega}} \exp(-\frac{\beta}{2}t)\varphi^{2(m-1)} \int_{\tilde{\Omega}} |\nabla\varphi|^2 \\
&\quad + \frac{m\beta}{2}\Big(\int_{\tilde{\Omega}} \exp(-\frac{\beta}{2}t)\varphi^{2(m-1)} \int_{\tilde{\Omega}} |\nabla\varphi|^2 \Big)^{\frac{1}{2}} \int_{\tilde{\Omega}} \exp(-\frac{\beta}{2}t)|\varphi|^m.
\end{aligned}$$

Then, if it holds that for some $C = C(m, \beta, L) > 0$,

$$\Big(\int_{\tilde{\Omega}} \exp(-\frac{\beta}{2}t)|\varphi|^m \Big)^{2/m} \leq C \int_{\tilde{\Omega}} |\nabla\varphi|^2, \tag{5}$$

and

$$\Big(\int_{\tilde{\Omega}} \exp(-\frac{\beta}{2}t)\varphi^{2(m-1)} \Big)^{1/(m-1)} \leq C \int_{\tilde{\Omega}} |\nabla\varphi|^2, \tag{6}$$

it follows that for some $C = C(m, \beta, L) > 0$,

$$\Big(\int_{\tilde{\Omega}} \exp(-\beta t)\varphi^{2m} \Big)^{1/m} \leq C \int_{\tilde{\Omega}} |\nabla\varphi|^2. \tag{7}$$

Note from(4) that (5) and (6) hold for $m = 2$. Therefore, we deduce by induction that (7) holds for any $m \geq 2$. This completes the proof. $\square$

We define $H(\tilde{\Omega})$ to be the completion of $C_0^\infty(\tilde{\Omega})$ with respect to the following norm

$$\|u\| \equiv (\int_{\tilde{\Omega}} |\nabla u|^2 dt dy)^{1/2}.$$

Then, from Proposition 2.1, we see that $H(\tilde{\Omega})$ is a Hilbert space. We note that, due to Sobolev embedding, $H(\Omega) = D_0^{1,2}(\Omega)$ is well defined for $N \geq 3$ and any domain $\Omega \subset \mathbb{R}^N$. Proposition 2.1 assures that the space $H(\tilde{\Omega})$ is also well defined for $N = 1, 2$ if there exists $L > 0$ such that $\tilde{\Omega} \subset (-L, \infty) \times \mathbb{R}^{N-1}$.

Now, we are interested in the existence of a ground state solution of (2). As we will see in the following, the existence depends on the shape of $\partial\tilde{\Omega}$. The following condition (referred as E-condition later) is a natural one, in a technical reason, for the existence of a ground state solution. The existence will be given in Proposition 2.3. We do not know whether this condition is optimal or not(see Proposition 2.7).

Definition 2.2. We say that $\tilde{\Omega}$ satisfies E-condition if there exists a fixed point $y^0 = (y_1^0, \cdots, y_{N-1}^0) \in \mathbb{R}^{N-1}$ such that for any $(t, y_1, \cdots, y_{N-1}) \in \tilde{\Omega}$ and $s_1, \cdots, s_{N-1} \in [-1, 1]$,

$$(t, y_1^0 + s_1|y_1 - y_1^0|, \cdots, y_{N-1}^0 + s_{N-1}|y_{N-1} - y_{N-1}^0|) \in \tilde{\Omega},$$

or if for each $T > 0$, $\tilde{\Omega} \cap \big((-L, T) \times \mathbb{R}^{N-1}\big)$ is bounded.

Proposition 2.3. *Let $p \in (1, (N+2)/(N-2))$ for $N \geq 3$, $p \in (1, \infty)$ for $N = 1, 2$ and $\beta > 0$. Suppose that a domain $\tilde{\Omega}$ satisfies E-condition. Then, there exists a minimizer $u \in H(\tilde{\Omega})$ of the following minimization problem*

$$J_{N,\beta}(\tilde{\Omega}) \equiv \inf \left\{ \|u\|^2 \ \Big| \int_{\tilde{\Omega}} \exp(-\beta t) u^{p+1} dt dy = 1, u \in H(\tilde{\Omega}) \right\}.$$

To prove Proposition 2.3, we prepare a lemma. We first consider an eigenvalue problem

$$\begin{aligned} \frac{d^2\phi}{dt^2} + \lambda \exp(-\beta t)\phi = 0 \quad &\text{on } (0, \infty), \\ \phi(0) = 0 & \\ \phi \in H((0, \infty)) & \end{aligned} \tag{8}$$

Let $s = \frac{2\sqrt{\lambda}}{\beta} \exp(-\frac{\beta}{2} t)$ and $w(s) = \phi(t)$. Then, it follows that

$$\frac{d^2 w}{ds^2} + \frac{1}{s} w_s + w = 0. \tag{9}$$

Note that for some $C > 0$,

$$|\phi(t)| = |\int_0^t \phi'(s)ds| \le \sqrt{t}(\int_0^\infty |\phi'|^2 dt)^{1/2} \le C\sqrt{t}.$$

This implies that for some $C > 0$,

$$|w(s)| \le C(1 + |\log s|)^{1/2}.$$

There are two kinds of solutions, Bessel functions of the first kind J_0 and the second kind N_0, for (9). Since $N_0(s) \approxeq \log s$ near 0, it follows that $w(s) = J_0(s)$. The Bessel function of the first kind $J_0(s)$ is given by $\sum_{n=0}^\infty \frac{(-1)^n}{(n!)^2}(\frac{s}{2})^{2n}$. Let $j_1 < j_2 < \cdots$ be the positive zeros of J_0. Then, it is well-known that

$$J_0(s) = \Pi_{n=1}^\infty (1 - \frac{s^2}{(j_n)^2}).$$

Thus, we have the following lemma.

Lemma 2.4. *The eigenvalues* $\{\lambda_{\beta,n}\}_{n=1}^\infty$ *of (8) are given by*

$$\lambda_{\beta,n} = \frac{(j_n)^2\beta^2}{4}, \quad n = 1, 2, \cdots.$$

The eigenfunction $\phi_{\beta,n}$ *corresponding to* $\lambda_{\beta,n}$ *is given by*

$$\phi_{\beta,n}(t) = J_0(\frac{2\sqrt{\lambda_{\beta,n}}}{\beta}\exp(-\frac{\beta}{2}t)), \quad t \in [0,\infty).$$

Proof Proposition 2.3. Let $\{v_n\}_n$ be a minimizing sequence of $J_{N,\beta}(\tilde{\Omega})$. Since $H(\tilde{\Omega})$ is the completion of $C_0^\infty(\tilde{\Omega})$, we can assume that $\{v_n\}_n \subset C_0^\infty(\tilde{\Omega})$. We take $T_n^1, T_n^2 > 0$ such that for each positive integer n,

$$\text{supp}(v_n) \subset \{(t,y) \in \tilde{\Omega} | \; -L < t < T_n^1, |y| < T_n^2\},$$

and that for each $i = 1, 2$, $T_1^i < T_2^i < \cdots$ and $\lim_{n\to\infty} T_n^i = \infty$. Define

$$D_n \equiv (-L, T_n^1) \times \{y \in \mathbb{R}^{N-1} | \; |y| \le T_n^2\}.$$

When $\tilde{\Omega}\cap\big((-L,T_n^1)\times\mathbb{R}^{N-1}\big)$ is bounded, we can take larger T_n^2 so that $\tilde{\Omega}\cap\big((-L,T_n^1)\times\mathbb{R}^{N-1}\big)\subset D_n$. Then, we consider a following minimization problem

$$I_n \equiv \inf\Big\{\|u\|^2 \ \Big|\int_{\tilde{\Omega}\cap D_n}\exp(-\beta t)u^{p+1}dtdy=1, u\in H_n\setminus\{0\}\Big\},$$

where $H_n\equiv H_0^{1,2}(\tilde{\Omega}\cap D_n)$. Since D_n is bounded, there exists a nonnegative minimizer u_n of I_n for each $n\geq 1$. It is easy to see that $I_n\to J_{N,\beta}(\tilde{\Omega})$ as $n\to\infty$.

When $\tilde{\Omega}\cap\big((-L,T_n^1)\times\mathbb{R}^{N-1}\big)$ is not bounded, from a Steiner symmetrization (refer [7]), we can assume that $u_n(t,y^0+z)$ is even with respect to each of the components of $z=(z_1,...,z_N-1)$ and is monotone decreasing in each of the components of z. Then, $\{u_n\}_n$ is also a minimizing sequence of $J_{N,\beta}(\tilde{\Omega})$, and $\lim_{n\to\infty}I_n=J_{N,\beta}(\tilde{\Omega})$.

Moreover, we see that

$$\begin{aligned}\Delta u_n+I_n\exp(-\beta t)(u_n)^p&=0 &&\text{in } \tilde{\Omega}\cap D_n,\\ u_n&>0 &&\text{in } \tilde{\Omega}\cap D_n,\\ u_n&=0 &&\text{on } \partial(\tilde{\Omega}\cap D_n).\end{aligned}\tag{10}$$

Note that

$$\begin{aligned}\Delta u_n+I_n\exp(-\beta L)(u_n)^p&\geq 0 &&\text{in } \tilde{\Omega}\cap D_n,\\ u_n&=0 &&\text{on } \partial(\tilde{\Omega}\cap D_n).\end{aligned}$$

Then, since $\{\|u_n\|_{L^{2N/(N-2)}}\}_n$ is bounded for $N\geq 3$, from a uniform L^∞−estimate [2, Proposition 3.5], we see that $\{\|u_n\|_{L^\infty}\}_n$ is bounded for $N\geq 3$. For $N=1$, it follows that for some $C>0$,

$$u_n(t)=\int_{-L}^t u_n'(s)ds\leq\sqrt{t+L}\Big(\int_{-L}^\infty|u_n'(s)|^2ds\Big)^{1/2}\leq C\sqrt{t}.$$

For $N=2$, we use the Green function on the upper half plane. Then there is a constant $C>0$,

$$u(t-L,x)\leq C\int_0^\infty\int_{-\infty}^\infty\log\frac{(t+s)^2+(x-y)^2}{(t-s)^2+(x-y)^2}\exp(-\beta s)u^p(y)dyds.$$

Since $\{\int_0^\infty \int_{-\infty}^\infty \exp(-\beta s)(u_n)^{p+1}(y)dyds\}_n$ is bounded, from Hölder's inequality, it follows that for some $C > 0$,

$$\begin{aligned}
&(u_n(t-L,x))^{p+1} \\
&\le C\int_0^\infty \int_{-\infty}^\infty \Big(\log \frac{(t+s)^2+(x-y)^2}{(t-s)^2+(x-y)^2}\Big)^{p+1} \exp(-\beta s)dyds \\
&\le C\exp(-\beta t)\int_{-t}^\infty \int_{-\infty}^\infty \Big(\log \frac{(s+2t)^2+(x-y)^2}{s^2+(x-y)^2}\Big)^{p+1} \exp(-\beta s)dyds \\
&= C\exp(-\beta t)\int_{-t}^\infty \int_{-\infty}^\infty \Big(\log(1+\frac{4t(s+t)}{s^2+(x-y)^2})\Big)^{p+1} \exp(-\beta s)dyds \\
&= C\exp(-\beta t)\int_{-t}^\infty \int_{-\infty}^\infty \Big(\log(1+\frac{4t(s+t)}{s^2+y^2})\Big)^{p+1} \exp(-\beta s)dyds.
\end{aligned}$$

