Periodic solutions of even Hamiltonian systems on the torus

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Abstract

We consider the Hamiltonian system (HS) $\quad -J\dot{z} = H_z(t,z) \quad$ where $H \in \mathcal{C}^2(\mathbf{R} \times \mathbf{R}^{2N}, \mathbf{R})$ is 2π-periodic in all variables, so (HS) induces a Hamiltonian system on the torus T^{2N}. In addition we assume that H is even in the z-variable. This implies the existence of 2^{2N} trivial stationary solutions of (HS). We are interested in the existence of nontrivial periodic solutions. Observe that the Arnol'd conjecture does not guarantee any more periodic solutions than those already given by the stationary solutions. Let w_0 be such a trivial stationary solution (considered as a 2π-periodic solution) and let $j(w_0)$ be its Maslov index. Under certain nondegeneracy assumptions we prove the existence of at least $|j(w_0)| - N$ pairs $\pm z(t)$ of nontrivial 2π-periodic solutions.

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1 Introduction

Hamiltonian systems with periodic nonlinearity have been studied extensively in the last decade. One of the most celebrated results in this direction is related to a conjecture, raised by Arnol'd in [A1,A2], concerning the number of fixed points of symplectic diffeomorphisms. Let M be a compact symplectic manifold with a symplectic form ω, that is, a closed nondegenerate 2-form. Let $h: \mathbf{R} \times M \to \mathbf{R}$ be a time dependent smooth function. Suppose that h is 2π-periodic in t. Then the Hamiltonian vector field associated with h_t is defined by

$$\omega(\cdot, X_t) = dh_t.$$

Consider the Hamiltonian system

$$\frac{d}{dt}\phi_t = X_t(\phi_t), \quad \phi_0 = Id,$$

which defines a family of symplectic diffeomorphisms. Then the Arnol'd conjecture states that $\phi_{2\pi}$ has at least as many fixed points as a function on M has critical points. There is a one-to-one correspondence between the fixed points of $\phi_{2\pi}$ and the periodic solutions of the following Hamiltonian system

$$\dot{z} = X_t(z(t)), \quad z(0) = z(2\pi).$$

Thus the problem becomes to estimate the number of periodic solutions of this system. The Arnol'd conjecture was first solved in 1983 by Conley and Zehnder in [CZ1] for the case $(M, \omega) = (T^{2N}, \omega_0)$, that is, for the $2N$-dimensional torus with its canonical symplectic form. The result reads as follows.

Theorem. *Suppose that $H \in \mathcal{C}^2(\mathbf{R} \times \mathbf{R}^{2N}, \mathbf{R})$ is 2π-periodic in all variables. Then the Hamiltonian system*

$$\dot{z} = JH_z(t, z), \tag{1.1}$$

where

$$J = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}$$

is the standard symplectic matrix, possesses at least $2N+1$ 2π-periodic orbits. If further, all of the 2π-periodic solutions have no Floquet multiplier equal to 1, then (1.1) has at least 2^{2N} 2π-periodic orbits.

If we further assume that H is even in terms of z, that is,

$$H(t, -z) = H(t, z), \qquad \text{for all } (t, z) \in \mathbf{R} \times \mathbf{R}^{2N}, \tag{1.2}$$

then it is not difficult to check that (1.1) possesses at least 2^{2N} constant orbits, namely the elements of

$$W = \{w \in \mathbf{R}^{2N} : \; w_i \in \{0, \pi\} \quad \text{for all } i = 1, \dots, 2N\}.$$

In this situation the Arnol'd conjecture is trivially true. For example, when we take

$$H(t, z) = \sum_{i=1}^{2N} a_i(t)(1 - \cos z_i),$$

then modulo 2π, (1.1) has exactly 2^{2N} constant solutions. Thus one question arises naturally here. Do there exist nontrivial periodic solutions of (1.1)? The goal of this paper is to address this question. We shall give an existence result for nontrivial periodic solutions of (1.1) when H is even. This will be stated and proved in §3. In §2 we first prove an abstract critical point theorem which is the basis of our main result.