Note that $\log(1+a) \le a$ for $a > 0$. Thus, for some constant $C > 0$ and $t > 0$,

$$\begin{aligned}
&\int_{-t}^\infty \int_{-\infty}^\infty \Big(\log(1+\frac{4t(s+t)}{s^2+y^2})\Big)^{p+1} \exp(-\beta s)dyds \\
&= \int_{(-t,\infty)\times(-\infty,\infty)\cap B(0,1)} \Big(\log(1+\frac{4t(s+t)}{s^2+y^2})\Big)^{p+1} \exp(-\beta s)dyds \\
&\quad + \int_{(-t,\infty)\times(-\infty,\infty)\setminus B(0,1)} \Big(\log(1+\frac{4t(s+t)}{s^2+y^2})\Big)^{p+1} \exp(-\beta s)dyds \\
&\le \int_0^1 \int_0^{2\pi} \Big(\log(1+\frac{4t(1+t)}{r^2})\Big)^{p+1} \exp(-\beta r\sin\theta)rd\theta dr \\
&\quad + (4t)^{p+1}\int_{-1}^1 \int_{s=\sqrt{1+y^2}}^\infty (s+t)^{p+1}\exp(-\beta s)dsdy \\
&\quad + (4t)^{p+1}\int_{-1}^1 \int_{s=-t}^{s=-\sqrt{1+y^2}} (s+t)^{p+1}\exp(-\beta s)dsdy \\
&\quad + (4t)^{p+1}\int_{|y|\ge 1} \int_{s=-t}^\infty \Big(\frac{(s+t)}{y^2}\Big)^{p+1}\exp(-\beta s)dsdy \\
&\le Ct(1+t) + Ct^{p+1}\int_{-t}^\infty \int_{s=-t}^\infty (s+t)^{p+1}\exp(-\beta s)ds \\
&\le C(t(1+t))^{p+1}\exp(\beta t).
\end{aligned}$$

This implies that there exists a constant $C > 0$, independent of $n \ge 1$, such that

$$u_n(t,x) \le C(1+t)^C, \qquad t > 0. \tag{11}$$

Having established some upper bounds we need an estimate from below for the L^∞

norm. From Proposition 2.2 and the fact that

$$\int_{\tilde{\Omega}\cap D_n} |\nabla u_n|^2 dxdt \le I_n \|u_n\|_{L^\infty}^{p-1} \int_{\tilde{\Omega}\cap D_n} \exp(-\beta t)(u_n)^2 dxdt,$$

we see that the set $\{\|u_n\|_{L^\infty}\}_n$ is bounded away from 0.

Next, we consider the convergence of u_n and we consider several cases. First, for $N = 1$, it is easy to see that

$$\lim_{T\to\infty} \int_T^\infty \exp(-\beta t)(u_n)^{p+1} dt = 0 \text{ uniformly with respect to } n = 1, 2, \cdots .$$

Since $\{u_n\}_n$ is bounded in H, u_n converges weakly to some u in H. Then,

$$\int_{\tilde{\Omega}} \exp(-\beta t) u^{p+1} dt = 1.$$

This implies that u is a minimizer of $J_{1,\beta}(\tilde{\Omega})$.

For the case $N \ge 2$, we claim that for sufficiently large $T > 0$,

$$\liminf_{n\to\infty} \sup\{u_n(t,y) |\ -L < t < T, (t,y) \in \tilde{\Omega} \cap D_n\} > 0.$$

Suppose that it is not true. Then, for any $T > 0$,

$$\liminf_{n\to\infty} \sup_{-L<t<T, (t,y)\in\tilde{\Omega}\cap D_n} u_n(t,y) = 0.$$

Taking a subsequence if necessary, we can assume that for sufficiently large $T > 0$,

$$\lim_{n\to\infty} \sup_{-L<t<T, (t,y)\in\tilde{\Omega}\cap D_n} u_n(t,y) = 0.$$

Let $(\phi_{\frac{\beta}{2},1}, \lambda_{\frac{\beta}{2},1})$ be a pair of the first eigenfunction and the first eigenvalue of

$$\frac{d^2\phi}{dt^2} + \lambda \exp(-\beta t/2)\phi = 0 \quad \text{ on } (0,\infty),$$

$$\phi(0) = 0$$

$$\phi \in H((0,\infty))$$

satisfying that for $t > 0$, $\phi_{\frac{\beta}{2},1}(t) > 0$, and $\lim_{t\to\infty} \phi_{\frac{\beta}{2},1}(t) = 1$. From the estimate (11) for $N = 2$ and and the boundedness of $\{\|u_n\|_{L^\infty}\}_n$ for $N \ge 3$, we see that

$\lim_{t\to\infty} \exp(-\frac{\beta}{2}t)(u_n(t,y))^{p-1} = 0$ uniformly with respect to $y \in \mathbb{R}^{N-1}$. Thus, there exists sufficiently large $T > 0$ such that

$$\begin{aligned}&\Delta\phi_{\frac{\beta}{2},1} + I_n \exp(-\beta t)(u_n)^{p-1}\phi_{\frac{\beta}{2},1}\\&= \phi_{\frac{\beta}{2},1}\exp(-\beta t)\left(I_n u_n^{p-1} - \exp(\beta t/2)\lambda_{\frac{\beta}{2},1}\right)\\&\le 0, \qquad t \ge T.\end{aligned}$$

From a comparison principle(refer [9]), we see that for each $n = 1, \cdots,$ and $t \ge T$,

$$u_n(t,y) \le \phi_{\frac{\beta}{2},1}(t) \max_{\{y|(T,y)\in\tilde{\Omega}\cap\tilde{D}_n\}} u_n(T,y)/\phi_{\frac{\beta}{2},1}(T).$$

This implies that

$$\lim_{n\to\infty} \sup_{(t,y)\in\tilde{\Omega}\cap D_n} u_n(t,y) = 0;$$

this contradicts that $\{\|u_n\|_{L^\infty}\}_n$ is bounded away from 0. This proves the claim.

Now, taking a subsequence if necessary, we can assume that u_n converges weakly to some u in $H(\tilde{\Omega})$ as $n \to \infty$. From the boundedness of $\{\|u_n\|_{L^\infty}\}_n$, we see that for some $\rho \in (0,1)$, $\{\|u_n\|_{C^{2,\rho}}\}_n$ is bounded. Thus,we can assume that u_n converges locally to u in C^2 as $n \to \infty$. Note that $\lim_{n\to\infty} I_n = J_{N,\beta}(\tilde{\Omega})$. Thus, this u satisfies the following equation

$$\begin{aligned}\Delta u + J_{N,\beta}(\tilde{\Omega})\exp(-\beta t)u^p = 0 &\qquad \text{in } \tilde{\Omega},\\ u = 0 &\qquad \text{on } \partial\tilde{\Omega}.\end{aligned}$$

From the preceeding claim, we see that if $\tilde{\Omega}$ satisfies the first assertion of $E-$condition in definition 2.2, for sufficiently large $T > 0$,

$$\liminf_{n\to\infty} \max_{t\in(-L,T),(t,y)\in\tilde{\Omega}\cap D_n} u_n(t,y) = \liminf_{n\to\infty} \max_{t\in(-L,T)} u_n(t,y^0) > 0,$$

and that if $\tilde{\Omega}$ satisfies the second assertion of $E-$condition in Definition 2.2, for sufficiently large $T > 0$,

$$\liminf_{n\to\infty} \max_{-L<t<T,(t,y)\in\tilde{\Omega}} u_n(t,y) = \liminf_{n\to\infty} \max_{-L<t<T,(t,y)\in\tilde{\Omega}\cap D_n} u_n(t,y) > 0.$$

Thus, it follows that $u \neq 0$. Let

$$\gamma \equiv \int_{\tilde{\Omega}} \exp(-\beta t) u^{p+1} dt dy \in (0,1].$$

Suppose that $\gamma < 1$. Then, taking $w = \gamma^{-1/(p+1)} u$, we see that

$$\int_{\tilde{\Omega}} \exp(-\beta t) w^{p+1} dt dy = 1,$$

and that

$$\begin{aligned} \Delta w + J_{N,\beta}(\tilde{\Omega}) \gamma^{(p-1)/(p+1)} \exp(-\beta t) w^p = 0 & \qquad \text{in } \tilde{\Omega}, \\ w > 0 & \qquad \text{in } \tilde{\Omega}, \\ w = 0 & \qquad \text{on } \partial\tilde{\Omega}. \end{aligned}$$

By integration by parts, we get

$$\int_{\tilde{\Omega}} |\nabla w|^2 dt dy = J_{N,\beta}(\tilde{\Omega}) \gamma^{(p-1)/(p+1)} < J_{N,\beta}(\tilde{\Omega}).$$

This contradicts the definition of $J_{N,\beta}(\tilde{\Omega})$. Thus, it follows that

$$\int_{\tilde{\Omega}} \exp(-\beta t) u^{p+1} dt dy = 1.$$

Since

$$\int_{\tilde{\Omega}} |\nabla u|^2 dt dy \leq \liminf_{n \to \infty} \int_{\tilde{\Omega}} |\nabla u_n|^2 dt dy = J_{N,\beta}(\tilde{\Omega}),$$

we conclude that u is a minimizer of $J_{N,\beta}(\tilde{\Omega})$. This completes the proof. □

For our applications to the Hénon equation (1), we are particularly interested in the case $\tilde{\Omega} = (0,\infty) \times \mathbb{R}^{N-1}$. In this case, we derive the following qualitative properties of the minimizers of $J_{N,\beta}((0,\infty) \times \mathbb{R}^{N-1})$.

Proposition 2.5. *Let u be a nonnegative minimizer of $J_{N,\beta}((0,\infty) \times \mathbb{R}^{N-1})$. Then,*

(i) For $N = 1$, u is a monotone increasing bounded function, and for any $c \in (0,\beta)$, there exists constants $C > 0$ such that

$$0 < \lim_{s \to \infty} u(s) - u(t) \leq C \exp(-ct).$$

(ii) For $N \geq 2$, for some $x_0 \in \mathbb{R}^{N-1}$, $u(t,x)$ depends only on t and $|x - x_0|$, and is monotone decreasing with respect to $|x - x_0|$.

(iii) For $N \geq 2$, $\lim_{|x|\to\infty} u(t,x) = 0$.

(iv) For $N \geq 3$, $\lim_{|(t,x)|\to\infty} u(t,x) = 0$.

(v) For $N \geq 4$, there exists $C > 0$ independent of $t > 0$ such that $u(t,x) \leq \frac{C}{|x|^{N-3}}$.

Proof. For simplicity of notations, let $J = J_{N,\beta}((0,\infty) \times \mathbb{R}^{N-1})$. First of all, we note that

$$\Delta u + J\exp(-\beta t)u^p = 0, \qquad u > 0 \text{ in } (0,\infty) \times \mathbb{R}^{N-1}$$
$$u = 0 \text{ on } \{0\} \times \mathbb{R}^{N-1}.$$

We prove (i) first. For $N = 1$, we see that

$$u(t) = \int_0^t u'(s)ds \leq \sqrt{t}(\int_0^t (u'(s))^2 ds)^{1/2} \leq C\sqrt{t}.$$

Denoting $w(t) = u'(t)$, we see that

$$w'' + \beta w' + pJ\exp(-\beta t)u^{p-1}w = 0 \text{ in } (0,\infty).$$

It is easy to see that $w(0) > 0$, and that u is monotone increasing and w is monotone decreasing. Since $\int_0^\infty w^2(t)dt < \infty$, it follows that $\lim_{t\to\infty} w(t) = 0$. Since $u(t) \leq C\sqrt{t}$, it follows that for $\phi(t) = \exp(-ct)$ with $c \in (0,\beta)$,

$$\phi'' + \beta\phi' + pJ\exp(-\beta t)u^{p-1}\phi < 0 \qquad \text{in} \qquad (T,\infty)$$

if $T > 0$ is sufficiently large. By a comparison principle(refer [9]), we see that for some $C > 0$,

$$w(t) \leq C\exp(-ct), \; t \in (0,\infty).$$

This and the monotonicity of u imply that for some $C > 0$,

$$0 < \lim_{s\to\infty} u(s) - u(t) \leq C\exp(-ct), \; t \in (0,\infty).$$

The proof of (ii) follows from using a rearrangement technique ([7]). We can show the monotonicity and the symmetry properties of u for $N \geq 2$.