2 A critical point theorem

Let X be a real Hilbert space and consider a functional $f \in \mathcal{C}^2(X, \mathbf{R})$. Suppose f can be written as

$$f(x) = \frac{1}{2}(Ax, x) + g(x)$$

and the following assumptions are satisfied.

(A_1) $A \in \mathcal{L}(X, X)$ is a self adjoint bounded linear operator such that 0 is isolated in the spectrum of A. Then X splits as $X \cong X^+ \oplus X^- \oplus X^0$ according to the spectral decomposition of A. We assume that $k := \dim X^0$ and $m := \dim X^-$ are finite.

(A_2) $g \in \mathcal{C}^1(X, \mathbf{R})$ satisfies $g(x) = o(\|x\|^2)$ as $\|x\| \to \infty$.

(A_3) There is a basis $e_1, \dots, e_k$ of X^0 such that

$$g\left(x + \sum_{i=1}^{k} l_i e_i\right) = g(x) \qquad \text{for all } l_1, \dots, l_k \in \mathbf{Z}.$$

(A_4) g is even: $g(-x) = g(x)$ for any $x \in X$.

(A_5) f satisfies the Palais-Smale condition.

By condition (A_3), f can be reduced to a functional defined on $M := (X^+ \oplus X^-) \times T^k$, where $T^k = X^0 / \bigoplus_{i=1}^k \mathbf{Z} e_i$. We still use $f\colon M \to \mathbf{R}$ to denote this functional. Observe that the antipodal action of $G = \mathbf{Z}/2$ on X induces an action of G on M and that f is G-invariant because of (A_4). There is a nontrivial fixed point set for this group action on M which is given by

$$M^G = \Big\{(0,0,w) \in M : w = \sum_{i=1}^k w_i e_i \ ,\ w_i \in \{0, \tfrac{1}{2}\}, \text{ for all } i = 1, \dots, k\Big\}.$$

Since M^G consists of isolated fixed points, each element of M^G must be a critical point of f. These are called the trivial critical points.

Theorem 2.1. *Let $x_0 \in M^G$ have Morse index n and suppose that in addition to (A_1)–(A_5) the following assumption holds.*

(A_6) *All trivial critical points $x \in M^G$ with $f(x) < f(x_0)$ are nondegenerate.*

Then there exist at least $n - (k + m)$ different pairs of nontrivial critical points of f with critical values less than $f(x_0)$.

It is not difficult to find examples which show that the lower bound $n - (k + m)$ is optimal. We conjecture that the result remains true without the technical assumption (A_6). Observe that we allow x_0 to be degenerate. Also one should remark that the result can be improved under additional conditions. For example, if x_0 is the minimum of $f|M^G$ then there exist at least $n - m$ pairs of nontrivial critical points of f lying in f^c where $c = f(x_0)$. This can be shown easily using standard methods like the genus because there are no fixed points of the G-action in $\{x \in M : f(x) < \min f(M^G)\}$. In the situation of Theorem 2.1 we were not able to work only with the genus or with the cohomological index for the group $\mathbf{Z}/2$ defined by Yang [Y] and Fadell and Rabinowitz [FR]. It is precisely the presence of the fixed points in the sublevel set f^c which forces us to use different methods and to look at the equivariant cohomology of f^c and not just at the genus or the index of f^c.

In order to prove Theorem 2.1 we need to recall Borel cohomology for the group $G = \mathbf{Z}/2$. For details we refer the reader to [tD; §III.1]. Let EG be a contractible free G-space, for instance the unit sphere of an infinite-dimensional normed vector space where G acts via the antipodal map. Then for any two G-spaces $B \subset A$ the Borel cohomology of the pair (A, B) is defined as

$$h^*(A,B) := H_G^*(A,B;\mathbf{F}_2) \ = H^*(EG \underset{G}{\times} A, EG \underset{G}{\times} B; \mathbf{F}_2)\,.$$