The decay property (iii) $\lim_{|x|\to\infty} u(t,x) = 0$ for $N \geq 2$ follows from an elliptic estimate. The decay property (iv) $\lim_{|(t,x)|\to\infty} u(t,x) = 0$ follows from elliptic estimates and the fact that by the Sobolev inequalities for $N \geq 3$, $\|u\|_{L^{2N/(N-2)}} < \infty$. Finally, for $N > 3$, we consider a function

$$\psi_\beta(t,x) \equiv \frac{\phi_{\beta,1}(t)}{|x|^{N-3}},$$

where $\phi_{\beta,1}$ is the first eigenfunction of (8) with the corresponding eigenvalue $\lambda_{\beta,1}$. Then, we see that for $x \neq 0$,

$$\Delta\psi_\beta + \exp(-\beta t)u^{p-1}\psi_\beta = (u^{p-1}(t,x) - \lambda_{\beta,1})\exp(-\beta t)\psi_\beta.$$

From above decay property of u, we see that for sufficiently large $x \in \mathbb{R}^{N-1}$, $u^{p-1}(t,x) - \lambda_{\beta,1} < 0$ Thus, by a comparison principle(refer [9]), we see that for some constant $C > 0$,

$$u(t,x) \leq C\psi_\beta(t,x) \leq \frac{C}{|x|^{N-3}}.$$

This finishes the proof of (v). $\square$

Finally, we close up this section with a symmetry property and a non-existence result of positive solutions for equation (2) for more general $\tilde{\Omega}$.

Proposition 2.6. *For $N \geq 2$, let $u \in H(\tilde{\Omega})$ be a solution of (2) satisfying*

$$\lim_{|(t,x)|\to\infty} u(t,x) = 0.$$

Suppose that for any $(t,x) \in \tilde{\Omega}$, $(t,y) \in \tilde{\Omega}$ if $|y| \leq |x|$. Then, for some $x_0 \in \mathbb{R}^{N-1}$, u depends only on t and $r \equiv |x - x_0|$, and $\frac{\partial u(t,r)}{\partial r} < 0$ for $r \neq 0$ and $t > 0$.

Proof. We sketch the proof here since it is standard by now to show the symmetry property of positive solution via a moving plane method [4]. Let $x_0 \in \mathbb{R}^{N-1} \setminus \{0\}$. For $\lambda > 0$, let $T_\lambda = \{(t,x) | \langle x_0, x\rangle = \lambda\}$, and $E_\lambda = \{(t,x) \in \tilde{\Omega} \mid \langle x_0, x\rangle \geq \lambda\}$. For $(t,x) \in E_\lambda$, we denote (t, x^λ) the reflection of (t,x) with respect to T_λ and define $u_\lambda(t,x) = u(t,x^\lambda)$. Define $\overline{\lambda} \equiv \sup\{\lambda \in \mathbb{R} \mid E_\lambda \neq \emptyset\} \in (-\infty, \infty]$.

Suppose that for λ sufficiently close to $\overline{\lambda}$,

$$w_\lambda(t,x) \equiv \min\{u_\lambda(t,x) - u(t,x), 0\} \neq 0.$$

Then, it is easy to deduce that

$$\int_{E_\lambda} |\nabla w_\lambda|^2 dtdx \leq \|pu^{p-1}\|_{L^\infty(E_\lambda)} \int_{E_\lambda} \exp(-\beta t)(w_\lambda)^2 dtdx. \tag{12}$$

Note that for $\overline{\lambda} = \infty$, $\lim_{\lambda\to\infty} \|pu^{p-1}\|_{L^\infty(E_\lambda)} = 0$, and that for $\overline{\lambda} \neq \infty$, the first eigenvalue $-\Delta$ on E_λ goes to ∞ as $\lambda \to \overline{\lambda}$. Thus, the inequality (12) contradicts Proposition 2.1. This implies that for λ sufficiently close to $\overline{\lambda}$, $w_\lambda > 0$ on E_λ. Define $\lambda_0 \equiv \inf\{\lambda \in \mathbb{R} | w_\lambda > 0 \text{ on } E_\lambda\} > -\infty$. Then, combining above arguments and the Hopf maximum principle, we conclude that $w_{\lambda_0} \equiv 0$, that is, u is symmetric for the reflection with respect to T_{λ_0}. Since it holds for any $x_0 \in \mathbb{R}^{N-1} \setminus \{0\}$, the symmetry and monotonicity properties of u follow. □

As in the proof above, for $x_0 \in \mathbb{R}^{N-1}$ and $\lambda \in \mathbb{R}$, we define let $T_\lambda = \{(t,x) | \langle x_0, x\rangle = \lambda\}$, and $E_\lambda = \{(t,x) \in \tilde{\Omega} \mid \langle x_0, x\rangle \geq \lambda\}$. For $(t,x) \in E_\lambda$, we denote (t,x^λ) the reflection of (t,x) with respect to T_λ, and define $E'_\lambda = \{(t,x^\lambda) \mid (t,x) \in E_\lambda\}$. Therefore, we obtain the following non-existence result.

Proposition 2.7. *Suppose that there exists $x_0 \in \mathbb{R}^{N-1}$ such that for any $\lambda \in \mathbb{R}$ with $E_\lambda \neq \emptyset$,*

$$E'_\lambda \cup E_\lambda \subsetneqq \tilde{\Omega}.$$

Then, there exists no solution $u \in H(\tilde{\Omega})$ for equation (2) satisfying $\lim_{|(t,x)|\to\infty} u(t,x) = 0$.

Proof. We sketch the proof here. Denote $\overline{\lambda} \equiv \sup\{\lambda \in \mathbb{R} \mid E_\lambda \neq \emptyset\}$ and $\underline{\lambda} \equiv \inf\{\lambda \in \mathbb{R} \mid E'_\lambda \subset \tilde{\Omega}\}$. From the fact that $E'_\lambda \cup E_\lambda \subsetneqq \tilde{\Omega}$ for any $\lambda \in \mathbb{R}$ with $E_\lambda \neq \emptyset$, we see that $\underline{\lambda} = -\infty$.

Suppose that there exists a solution $u \in H(\tilde{\Omega})$ of (2) satisfying $\lim_{|(t,x)|\to\infty} u(t,x) = 0$. Then, as in the proof of Proposition 2.6, we see that for λ sufficiently close to $\overline{\lambda}$,

$$u(t,x^{\lambda}) > u(t,x), \qquad x \in E_{\lambda}.$$

Then, since $E'_{\lambda} \cup E_{\lambda} \subsetneqq \tilde{\Omega}$ for any $\lambda \in \mathbb{R}$ with $E_{\lambda} \neq \emptyset$, by the same argument as in the proof of Proposition 2.6, it follows that for $\lambda < \overline{\lambda}$,

$$u(t,x^{\lambda}) > u(t,x), \qquad x \in E_{\lambda}.$$

This contradicts that $\lim_{|(t,x)|\to\infty} u(t,x) = 0$. This proves the claim. $\square$

3. Asymptotic profile of least energy radial solutions on unit ball

In this section, we consider the limiting behaviour of the least energy radial solutions, i.e., the minimizers of $I^{ral,\alpha}(B(0,1))$. We consider both the asymptotics of limiting energy and limiting profile. Let $\Omega = B(0,1) \equiv \{x \in \mathbb{R}^N |\ |x| < 1\}$, and $H_{rad} \equiv \{u \in H_0^{1,2}(\Omega) | u(x) = u(|x|)\}$. We denote $J_{N,\beta} \equiv J_{N,\beta}((0,\infty) \times \mathbb{R}^{N-1})$. Then, we consider the following minimization problem

$$I^{rad,\alpha} \equiv \inf\Big\{ \int_{\Omega} |\nabla u|^2 dx \ \Big| \int_{\Omega} |x|^{\alpha} u^{p+1} dx = 1, u \in H_{rad} \Big\}. \tag{13}$$

In [8], Ni proved that the above minimization problem has a positive minimizer u^{α}_{rad} for $1 \le p < (N+2+2\alpha)/(N-2)$. Moreover, from the Pohozaev identity, we can show that there is no solution of equation (1) with $\Omega = B(0,1)$ for $p \ge (N+2+2\alpha)/(N-2)$. This u^{rad}_{α} satisfies the following equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{N-1}{r}\frac{\partial u}{\partial r} + I^{rad,\alpha}|x|^{\alpha}u^p = 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega.$$

In [10], it was shown that for $N \ge 2$, $\lim_{\alpha\to\infty}(\frac{N}{\alpha+N})^{\frac{p+3}{p+1}} I^{rad,\alpha} \in (0,\infty)$. We will examine the exact values of the limits $I^{rad,\alpha}$ and our analysis applies to $N = 1$ too. This analysis in turn will be used to find a fine asymptotic behaviour of the minimizers u^{rad}_{α} for $I^{rad,\alpha}$ as $\alpha \to \infty$. We have the following asymptotic result for $I^{rad,\alpha}$ and u^{rad}_{α}.

Theorem 3.1. *Let $N \geq 1$ and $p \geq 1$. Then*

$$\lim_{\alpha\to\infty}(\frac{N}{\alpha+N})^{\frac{p+3}{p+1}} I^{rad,\alpha} = |S^{N-1}|^{(p-1)/(p+1)} J_{1,N},$$

where $|S^{N-1}|$ is the volume of $(N-1)-$dimensional sphere S^{N-1}. For any $r \in (0,1]$, $\frac{du_\alpha^{rad}(r)}{dr} < 0$. Furthermore, under the following transformation

$$v_\alpha^{rad}(t) \equiv |S^{N-1}|^{\frac{1}{p+1}}(\frac{N}{\alpha+N})^{\frac{1}{p+1}} u_\alpha^{rad}(\exp(-\frac{N}{\alpha+N}t))$$

$v_\alpha^{rad}(t)$ converges uniformly on $(0,\infty)$ to a minimizer of $J_{1,N}$ as $\alpha \to \infty$.

For a nonnegative minimizer u_α^{rad} of I_α^{rad}, a scaled function $\omega_\alpha^{rad} \equiv (I_\alpha^{rad})^{\frac{1}{p-1}} u_\alpha^{rad}$ is a least energy solution(a mountain pass solution) of (1) in the class of radial functions H_{rad}. Proposition 2.3 implies that there exists a least energy solution (a mountain pass solution) of $\omega_{N,\beta}$ in $H((0,\infty)\times\mathbb{R}^{N-1})$ of the equation

$$\begin{aligned} \Delta u + \exp(-\beta t)u^p = 0 \qquad & \text{in } (0,\infty)\times\mathbb{R}^{N-1}, \\ u > 0 \qquad & \text{in } (0,\infty)\times\mathbb{R}^{N-1} \\ u = 0 \qquad & \text{on } \partial\{0\}\times\mathbb{R}^{N-1}. \end{aligned} \tag{14-(N, β)}$$

For the minimizer $u_{N,\beta}$ of $J_{N,\beta} \equiv J_{N,\beta}((0,\infty)\times\mathbb{R}^{N-1})$, the least energy solution $\omega_{N,\beta}$ of (14-(N,β)) is given by $(J_{N,\beta})^{\frac{1}{p-1}} u_{N,\beta}$. Then, we have the following equivalent version of Theorem 3.1 for ω_α^{rad} and its energy.

Theorem 3.1-E. *Let $N \geq 1$ and $p \geq 1$. Then*

$$\begin{aligned} &\lim_{\alpha\to\infty}(\frac{N}{\alpha+N})^{\frac{p+3}{p-1}} \int_{B(0,1)} \frac{1}{2}|\nabla\omega_\alpha^{rad}|^2 - \frac{1}{p+1}|x|^\alpha(\omega_\alpha^{rad})^{p+1}dx \\ &= |S^{N-1}| \int_0^\infty \frac{1}{2}|\nabla\omega_{1,N}|^2 - \frac{1}{p+1}\exp(-Nt)(\omega_{1,N})^{p+1}dt, \end{aligned}$$

where $|S^{N-1}|$ is the volume of $(N-1)-$dimensional sphere S^{N-1}. Furthermore, the following transformed solution

$$W_\alpha^{rad}(t) \equiv |S^{N-1}|^p(\frac{N}{\alpha+N})^{\frac{2}{p-1}} \omega_\alpha^{rad}(\exp(-\frac{N}{\alpha+N}t))$$

converges uniformly on $(0,\infty)$ *to a least energy solution* $\omega_{1,N}$ *of (14-(1,N)) as* $\alpha \to \infty$.

Proof of Theorem 3.1. It is easy to see that there are no local minimum points of u_α^{rad} in $(0,1)$. Suppose that there exists $r_0 \in (0,1)$ satisfying $u_\alpha^{rad}(0) < u_\alpha^{rad}(r)$ for any $r \in (0, r_0)$. Then , defining

$$w_\alpha^{rad} = \begin{cases} u_\alpha^{rad}(r) & \text{for } r \in (r_0, 1), \\ u_\alpha^{rad}(r_0) & \text{for } r \in [0, r_0], \end{cases}$$

we see that

$$\int_\Omega |\nabla u_\alpha^{rad}|^2 dx > \int_\Omega |\nabla w_\alpha^{rad}|^2 dx$$

and

$$\int_\Omega |x|^\alpha (u_\alpha^{rad})^{p+1} dx < \int_\Omega |x|^\alpha (w_\alpha^{rad})^{p+1} dx.$$

This implies that

$$\frac{\int_\Omega |\nabla w_\alpha^{rad}|^2 dx}{\left(\int_\Omega |x|^\alpha (w_\alpha^{rad})^{p+1} dx\right)^{2/(p+1)}} < \frac{\int_\Omega |\nabla u_\alpha^{rad}|^2 dx}{\left(\int_\Omega |x|^\alpha (u_\alpha^{rad})^{p+1} dx\right)^{2/(p+1)}};$$

this contradicts that u_α^{rad} is a minimizer of $I^{rad,\alpha}$. Thus we see that u_{rad}^α is monotone decreasing on $[0,1]$.

We transform u_α^{rad} as follows: for $t \in (0,\infty)$,

$$v_\alpha^{rad}(t) \equiv |S^{N-1}|^{\frac{1}{p+1}} (\frac{N}{\alpha+N})^{\frac{1}{p+1}} u_\alpha^{rad}(\exp(-\frac{N}{\alpha+N}t)).$$

Then, direct calculations show

$$\int_\Omega |\nabla u_\alpha^{rad}|^2 dx = |S^{N-1}|^{\frac{p-1}{p+1}} (\frac{\alpha+N}{N})^{\frac{p+3}{p+1}} \int_0^\infty \exp(-\frac{N(N-2)t}{\alpha+N}) |\frac{dv_\alpha^{rad}}{dt}|^2 dt,$$

and

$$\int_\Omega |x|^\alpha (u_\alpha^{rad})^{p+1} dx = \int_0^\infty \exp(-Nt)(v_\alpha^{rad})^{p+1} dt.$$

Thus, we see that for any $p \geq 1$,

$$(\frac{N}{\alpha+N})^{\frac{p+3}{p+1}} I^{rad,\alpha} = |S^{N-1}|^{(p-1)/(p+1)} \int_0^\infty \exp(-\frac{N(N-2)t}{\alpha+N}) |\frac{dv_\alpha^{rad}}{dt}|^2 dt$$

and

$$\int_0^\infty \exp(-Nt)(v_\alpha^{rad})^{p+1}dt = 1.$$

Moreover, it follows that

$$\frac{d}{dt}\Big(\exp(-\frac{N(N-2)t}{\alpha+N})\frac{dv_\alpha^{rad}}{dt}\Big) + H_\alpha \exp(-Nt)(v_\alpha^{rad})^p = 0 \quad \text{on} \quad (0,\infty)$$

$$\lim_{t\to\infty}\frac{v_\alpha^{rad}(t)}{dt} = v_\alpha^{rad}(0) = 0,$$

where $H_\alpha \equiv |S^{N-1}|^{-(p-1)/(p+1)}(\frac{N}{\alpha+N})^{\frac{p+3}{p+1}} I^{rad,\alpha}$. Since $\lim_{\alpha\to\infty}\exp(-\frac{N(N-2)t}{\alpha+N}) = 1$ uniformly on each compact subset of $[0,\infty)$, it follows that

$$\lim_{\alpha\to\infty}(\frac{N}{\alpha+N})^{\frac{p+3}{p+1}} I^{rad,\alpha} \le |S^{N-1}|^{(p-1)/(p+1)} J_{1,N}. \tag{15}$$

This implies that $\limsup_{\alpha\to\infty} H_\alpha \le J_{1,N}$.

For sufficiently large $\alpha > 0$, we see that

$$\int_0^\infty \exp(-Nt)|\frac{dv_\alpha^{rad}}{dt}|^2 dt \le \int_0^\infty \exp(-\frac{N(N-2)t}{\alpha+N})|\frac{dv_\alpha^{rad}}{dt}|^2 dt$$
$$= H_\alpha \int_0^\infty \exp(-Nt)(v_\alpha^{rad})^{p+1}dt \le H_\alpha \|v_\alpha^{rad}\|_{L^\infty}^{p-1} \int_0^\infty \exp(-Nt)(v_\alpha^{rad})^2 dt.$$

Then, from the inequality (3), we see that $\{\|v_\alpha^{rad}\|_{L^\infty}\}_\alpha$ is bounded away from 0.

From now, we will show that $\{\|v_\alpha^{rad}\|_{L^\infty}\}_\alpha$ is bounded. Defining $W_\alpha \equiv \frac{dv_\alpha^{rad}}{dt}$, we see from the equation for v_α^{rad} that

$$\frac{d^2W_\alpha}{dt^2} + \frac{N(\alpha-N+4)}{\alpha+N}\frac{dW_\alpha}{dt}$$
$$+ \Big(pH_\alpha \exp(-\frac{N(\alpha+2)}{\alpha+N}t)(v_\alpha^{rad})^{p-1} - \frac{N^2(\alpha+2)(N-2)}{(\alpha+N)^2}\Big)W_\alpha = 0.$$

Note that $v_\alpha^{rad}(0) = 0, \lim_{t\to\infty} W_\alpha(t) = 0$. From (15), we deduce from Cauchy's inequality that for some constant $C > 0$, independent of α,

$$v_\alpha^{rad}(t) = \int_0^t W_\alpha(s)ds = \int_0^t \exp(\frac{N(N-2)}{2(\alpha+N)}s)\exp(-\frac{N(N-2)}{2(\alpha+N)}s)W_\alpha ds$$
$$\le \begin{cases} C\sqrt{\frac{\alpha+N}{N(N-2)}}\sqrt{\exp(\frac{N(N-2)}{\alpha+N}t)-1} & \text{for } N>2, \\ C\sqrt{t} & \text{for } N=2, \\ C & \text{for } N=1. \end{cases}$$

Note that for any $\gamma, c, t > 0$,

$$\exp(-ct)\gamma(\exp(\frac{t}{\gamma}) - 1) \leq t \exp((\frac{1}{\gamma} - c)t).$$

Thus we see that

$$\lim_{|(\alpha,t)|\to\infty} \Big(pH_\alpha \exp(-\frac{N(\alpha+2)}{\alpha+N}t)(v_\alpha^{rad})^{p-1} - \frac{N^2(\alpha+2)(N-2)}{(\alpha+N)^2}\Big) = 0.$$

It is standard to show that for each $T > 0$, $\{\|W_\alpha\|_{L^\infty(0,T)}\}_\alpha$ is bounded. For $a \in (0, N)$, we denote $\phi(t) \equiv \exp(-at)$, Then, we see that for sufficiently large $\alpha > 0$ and $t > 0$,

$$\begin{aligned} &\frac{d^2\phi}{dt^2} + \frac{N(\alpha - N + 4)}{\alpha + N}\frac{d\phi}{dt} \\ &+ \Big(pH_\alpha \exp(-\frac{N(\alpha+2)}{\alpha+N}t)(v_\alpha^{rad})^{p-1} - \frac{N^2(\alpha+2)(N-2)}{(\alpha+N)^2}\Big)\phi \leq 0. \end{aligned}$$

Then it follows from the comparison principle that for any given $a \in (0, N)$, there exists some $C > 0$, independent of $\alpha > 0$, satisfying

$$W_\alpha(t) \leq C\exp(-at), \qquad t \geq 0. \tag{16}$$

Then, since $v_\alpha^{rad}(t) = \int_0^t W_\alpha(s)ds$, it follows that $\{\|v_\alpha^{rad}\|_{L^\infty}\}_\alpha$ is bounded, and that for any $a \in (0, N)$, there exists some $C > 0$ satisfying

$$\lim_{s\to\infty} v_\alpha^{rad}(s) - v_\alpha^{rad}(t) \leq C\exp(-at), \qquad t > 0. \tag{17}$$

Now, from the elliptic estimates ([5]), we deduce that for each $T > 0$ and $\gamma \in (0, 1)$, $\{|v_\alpha^{rad}|_{C^{2,\gamma}(0,T)}\}_\alpha$ is bounded. If $\liminf_{\alpha\to\infty} \|v_\alpha^{rad}\|_{L^\infty((0,T))} = 0$ for sufficiently large $T > 0$, from the boundedness of $\{\|v_\alpha^{rad}\|_{L^\infty}\}_\alpha$, it follows that

$$\liminf_{\alpha\to\infty} \int_0^\infty \exp(-Nt)(v_\alpha^{rad})^{p+1}dt = 0;$$

this contradicts that for any $\alpha > 0$, $\int_0^\infty \exp(-Nt)(v_\alpha^{rad})^{p+1}dt = 1$. Thus, we deduce that for some $H \in (0, J_{1,N}]$, the solution v_α^{rad} converges in $C^2_{loc}(0,\infty)$ to a solution $v \in H((0,\infty))$ of

$$\frac{d^2v}{dt^2} + H\exp(-Nt)v^p = 0, \quad v > 0 \text{ in } (0,\infty)$$

$$v(0) = 0.$$

Since $\int_0^\infty \exp(-Nt)v^{p+1}dt = 1$, it follows that $H = \int_0^\infty |\frac{dv}{dt}|^2 dt \ge J_{1,N}$. Therefore, it follows that

$$\lim_{\alpha\to\infty} (\frac{N}{\alpha+N})^{\frac{p+3}{p+1}} I^{rad,\alpha} \ge |S^{N-1}|^{(p-1)/(p+1)} J_{1,N}.$$

Thus we get

$$\lim_{\alpha\to\infty} (\frac{N}{\alpha+N})^{\frac{p+3}{p+1}} I^{rad,\alpha} = |S^{N-1}|^{(p-1)/(p+1)} J_{1,N}.$$