Here $EG \underset{G}{\times} A := (EG \times A)/G$ is the orbit space of $EG \times A$ with respect to the diagonal G-action. $H^*(-;\mathbf{F}_2)$ denotes Čech cohomology with coefficients in the field $\mathbf{F}_2$ of two elements. The cup product in cohomology turns $h^*(A)$ into a ring with $1_A \in h^0(A)$ and $h^*(A,B)$ into a module over $h^*(A)$. Using the ring homomorphism $p_A^* : h^*(\mathrm{pt}) \to h^*(A)$ induced by the constant map $p_A : A \to \mathrm{pt}$ each $h^*(A,B)$ is a module over the coefficient ring $R := h^*(\mathrm{pt})$. We write

$$\alpha\xi = \alpha \cdot \xi := p_A^*(\alpha) \cup \xi\ , \qquad \alpha \in R,\ \xi \in h^*(A,B)$$

for this product and similarly for the ring product. We need to know the coefficient ring R. The space $EG \underset{G}{\times} \mathrm{pt} \cong EG/G =: BG$ is the classifying space for G. In our case this is the infinite-dimensional projective space $BG = \mathbf{R}P^\infty$. Therefore the coefficient ring of h^* is the polynomial ring $R \cong \mathbf{F}_2[\omega]$ with $\omega \in h^1(\mathrm{pt})$. If $B \subset A$ are free G-spaces then

$$h^*(A,B) \cong H^*(A/G, B/G; \mathbf{F}_2). \tag{2.2}$$

In particular, if G acts on the sphere S^{n-1} via the antipodal map then

$$h^*(S^{n-1}) \cong H^*(S^{n-1}/G; \mathbf{F}_2) = H^*(\mathbf{R}P^{n-1}; \mathbf{F}_2) \cong \mathbf{F}_2[\omega]/\omega^n$$

and the homomorphism (induced by the inclusion $S^{n-1} \hookrightarrow B^n$ or by the constant map)

$$R = h^*(\mathrm{pt}) \cong h^*(B^n) \to h^*(S^{n-1})$$

is the quotient map $\mathbf{F}_2[\omega] \to \mathbf{F}_2[\omega]/\omega^n$.

Now we begin with the proof of Theorem 2.1. First we shall define our critical values. We may assume that $x_0 = 0$ and $f(0) = 0$. We set $B^k := \{x \in T^k : \|x\| \le 1/4\}$ and look at the following sequence of homomorphisms induced by inclusions:

$$h^k(T^k, T^k - 0) \xrightarrow{i_1^*} h^k(B^k, B^k - 0) \xrightarrow{i_2^*} h^k(B^k) \to h^k(B^k - 0) \cong h^k(S^{k-1}) \cong 0.$$

The first map i_1^* is an excision isomorphism. Moreover, i_2^* is onto by exactness. Thus there exists $\xi \in h^k(T^k, T^k - 0)$ such that

$$i_2^* \circ i_1^*(\xi) = \omega^k \cdot 1_{B^k}.$$

Let $\eta := \xi|_{T^k} \in h^k(T^k)$ be the restriction of ξ, that is, the image of ξ under the map $h^k(T^k, T^k - 0) \to h^k(T^k)$ induced by the inclusion. We shall use similar notation below. Writing $p : M \to T^k$ for the projection we define

$$\begin{aligned} c_i &:= \inf\{c \in \mathbf{R} : \omega^i p^*(\eta)|_{f^c} \neq 0\} \\ &= \sup\{c \in \mathbf{R} : \omega^i p^*(\eta)|_{f^c} = 0\}\,. \end{aligned}$$

Here $\omega^i p^*(\eta) \in h^{k+i}(M)$ and $\omega^i p^*(\eta)|_{f^c} \in h^{k+i}(f^c)$. Clearly, f^c changes topology as c passes c_i. Thus the deformation lemma shows that c_i is a critical value of f provided $c_i \in \mathbf{R}$, which is not so clear. It is also not clear that there are nontrivial critical points at the level c_i. Theorem 2.1 is a consequence of the following two lemmas.