Moreover, from (17), it follows that v_α^{rad} converges uniformly to a minimizer of $J_{1,N}$. This completes the proof. □

4. Asymptotic profile of least energy solutions on the unit ball

In this section, we turn to the least energy solutions of the Hénon equation (1). We will study both the asymptotic energy and asymptotic profile of the ground states, Let us consider the following minimization problem

$$I^{all,\alpha} \equiv \inf \left\{ \|u\|^2 \;\Big|\; \int_\Omega |x|^\alpha u^{p+1} dx = 1, u \in H_0^{1,2}(B(0,1)) \right\}. \tag{18}$$

For $p \in [1, (N+2)/(N-2))$, there exists a positive minimizer u_α^{all} of (18). This u_α^{all} satisfies the following equation

$$\Delta u + I^{all,\alpha}|x|^\alpha u^p = 0, \quad u > 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega.$$

When $N \ge 2$, for $x \in \mathbb{R}^N$, we take polar coordinates $x = (r,\theta)$ with $r = |x| \in [0,\infty), \theta = x/|x| \in S^{N-1}$, and denote $u(x) = u(r,\theta)$. For the sake of convenience, we denote $S_\alpha^{N-1} \equiv \frac{\alpha+N}{N} S^{N-1}$. For each $y \in \mathbb{R}^{N-1} \equiv \mathbb{R}^{N-1} \times \{0\} \subset \mathbb{R}^N$, there exists a unique $\psi_\alpha(y) \in S_\alpha^{N-1} \backslash \{(0,\cdots,0,\frac{\alpha+N}{N})\}$ such that $t(y)\psi_\alpha(y) + (1-t(y))(0,\cdots,0,\frac{\alpha+N}{N}) = y$ for some $t(y) > 0$ depending upon α. Then, the map $(\psi_\alpha)^{-1} : S_\alpha^{N-1} \setminus \{(0,\cdots,0,\frac{\alpha+N}{N})\} \to \mathbb{R}^{N-1}$ is a stereographic projection. Also when $N \ge 2$, by a rearrangement technique ([7]), we can assume that $u_\alpha^{all}(x) = u_\alpha^{all}(g \cdot x)$ for $g \in O(N-1) \otimes I \subset O(N)$ (i.e., u is radially symmetric with respect to the first $N-1$ variables), and that for fixed $r \in (0,1)$, $u_\alpha^{all}(r,\theta)$ decreases strictly as $|\theta - (0,\cdots,0,-1)|$ increases. If $N = 1$, we can assume that $\frac{du_\alpha^{all}(0)}{dx} \ge 0$.

Then, we have the following results on the asymptotic behaviours of the least energy I_α^{all} and the minimizer u_α^{all}.

Theorem 4.1. *Let $p \in (1, 2^* - 1)$. Then*

$$\lim_{\alpha\to\infty} (\frac{N}{\alpha+N})^{\frac{N+2-(N-2)p}{p+1}} I^{all,\alpha} = J_{N,N}.$$

Moreover, the following transformed solution

$$V_\alpha^{all}(t,y) \equiv \begin{cases} (\frac{N}{\alpha+N})^{N/(p+1)} u_\alpha^{all}(\exp(-\frac{N}{\alpha+N}t), \frac{N}{\alpha+N}\psi_\alpha(y)) & for\ N \geq 3 \\ (\frac{2}{\alpha+2})^{2/(p+1)} u_\alpha^{all}(\exp(-\frac{2}{\alpha+2}t), \frac{2}{\alpha+2}y) & for\ N = 2 \\ (\frac{1}{\alpha+1})^{1/(p+1)} u_\alpha^{all}(\exp(-\frac{t}{\alpha+1})) & for\ N = 1 \end{cases}$$

with $t \in [0,\infty), y \in \mathbb{R}^{N-1}$, converges to a minimizer of $J_{N,N}$ uniformly for $N \geq 3$ and locally uniformly for $N = 1, 2$ as $\alpha \to \infty$, And, for $N = 1$, the following transformed solution $(\frac{1}{\alpha+1})^{1/(p+1)} u_\alpha^{all}(-\exp(-\frac{t}{\alpha+1}))$ converges locally uniformly to 0 as $\alpha \to \infty$.

For a nonnegative minimizer u_α^{all} of I_α^{all}, a scaled function $\omega_\alpha^{all} \equiv (I_\alpha^{all})^{\frac{1}{p-1}} u_\alpha^{all}$ is a least energy solution(a mountain pass solution) of (1) in the whole class of functions in $H_0^{1,2}(B(0,1))$. Then, we have the following equivalent version of Theorem 4.1 as for ω_α^{rad} and its energy in Theorem 3.1-E.

Theorem 4.1-E. *Let $p \in (1, 2^* - 1)$. Then*

$$\lim_{\alpha\to\infty} (\frac{N}{\alpha+N})^{\frac{N+2-(N-2)p}{p-1}} \int_{B(0,1)} \frac{1}{2}|\nabla\omega_\alpha^{all}|^2 - \frac{1}{p+1}|x|^\alpha(\omega_\alpha^{all})^{p+1}dx$$
$$= \int_0^\infty \frac{1}{2}|\nabla\omega_{N,N}|^2 - \frac{1}{p+1}\exp(-Nt)(\omega_{N,N})^{p+1}dt$$

for some $\omega_{N,N}$ being a least energy solution of (14-(N,N)). Moreover, the following transformed solution

$$W_\alpha^{all}(t,y) \equiv \begin{cases} (\frac{N}{\alpha+N})^{\frac{2}{p-1}} \omega_\alpha^{all}(\exp(-\frac{N}{\alpha+N}t), \frac{N}{\alpha+N}\psi_\alpha(y)) & for\ N \geq 3 \\ (\frac{2}{\alpha+2})^{\frac{2}{p-1}} \omega_\alpha^{all}(\exp(-\frac{2}{\alpha+2}t), \frac{2}{\alpha+2}y) & for\ N = 2 \\ (\frac{1}{\alpha+1})^{\frac{2}{p-1}} \omega_\alpha^{all}(\exp(-\frac{t}{\alpha+1})) & for\ N = 1 \end{cases}$$

with $t \in [0,\infty), y \in \mathbb{R}^{N-1}$, converges to $\omega_{N,N}$ uniformly for $N \geq 3$ and locally uniformly for $N = 1,2$ as $\alpha \to \infty$, And, for $N = 1$, the following transformed solution $(\frac{1}{\alpha+1})^{2/(p-1)}\omega_\alpha^{all}(-\exp(-\frac{t}{\alpha+1}))$ converges locally uniformly to 0 as $\alpha \to \infty$.

Proof of Theorem 4.1. We take polar coordinates $x = (r,\theta)$ with $r \in [0,\infty), \theta \in S^{N-1}$, and denote $u(x) = u(r,\theta)$. We first consider the following transformation

$$v_\alpha^{all}(t,\phi) \equiv (\frac{\alpha+N}{N})^{-N/(p+1)} u_\alpha^{all}(\exp(-\frac{N}{\alpha+N}t), \frac{N}{\alpha+N}\phi),$$

where $t \in [0,\infty)$ and $\phi \in S_\alpha^{N-1}$. For the sake of convenience, we denote $d_\alpha\sigma$ the volume element of S_α^{N-1}. Then, from some direct calculations, we get for $N \geq 2$

$$I^{all,\alpha} = \int_\Omega |\nabla u_\alpha^{all}|^2 dx =$$
$$(\frac{\alpha+N}{N})^{\frac{N+2-(N-2)p}{p+1}} \int_{(0,\infty)\times S_\alpha^{N-1}} \exp(-\frac{N(N-2)t}{\alpha+N})\Big(|\frac{\partial v_\alpha^{all}}{\partial t}|^2 + |\nabla_{S_\alpha} v_\alpha^{all}|^2\Big) dt d_\alpha\sigma,$$

$$1 = \int_\Omega |x|^\alpha (u_\alpha^{all})^{p+1} dx = \int_{(0,\infty)\times S_\alpha^{N-1}} \exp(-Nt)(v_\alpha^{all})^{p+1} dt d_\alpha\sigma,$$

where ∇_{S_α} is the gradient on S_α^{N-1}. Thus we see that $K_{\alpha,N} \equiv I^{all,\alpha}(\frac{N}{\alpha+N})^{\frac{N+2-(N-2)p}{p+1}}$,

$$\frac{\partial^2 v_\alpha^{all}}{\partial t^2} - \frac{N(N-2)}{\alpha+N}\frac{\partial v_\alpha^{all}}{\partial t} + \Delta_{S_\alpha^{N-1}} v_\alpha^{all} + K_{\alpha,N}\exp(-\frac{N(\alpha+2)t}{\alpha+N})(v_\alpha^{all})^p = 0 \qquad (19)$$

in $(0,\infty)\times S_\alpha$ and $v_\alpha^{all} = 0$ on $\{0\}\times S_\alpha^{N-1}$. A direct computation shows that (19) also holds for $N = 1$.

For each $\varphi \in C_0^\infty((0,\infty)\times\mathbb{R}^{N-1})$, we define a function $w_\alpha \in C_0^\infty(B(0,1))$

$$w_\alpha(r,\theta) \equiv \begin{cases} (\frac{\alpha+N}{N})^{N/(p+1)}\varphi(-\frac{\alpha+N}{N}\log r, (\psi_\alpha)^{-1}(\frac{\alpha+N}{N}\theta)) & \text{for } N \geq 2 \\ (\frac{\alpha+N}{N})^{N/(p+1)}\varphi(-\frac{\alpha+N}{N}\log r) & \text{for } N = 1. \end{cases}$$

Then, since $\varphi \in C_0^\infty((0,\infty)\times\mathbb{R}^{N-1})$, it is not difficult to deduce that

$$\frac{\int_\Omega |\nabla w_\alpha|^2 dx}{\Big(\int_\Omega |x|^\alpha |w_\alpha|^{p+1} dx\Big)^{2/(p+1)}}$$
$$= (\frac{\alpha+N}{N})^{\frac{N+2-(N-2)p}{p+1}} \frac{\int_{(0,\infty)\times\mathrm{R}^{N-1}} \exp(-\frac{N(N-2)t}{\alpha+N})\Big(|\frac{\partial\varphi}{\partial t}|^2 + |\nabla_y\varphi|^2\Big) dt dy + O(\frac{1}{\alpha})}{\Big(\int_{(0,\infty)\times\mathrm{R}^{N-1}} \exp(-Nt)|\varphi|^{p+1} dt dy + O(\frac{1}{\alpha})\Big)^{2/(p+1)}}$$

as $\alpha \to \infty$. Furthermore, since $\varphi \in C_0^\infty((0,\infty)\times\mathbb{R}^{N-1})$, it follows that

$$\lim_{\alpha\to\infty} \frac{\int_{(0,\infty)\times\mathrm{R}^{N-1}} \exp(-\frac{N(N-2)t}{\alpha+N})|\frac{\partial\varphi}{\partial t}|^2 + |\nabla_y\varphi|^2 dtdy + O(\frac{1}{\alpha})}{\Big(\int_{(0,\infty)\times\mathrm{R}^{N-1}} \exp(-Nt)|\varphi|^{p+1}dtdy + O(\frac{1}{\alpha})\Big)^{2/(p+1)}}$$
$$= \frac{\int_{(0,\infty)\times\mathrm{R}^{N-1}} |\nabla\varphi|^2 dtdy}{\Big(\int_{(0,\infty)\times\mathrm{R}^{N-1}} \exp(-Nt)|\varphi|^{p+1}dtdy\Big)^{2/(p+1)}}.$$

This implies that

$$\lim_{\alpha\to\infty} \Big(\frac{N}{\alpha+N}\Big)^{\frac{N+2-(N-2)p}{p+1}} I^{all,\alpha} \le J_{N,N}. \tag{20}$$

Next, by a similar argument as in the proof of Theorem 3.1, we deduce using (3) that for $N \ge 2$, $\{\|v_\alpha^{all}\|_{L^\infty}\}_\alpha$ is bounded away from 0. For $N = 1$, since $v_\alpha^{all}(t) = (\alpha+1)^{\frac{-1}{p+1}} u_\alpha^{all}(\exp(\frac{-t}{\alpha+1}))$, we see that

$$\frac{d}{dt}\Big(\exp(\frac{t}{\alpha+1})\frac{dv_\alpha^{all}}{dt}\Big) + I^{all,\alpha}(\alpha+1)^{-\frac{p+3}{p+1}}\exp(-t)(v_\alpha^{all})^p = 0 \quad \text{on } (0,\infty), \tag{21}$$

and that

$$v_\alpha^{all}(0) = 0, \qquad \lim_{t\to\infty}\exp(\frac{t}{\alpha+1})\frac{dv_\alpha^{all}(t)}{dt} = -\frac{du_\alpha^{all}(0)}{dx}\frac{1}{(\alpha+1)^{\frac{p+2}{p+1}}}.$$

We assumed that $\frac{du_\alpha^{all}}{dx}(0) \ge 0$. Then, since $\frac{d^2u_\alpha^{all}}{dx^2}(x) \le 0$ for $|x| < 1$, the unique maximum point of u_α^{all} is located in $[0,1)$. From Theorem 3.1 and (20), we see that u_α^{all} is not symmetric, that is, $u_\alpha^{all}(x) \ne u_\alpha^{all}(-x)$ for some $x \in (0,1)$. Thus, from the uniqueness of a solution for the initial value problem of ordinary differential equations, we deduce that $\frac{du_\alpha^{all}}{dx}(0) \ne 0$; then $\frac{du_\alpha^{all}}{dx}(0) > 0$. Multiplying v_α^{all} on both sides of (21) and integrating by parts, we get

$$\int_0^\infty \exp(\frac{t}{\alpha+1})\Big|\frac{dv_\alpha^{all}}{dt}\Big|^2 dt - \lim_{t\to\infty}\exp(\frac{t}{\alpha+1})v_\alpha^{all}(t)\frac{dv_\alpha^{all}(t)}{dt}$$
$$= \int_0^\infty \exp(\frac{t}{\alpha+1})\Big|\frac{dv_\alpha^{all}}{dt}\Big|^2 dt + \frac{du_\alpha^{all}(0)}{dx}\frac{1}{(\alpha+1)^{\frac{p+2}{p+1}}}$$
$$= I^{all,\alpha}(\alpha+1)^{-\frac{p+3}{p+1}}\int_0^\infty \exp(-t)(v_\alpha^{all})^{p+1}dt.$$

Then, it follows that

$$\int_0^\infty \exp(-t)\Big|\frac{dv_\alpha^{all}}{dt}\Big|^2 dt \le \int_0^\infty \Big|\frac{dv_\alpha^{all}}{dt}\Big|^2 dt \le \int_0^\infty \exp(\frac{t}{\alpha+1})\Big|\frac{dv_\alpha^{all}}{dt}\Big|^2 dt$$
$$\le I^{all,\alpha}(\alpha+1)^{-\frac{p+3}{p+1}}\|v_\alpha^{all}\|_{L^\infty}^{p-1}\int_0^\infty \exp(-t)(v_\alpha^{all})^2 dt. \tag{22}$$

Then, from (3) and (20), we deduce that for $N=1$, $\{\|v_\alpha^{all}\|_{L^\infty}\}_\alpha$ is bounded away from 0. Prior to proceeding further, we prepare some lemmas.

We consider the L^∞ bound first.

Lemma 1. *For each $N \neq 2$, $\{\|v_\alpha^{all}\|_{L^\infty}\}_\alpha$ is bounded.*

Proof. We prove the claim for the cases $N \ge 3$ and $N = 1$ separately.

Let $N \ge 3$. First of all, we note that for $U_\alpha \equiv (I^{all,\alpha})^{1/(p-1)}u_\alpha^{all}$,

$$\Delta U_\alpha + (U_\alpha)^p \ge 0, \qquad U_\alpha > 0 \quad \text{in } B$$
$$U_\alpha = 0 \quad \text{on } \partial B.$$

Then, by an uniform estimate [2, Proposition 3.5], we see that for $N \ge 3$ and some $C > 0$, independent of α,

$$\|U_\alpha\|_{L^\infty} \le C\|U_\alpha\|_{L^{2N/(N-2)}}^{4/(N+2-p(N-2))}.$$

Thus, from the Sobolev inequality and (20), it follows that for some $C > 0$,

$$\|u_\alpha^{all}\|_{L^\infty} = (I^{all,\alpha})^{-1/(p-1)}\|U_\alpha\|_{L^\infty} \le C(I^{all,\alpha})^{\frac{2}{N+2-p(N-2)}-\frac{1}{p+1}} \le C\big(\frac{\alpha+N}{N}\big)^{N/(p+1)}.$$

Thus, for $N \ge 3$, $\{\|v_\alpha^{all}\|_{L^\infty}\}_\alpha$ is bounded.

Let $N = 1$. Defining $W_\alpha \equiv \frac{dv_\alpha^{all}}{dt}$, we see that for $K_{\alpha,1} \equiv I^{all,\alpha}(\alpha+1)^{\frac{-(p+3)}{p+1}}$,

$$\frac{d^2W_\alpha}{dt^2} + \frac{\alpha+3}{\alpha+1}\frac{dW_\alpha}{dt} + \Big(\frac{\alpha+2}{(\alpha+1)^2} + K_{\alpha,1}p\exp(-\frac{\alpha+2}{\alpha+1}t)(v_\alpha^{all})^{p-1}\Big)W_\alpha = 0$$

and

$$W_\alpha(0) > 0, \quad \lim_{t\to\infty} W_\alpha(t) = -\lim_{t\to\infty}\exp(-\frac{t}{\alpha+1})\frac{1}{\alpha+1}\frac{du_\alpha^{all}(0)}{dx} = 0.$$

As in Theorem 3.1, we deduce that for any $a \in (0,1)$, there exists some $C > 0$, independent of $\alpha > 0$, satisfying

$$W_\alpha(t) \leq C \exp(-at), \qquad t \geq 0.$$

This implies that

$$v_\alpha^{all}(t) = \int_0^t W_\alpha(s)ds \leq \frac{C}{a}, \qquad t > 0.$$

This completes the proof. $\square$

Lemma 2. *For $N = 2$, there exists a constant $C > 0$, independent of $\alpha > 0$, such that*

$$v_\alpha^{all}(t,\phi) \leq C \exp(\frac{2pt}{p^2-1}), \quad t > 0, \ -\frac{\alpha+2}{2}\pi \leq \phi \leq \frac{\alpha+2}{2}\pi.$$

Proof. Let $N = 2$ and $K_{\alpha,2} \equiv I^{all,\alpha}(\frac{2}{\alpha+2})^{\frac{4}{p+1}}$, Then, the v_α^{all} satisfies the following equation

$$\begin{aligned} \frac{\partial^2 v_\alpha^{all}}{\partial t^2} + \frac{\partial^2 v_\alpha^{all}}{\partial y^2} + K_{\alpha,2}\exp(-2t)(v_\alpha^{all})^p &= 0 \qquad \text{in } (0,\infty) \times (-\frac{\alpha+2}{2}\pi, \frac{\alpha+2}{2}\pi) \\ \frac{\partial v_\alpha^{all}}{\partial y} &= 0 \qquad \text{on } (0,\infty) \times \{-\frac{\alpha+2}{2}\pi, \frac{\alpha+2}{2}\pi\} \\ v_\alpha^{all} &= 0 \qquad \text{on } \{0\} \times (-\frac{\alpha+2}{2}\pi, \frac{\alpha+2}{2}\pi). \end{aligned}$$

Let $T > 2$, and ϕ a smooth function such that

$$\phi(t,y) = \begin{cases} 1 & \text{for } \sqrt{(t-T)^2 + y^2} \leq 1 \\ 0 & \text{for } \sqrt{(t-T)^2 + y^2} \geq 2 \end{cases}$$

and $0 \leq \phi \leq 1$ on $\mathbb{R}^N$. For the sake of convenience, we denote $v = v_\alpha^{all}$ in the followings. Then, for any $\alpha > 0$, multiplying $v^{2\alpha+1}\phi^2$ to above equation for v and integrating by parts, we deduce that

$$\begin{aligned} &\int |\nabla v^{\alpha+1}\phi|^2 dtdy \\ &\leq \int v^{2\alpha+2}|\nabla\phi|^2 dtdy + K_{\alpha,2}(\alpha+1)\int \exp(-2t) v^{2\alpha+2} v^{p-1}\phi^2 dtdy. \end{aligned} \tag{23}$$

Denoting $w = \exp(-\frac{2t}{p})v\phi^{\frac{2}{p}}$, we see from Hölder's inequality that for any $K > 0$,

$$\int \exp(-2t)v^{2\alpha+2}v^{p-1}\phi^2 dtdy = \int w^{p-1}\exp(-\frac{2t}{p})v^{2\alpha+2}\phi^{\frac{2}{p}}dtdy$$
$$\leq \Big(\int_{\{(t,y)|w(t,y)\geq K\}} \exp(-2t)v^p\phi^2 dtdy\Big)^{\frac{p-1}{p}}\Big(\int \exp(-2t)v^{p(2\alpha+2)}\phi^2 dtdy\Big)^{\frac{1}{p}}$$
$$+ K^{p-1}\int \exp(-\frac{2t}{p})v^{2\alpha+2}\phi^{\frac{2}{p}}dtdy \tag{24}$$

It is easy to see that

$$\int_{\{(t,y)|w(t,y)\geq K\}} \exp(-2t)v^p\phi^2 dtdy$$
$$\leq \Big|\{(t,y)|w(t,y)\geq K\}\Big|^{\frac{1}{p+1}}\Big(\int \exp(-2\frac{(p+1)t}{p})v^{p+1}\phi^{\frac{2(p+1)}{p}}dtdy\Big)^{\frac{p}{p+1}}$$
$$\leq \Big|\{(t,y)|w(t,y)\geq K\}\Big|^{\frac{1}{p+1}}\Big(\int \exp(-2t)v^{p+1}\phi^2 dtdy\Big)^{\frac{p}{p+1}} \tag{25}$$

and

$$\Big|\{(t,y)|w(t,y)\geq K\}\Big| \leq K^{-(p+1)}\int \exp(-2\frac{(p+1)t}{p})v^{p+1}\phi^{\frac{2(p+1)}{p}}dtdy$$
$$\leq K^{-(p+1)}\int \exp(-2t)v^{p+1}\phi^2 dtdy.$$

Thus, it follows that

$$\int_{\{(t,y)|w(t,y)\geq K\}} \exp(-2t)v^p\phi^2 dtdy \leq \frac{1}{K}\int \exp(-2t)v^{p+1}\phi^2 dtdy. \tag{26}$$

Combining (20),(23-6) and Proposition 2.1, we see that for some $C > 0$, independent of α, ϕ and v,

$$\Big(\int \exp(-2t)v^{p(2\alpha+2)}\phi^{2p}dtdy\Big)^{\frac{1}{p}}$$
$$\leq C\int v^{2\alpha+2}|\nabla\phi|^2 dtdy + +C(\alpha+1)K^{p-1}\int \exp(-\frac{2t}{p})v^{2\alpha+2}\phi^{\frac{2}{p}}dtdy$$
$$+ C(\alpha+1)\Big(\frac{1}{K}\int \exp(-2t)v^{p+1}\phi^2 dtdy\Big)^{\frac{p-1}{p}}\Big(\int \exp(-2t)v^{p(2\alpha+2)}\phi^2 dtdy\Big)^{\frac{1}{p}}$$

We take $K > 0$ so that

$$C(\alpha+1)\Big(\frac{1}{K}\int \exp(-2t)v^{p+1}\phi^2 dtdy\Big)^{\frac{p-1}{p}} = 1/2.$$

Then, since $\int_{-\frac{(\alpha+N)\pi}{N}}^{\frac{(\alpha+N)\pi}{N}}\int_0^\infty \exp(-2t)v^{p+1}dtdy = 1$, it follows that

$$\begin{aligned}
&\Big(\int \exp(-2t)v^{p(2\alpha+2)}\phi^{2p}dtdy\Big)^{\frac{1}{p}}\\
&\le 2C\int v^{2\alpha+2}|\nabla\phi|^2dtdy\\
&\quad + (2C(\alpha+1))^{1-p}\Big(\int \exp(-2t)v^{p+1}\phi^2dtdy\Big)^{p-1}\int \exp(-\frac{2t}{p})v^{2\alpha+2}\phi^{\frac{2}{p}}dtdy\\
&\le 2C\int v^{2\alpha+2}|\nabla\phi|^2dtdy\\
&\quad + (2C(\alpha+1))^{1-p}\int \exp(-\frac{2t}{p})v^{2\alpha+2}\phi^{\frac{2}{p}}dtdy.
\end{aligned}$$

We take a smooth function ϕ_i such that

$$\phi_i(t,y) = \begin{cases} 1 & \text{for } \sqrt{(T-t)^2+y^2} \le 1+2^{-i} \\ 0 & \text{for } \sqrt{(T-t)^2+y^2} \ge 1+2^{-i+1} \end{cases}$$

and $0 \le \phi_i \le 1, |\nabla\phi_i| \le 2^{i+1}$. Then, substituting ϕ and $2\alpha+2$ by ϕ_i and $(p+1)p^{i-1}$ respectively in above inequality, we see that

$$\begin{aligned}
&\Big(\int_{B((T,0),1+2^i)} \exp(-2t)v^{p^i(p+1)}dtdy\Big)^{\frac{1}{p}}\\
&\le 2C4^{i+1}\exp(2T+4)\int_{B((T,0),1+2^{i-1})} \exp(-2t)v^{p^{i-1}(p+1)}dtdy\\
&\quad + (2C(\alpha+1))^{1-p}\exp(\frac{p-1}{p}(2T+4))\int_{B((T,0),1+2^{i-1})} \exp(-2t)v^{p^{i-1}(p+1)}dtdy.
\end{aligned}$$

Then, we deduce that for some $D > 0$, independent of i and $T > 0$,

$$\begin{aligned}
&\Big(\int_{B((T,0),1+2^i)} \exp(-2t)v^{p^i(p+1)}dtdy\Big)^{\frac{1}{p^i(p+1)}}\\
&\le D\exp(2T\sum_{j=1}^{i}\frac{1}{p^{j-1}(p+1)})\Big(\int_{B((T,0),2)} \exp(-2t)v^{p+1}dtdy\Big)^{\frac{1}{p+1}}\\
&\le D\exp(2T\frac{p}{p^2-1}).
\end{aligned}$$

Then, taking $i \to \infty$ in above inequality, we see that for some $D > 0$,

$$v(T,0) \le D\exp(2T\frac{p}{p^2-1}), \qquad T > 2.$$

This proves the claim. $\square$

Lemma 3. *Let $N \geq 3$, and let $A_\alpha := \lim_{t\to\infty} v_\alpha^{all}(t,\phi)$. Then $\lim_{\alpha\to\infty} A_\alpha = 0$.*

Proof. Denoting $U_\alpha \equiv (I^{all,\alpha})^{1/(p-1)} u_\alpha^{all}$, we see that

$$\Delta U_\alpha + |x|^\alpha (U_\alpha)^p = 0 \text{ in } B(0,1),$$

and that

$$\int_{B(0,1)} |x|^\alpha (U_\alpha)^{p+1}(x)dx = (I^{all,\alpha})^{(p+1)/(p-1)}.$$

Let $G(x,y)$ be the Green function of $-\Delta$ on $B(0,1)$. Then, we see that

$$U_\alpha(x) = -\int_{B(0,1)} G(y,x)|y|^\alpha (U_\alpha)^p(y)dy.$$

Note that $G(x,0) = C(1/|x|^{N-2} - 1)$ for some $C > 0$. Applying Hölder's inequality, we deduce that for some constant $C > 0$,

$$\begin{aligned} U_\alpha(0) &\leq C\Big(\int_{B(0,1)} (G(x,0))^{p+1}|x|^\alpha dx\Big)^{1/(p+1)} \Big(\int_{B(0,1)} |x|^\alpha (U_\alpha)^{p+1} dx\Big)^{p/(p+1)} \\ &\leq C(\alpha + C)^{-1/(p+1)} (I^{all,\alpha})^{p/(p-1)}. \end{aligned}$$

Thus, it follows that

$$\begin{aligned} \lim_{t\to\infty} v_\alpha^{all}(t,\phi) &= \big(\frac{N}{\alpha+N}\big)^{N/(p+1)} u_\alpha^{all}(0,\theta) \leq C I^{all,\alpha} \big(\frac{N}{\alpha+N}\big)^{N/(p+1)+1/(p+1)} \\ &\leq C\big(\frac{\alpha+N}{N}\big)^{\frac{N+2-p(N-2)}{p+1} - \frac{N+1}{p+1}} = C\big(\frac{\alpha+N}{N}\big)^{\frac{1-p(N-2)}{p+1}}. \end{aligned}$$

Since $\frac{1-p(N-2)}{p+1} < 0$ for $p > 1$ and $N \geq 3$, the claim follows. $\square$

Lemma 4. *Let $N \geq 3$. Suppose that there exists $T_\alpha > 0$ satisfying $\lim_{\alpha\to\infty} T_\alpha = \infty$ and $\lim_{\alpha\to\infty} \sup_{\phi \in S_\alpha^{N-1}} v_\alpha^{all}(T_\alpha, \phi) = 0$. Then, it follows that*

$$\lim_{\alpha\to\infty} \sup\{v_\alpha^{all}(t,\phi) \Big| t \geq T_\alpha, \phi \in S_\alpha^{N-1}\} = 0.$$

Proof. Let $(\phi_{\frac{N}{2},1}, \lambda_{\frac{N}{2},1})$ be a pair of the first eigenfunction and the first eigenvalue of

$$\frac{d^2\phi}{dt^2} + \lambda \exp(-Nt/2)\phi = 0 \quad \text{ on } (0,\infty),$$

$$\phi(0) = 0$$

$$\phi \in H((0,\infty))$$

satisfying that for $t>0$, $\phi_{\frac{N}{2},1}(t)>0$, and $\lim_{t\to\infty}\phi_{\frac{N}{2},1}(t)=1$. Note that $\frac{d\phi_1(t)}{dt}>0$ for $t>0$. Let $K_{\alpha,N}\equiv I^{all,\alpha}(\frac{N}{\alpha+N})^{\frac{N+2-(N-2)p}{p+1}}$. From the boundedness of $\{\|v_\alpha^{all}\|_{L^\infty}\}_\alpha$ for $N\geq 3$, we see that for sufficiently large $T>0$,

$$\begin{aligned}
&(\phi_{\frac{N}{2},1})_{tt}-\frac{N(N-2)}{\alpha+N}(\phi_{\frac{N}{2},1})_t+\Delta_{S_\alpha^{N-1}}\phi_{\frac{N}{2},1}+K_{\alpha,N}\exp(-\frac{N(\alpha+2)t}{\alpha+N})(v_\alpha^{all})^{p-1}\phi_{\frac{N}{2},1}\\
&\leq \phi_{\frac{N}{2},1}\exp(-\frac{N(\alpha+2)t}{\alpha+N})\big(K_{\alpha,N}(v_\alpha^{all})^{p-1}-\exp(\frac{N(\alpha+4-N)t}{2\alpha+2N})\lambda_{\frac{N}{2},1}\big)\\
&\leq 0, \qquad t\geq T.
\end{aligned}$$

From a comparison principle(refer [9]), (19) and Lemma 3, we deduce as in Proposition 2.3 that that

$$\lim_{\alpha\to\infty}\sup\{v_\alpha^{all}(t,\phi)\Big|t\geq T_\alpha,\phi\in S_\alpha^{N-1}\}=0.$$

This completes the proof of Lemma 4. □

Now we consider the limit of v_α^{all}. Note that $\int_{(0,T)\times S_\alpha^{N-1}}(v_\alpha^{all})^{p+1}d_\alpha\sigma dt\leq\exp(NT)$ for each $T>0$. Then, from elliptic estimates[5], we deduce that there exists $\gamma\in(0,1)$ such that $\{|v_\alpha^{all}|_{C^{2,\gamma}((0,T)\times S_\alpha^{N-1})}\}_\alpha$ is bounded for any $T<\infty$. Thus, for some $K\in[0,J_{N,N}]$, $v_\alpha(t,y)=v_\alpha^{all}(t,\psi_\alpha(y))$ converges in C^2_{loc} to some w satisfying

$$\begin{aligned}
\Delta w+K\exp(-Nt)w^p=0 \quad &\text{in} \quad (0,\infty)\times\mathbb{R}^{N-1} \qquad (27)\\
w=0 \quad &\text{on} \quad \{0\}\times\mathbb{R}^{N-1}.
\end{aligned}$$

Furthermore, it follows that

$$\int_{(0,\infty)\times\mathrm{R}^{N-1}}|\nabla w|^2dtdy\leq J_{N,N} \quad \text{and} \quad \int_{(0,\infty)\times\mathrm{R}^{N-1}}\exp(-Nt)w^{p+1}dtdy\leq 1.$$

Then, we see the following result.

Lemma 5. *For each* $N\geq 1$, $w>0$ *in* $(0,\infty)\times\mathbb{R}^{N-1}$.

Proof. To the contrary, suppose that $w=0$.

First, consider the cases $N\geq 3$. From Lemma 1, we see that

$$\lim_{\alpha\to\infty}\|\exp(-\frac{Nt}{2})v_\alpha^{all}\|_{L^\infty((0,\infty)\times S_\alpha)}=0.$$

Then, for sufficiently large $\alpha > 0$ and $N > 2$, we see that

$$\begin{aligned}
&\int_{(0,\infty)\times S_\alpha^{N-1}} \exp(-\frac{Nt}{2})\Big(|\frac{\partial v_\alpha^{all}}{\partial t}|^2 + |\nabla_{S_\alpha} v_\alpha^{all}|^2\Big) dt d_\alpha\sigma \\
&\le \int_{(0,\infty)\times S_\alpha^{N-1}} \exp(-\frac{N(N-2)t}{\alpha+N})\Big(|\frac{\partial v_\alpha^{all}}{\partial t}|^2 + |\nabla_{S_\alpha} v_\alpha^{all}|^2\Big) dt d_\alpha\sigma \\
&= K_{\alpha,N}\int_{(0,\infty)\times S_\alpha^{N-1}} \exp(-Nt)(v_\alpha^{all})^{p+1} dt d_\alpha\sigma \\
&\le K_{\alpha,N}\|\exp(-\frac{Nt}{2})(v_\alpha^{all})^{p-1}\|_{L^\infty((0,\infty)\times S_\alpha)}\int_{(0,\infty)\times S_\alpha^{N-1}} \exp(-\frac{N}{2}t)(v_\alpha^{all})^2 dt d_\alpha\sigma,
\end{aligned}$$

where $K_{\alpha,N} = I^{all,\alpha}(\frac{\alpha+N}{N})^{\frac{N+2-(N-2)p}{p+1}}$. On the other hand, integrating both sides of (3) on S_α with respect to y, we see that for some $C > 0$,

$$\begin{aligned}
&\int_{(0,\infty)\times S_\alpha^{N-1}} \exp(-\frac{N}{2}t)(v_\alpha^{all})^2 dt d_\alpha\sigma \\
&\le C\int_{(0,\infty)\times S_\alpha^{N-1}} \exp(-\frac{Nt}{2})\Big(|\frac{\partial v_\alpha^{all}}{\partial t}|^2 + |\nabla_{S_\alpha} v_\alpha^{all}|^2\Big) dt d_\alpha\sigma.
\end{aligned}$$

Since $\lim_{\alpha\to\infty} K_{\alpha,N} \le J_{N,N}$, this is a contradiction.

Secondly, consider the case $N = 2$. Note that a function $\psi = \frac{\partial v_\alpha^{all}}{\partial\phi}$ satisfies

$$\begin{aligned}
\Delta\psi + K_{\alpha,2}\exp(-2t)(v_\alpha^{all})^{p-1}\psi = 0 \qquad & \text{in } (0,\infty)\times(-\frac{\alpha+N}{N}\pi, \frac{\alpha+N}{N}\pi) \\
\psi = 0 \qquad & \text{on } \partial((0,\infty)\times(-\frac{\alpha+N}{N}\pi, \frac{\alpha+N}{N}\pi))
\end{aligned}$$

It is obvious that

$$\lim_{t\to\infty}\sup\{\psi(t,y)|y\in(-\frac{\alpha+N}{N}\pi, \frac{\alpha+N}{N}\pi)\} = 0.$$

From Lemma 2, we see that

$$\begin{aligned}
&\lim_{t\to\infty}\sup\{\exp(-2t)(v_\alpha^{all})^{p-1}(t,y)|y\in(-\frac{\alpha+N}{N}\pi, \frac{\alpha+N}{N}\pi)\} \\
&= \lim_{t\to\infty}\sup\{\exp(-\frac{(2p+1)t}{p+1})(v_\alpha^{all})^{p-1}(t,y)|y\in(-\frac{\alpha+N}{N}\pi, \frac{\alpha+N}{N}\pi)\} \\
&= 0.
\end{aligned}$$

Then, it is standard to see from elliptic estimates(refer [5]) that $\{\|\psi\|_{C^2}\}$ is bounded. Then, by integration by parts, we see that

$$\int_0^\infty \int_{-\frac{\alpha+N}{N}\pi}^{\frac{\alpha+N}{N}\pi} |\nabla\psi|^2 dtdy = K_{\alpha,2}\int_0^\infty \int_{-\frac{\alpha+N}{N}\pi}^{\frac{\alpha+N}{N}\pi} \exp(-2t)(v_\alpha^{all})^{p-1}\psi^2 dtdy < \infty.$$

Thus, it follows that $\psi \in H((0,\infty)\times\mathbb{R})$. Moreover, since

$$\int_0^\infty \int_{-\frac{\alpha+N}{N}\pi}^{\frac{\alpha+N}{N}\pi} \exp(-2t)(v_\alpha^{all})^{p-1}\psi^2 dtdy$$
$$\leq \|\exp(-\frac{(2p+1)t}{p+1})(v_\alpha^{all})^{p-1}\|_{L^\infty}\int_0^\infty \int_{-\frac{\alpha+N}{N}\pi}^{\frac{\alpha+N}{N}\pi} \exp(-\frac{t}{p+1})\psi^2 dtdy.$$

it follows that

$$\int_0^\infty \int_{-\frac{\alpha+N}{N}\pi}^{\frac{\alpha+N}{N}\pi} |\nabla\psi|^2 dtdy$$
$$\leq K_{\alpha,2}\|\exp(-\frac{(2p+1)t}{p+1})(v_\alpha^{all})^{p-1}\|_{L^\infty}\int_0^\infty \int_{-\frac{\alpha+N}{N}\pi}^{\frac{\alpha+N}{N}\pi} \exp(-\frac{t}{p+1})\psi^2 dtdy.$$

Note that $\lim_{\alpha\to\infty} K_{\alpha,2} \leq J_{2,2}$ and $\lim_{\alpha\to\infty}\|\exp(-\frac{(2p+1)t}{p+1})(v_\alpha^{all})^{p-1}\|_{L^\infty} = 0$. This contradicts (4).

Finally, we consider the case $N = 1$. As in (22), we deduce that

$$\int_0^\infty \exp(-\frac{t}{2})\left|\frac{dv_\alpha^{all}}{dt}\right|^2 dt \leq \int_0^\infty \exp(\frac{t}{\alpha+1})\left|\frac{dv_\alpha^{all}}{dt}\right|^2 dt$$
$$\leq K_{\alpha,1}\|\exp(-\frac{t}{2})v_\alpha^{all}\|_{L^\infty}^{p-1}\int_0^\infty \exp(-\frac{t}{2})(v_\alpha^{all})^2 dt.$$

Since $w = 0$, $\lim_{\alpha\to\infty} K_{\alpha,1} \leq J_{1,1}$ and $\{\|v_\alpha^{all}\|_{L^\infty}\}_\alpha$ is bounded, it follows that

$$\lim_{\alpha\to\infty} K_{\alpha,1}\|\exp(-\frac{t}{2})(v_\alpha^{all})^{p-1}\|_{L^\infty} = 0.$$

This contradicts (4).

Therefore, we conclude that $w > 0$ in $(0,\infty)\times\mathbb{R}^{N-1}$. The proof of Lemma 5 is finished. $\square$

If $K = 0$ in (27), the limit function w is harmonic. Then, it is easy to see that $w = at$ for some $a > 0$. This contradicts that $\int_0^\infty\int_{\mathbb{R}^{N-1}} |\nabla w| dtdy \leq \infty$. Thus, we have $K > 0$.

Now, let $\gamma = \int_0^\infty \int_{\mathbb{R}^{N-1}} \exp(-Nt) w^{p+1} dt dy \in (0,1]$ and $W = \gamma^{-1/(p+1)} w$. Then, we see that $\int_0^\infty \int_{\mathbb{R}^{N-1}} \exp(-Nt) W^{p+1} dt dy = 1$, and that

$$\Delta W + K\gamma^{(p-1)/(p+1)} \exp(-Nt) W^p = 0 \qquad \text{in} \quad (0,\infty) \times \mathbb{R}^{N-1}.$$

This implies that $\int_0^\infty \int_{\mathbb{R}^{N-1}} |\nabla W|^2 dt dy = K\gamma^{(p-1)/(p+1)}$. Thus, it follows that

$$K\gamma^{(p-1)/(p+1)} \geq J_{N,N}.$$

Since $J_{N,N} \geq K$ by (20) and $K \geq K\gamma^{(p-1)/(p+1)}$, it follows that $K = J_{N,N}$ and $\gamma = 1$. Therefore, the function w is a minimizer of $J_{N,N}$, and

$$\lim_{\alpha \to \infty} I^{all,\alpha} \left(\frac{N}{\alpha + N}\right)^{\frac{N+2-(N-2)p}{p+1}} = J_{N,N}.$$

To complete the proof of Theorem 4.1, it suffices to show that V_α^{all} (defined in the statement of Theorem 4.1) converges uniformly to w for $N \geq 3$. It is standard to see that for each $T > 0$, $\lim_{|y|,\alpha\to\infty} V_\alpha^{all}(t,y) = 0$ uniformly for $t \in (0,T)$. Note that $\lim_{\alpha\to\infty} \sup_{\phi \in S_\alpha^{N-1}} v_\alpha^{all}(T,\phi) \leq w(T,0)$. Then, since $\lim_{t\to\infty} \sup_{y \in \mathbb{R}^{N-1}} w(t,y) = 0$ for $N \geq 3$ (Proposition 2.5), by Lemma 4,

$$\lim_{t\to\infty,\alpha\to\infty} \sup_{y\in\mathbb{R}^{N-1}} V_\alpha^{all}(t,y) = 0.$$

Thus, $V_\alpha^{all}(t,y)$ converges uniformly to w for $N \geq 3$. For the convergence of $v_\alpha^{all,-} \equiv (\frac{1}{\alpha+1})^{1/(p+1)} u_\alpha^{all}(-\exp(-\frac{t}{\alpha+1}))$, we note that

$$1 = \int_0^\infty \exp(-t) w^{p+1} = \lim_{\alpha\to\infty} \int_0^\infty \exp(-t)(v_\alpha^{all})^{p+1} dt = \lim_{\alpha\to\infty} \int_0^1 |x|^\alpha (u_\alpha^{all})^{p+1} dx$$

and

$$\int_{-1}^1 |x|^\alpha (u_\alpha^{all})^{p+1} dx = \int_0^\infty \exp(-t)(v_\alpha^{all})^{p+1} dt + \int_0^\infty \exp(-t)(v_\alpha^{all,-})^{p+1} dt.$$

Thus,

$$\lim_{\alpha\to\infty} \int_0^\infty \exp(-t)(v_\alpha^{all,-})^{p+1} dt = 0.$$

Then, the convergence of $v_\alpha^{all,-}$ comes from standard elliptic estimates [5]. This completes the proof of Theorem 4.1. □

5. Some final remarks

First, as a corollary of Theorems 3.1, 3.1-E, 4.1 and 4.1-E, we obtain symmetry breaking of least energy solutions of the Hénon equation (1). For $N \geq 2$, this was proved in [10].

Corollary 5.1. *For $N \geq 1$ and $p \in (1, 2^* - 1)$ fixed, a minimizer u_α^{all} of $I^{all,\alpha}$ and a least energy solution ω_α^{all} of (1) is not radially symmetric if $\alpha > 0$ is sufficiently large.*

As it can be seen in Theorems 3.1 and 3.1-E, the behaviour of u_α^{rad} and ω_α^{rad} as $\alpha \to \infty$ is rather completely understood. On the other hand, the behaviour of u_α^{all} and ω_α^{all} as $\alpha \to \infty$ is not quite completely understood. The followings are interesting problems which need further study:

1. What is the exact growth rate of $u_\alpha^{all}(0)$ for $N \geq 2$ as $\alpha \to \infty$? Through the Proof of Theorem 4.1, we showed that if $N \geq 3$, there exists some constant $C > 0$ satisfying

$$(\frac{\alpha+N}{N})^{\frac{-N}{p+1}} u_\alpha^{all}(0) \leq C(\frac{\alpha+N}{N})^{\frac{1-p(N-2)}{p+1}} \to 0 \quad \text{as } \alpha \to 0$$

and

$$1/C \leq \lim(\frac{\alpha+N}{N})^{\frac{-N}{p+1}} \max_{x \in B(0,1)} u_\alpha^{all}(x) \leq C.$$

From Harnack inequality, we see that for any fixed $x \in B(0,1)$, the growth rate of $u_\alpha^{all}(x)$ is the same with that of $u_\alpha^{all}(0)$.

2. Can we obtain finer convergence of u_α^{all} for $N = 2$? Main difficulties in the case $N = 2$ come from the fact that for $N = 2$, in contrast to the cases $N \geq 3$, there is no appropriate inequality of Sobolev type which is independent of domains.

3. There is a unique maximum point x_α of u_α^{all} for $N = 1$. Then, what is the asymptotic behaviour of x_α as $\alpha \to \infty$?

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