Lemma 2.3. $-\infty < c_m \le c_{m+1} \le \ldots \le c_{n-k-1} < f(0) = 0$.

Lemma 2.4. *Suppose $c = c_i = \ldots = c_{i+j}$ for some $i \ge m$ and $0 \le j \le n-k-1-i$. Let*

$$A_c := K_c - K_c^G = \{x \in M - M^G : f'(x) = 0, f(x) = c\}$$

be the set of nontrivial critical points at the level c. Then $\omega^j \cdot 1_{A_c} \in h^j(A_c)$ is not zero.

Lemma 2.4 implies that $H^j(A_c/G;\mathbf{F}_2) \cong h^j(A_c) \ne 0$, hence $\dim(A_c) = \dim(A_c/G) \ge j$. Here dim means covering dimension (cf. [HW]). Consequently $A_c \ne \emptyset$ and, if $j > 0$ then A_c has infinitely many elements.

Proof of Lemma 2.3. From assumptions (A_1) and (A_2) it follows that $f|_{X^+\times T^k}$ is bounded below. Thus

$$f^c \subset M - (X^+ \times T^k) \quad \text{for } c < c_- \text{ small enough.}$$

We claim that

$$\omega^m p^*(\eta)|_{M-(X^+\times T^k)} = 0 \tag{2.5}$$

hence $\omega^m p^*(\eta)|_{f^c} = 0$. This implies $c_m \ge c_- > -\infty$. In order to see (2.5) we set $SX^- = \{x \in X^- : \|x\| = R\}$ where $R > 0$ is large enough so that $SX^- \times T^k \subset M - (X^+ \times T^k)$. Such an R exists because of (A_1) and (A_2). Now we observe that the inclusion $SX^- \times T^k \hookrightarrow M - (X^+ \times T^k)$ is an equivariant homotopy equivalence and $SX^- \cong S^{m-1}$. Therefore it suffices to show that $\omega^m p^*(\eta)|_{SX^-\times T^k}$ is zero. This is clear, however, since G acts freely on $SX^- \times T^k$, hence

$$h^{k+m}(SX^- \times T^k) \cong H^{k+m}((SX^- \times T^k)/G;\mathbf{F}_2) \cong 0\,.$$

Here we used (2.2) and the fact that $\dim(SX^- \times T^k)/G = k+m-1$.

It remains to prove that $c_{n-k-1} < f(0) = 0$. The Morse index of f at 0 is n by assumption. Thus there exists a G-invariant subset $S \subset U_{1/4}(0) \subset M$ with $S \cong S^{n-1}$ and $S \subset f^{-\varepsilon}$ for some $\varepsilon > 0$. We claim that

$$\omega^{n-k-1} p^*(\eta)|_S \ne 0. \tag{2.6}$$

Then $\omega^{n-k-1} p^*(\eta)|_{f^{-\varepsilon}} \ne 0$, hence $c_{n-k-1} \le -\varepsilon < 0$. For the proof of (2.6) we observe first that $p(S) \subset B^k = \{x \in T^k : \|x\| \le 1/4\}$. So there is a commutative diagram

$$\begin{array}{ccc} h^{n-1}(T^k) & \xrightarrow{p^*} & h^{n-1}(M) \\ \downarrow & & \downarrow \\ h^{n-1}(B^k) & \xrightarrow{(p|_S)^*} & h^{n-1}(S) \end{array}$$

By definition, $\eta|_{B^k} = \omega^k \cdot 1_{B^k}$, hence $\omega^{n-k-1}\eta|_{B^k} = \omega^{n-1} \cdot 1_{B^k}$. This implies

$$\omega^{n-k-1}p^*(\eta)|_S = (p|_S)^*(\omega^{n-k-1}\eta) = \omega^{n-1} \cdot 1_S \neq 0$$

because

$$h^*(S) \cong h^*(S^{n-1}) \cong \mathbf{F}_2[\omega]/\omega^n \,.$$

This proves Lemma 2.3. □

Proof of Lemma 2.4. By (A_6) the set K_c of critical points of f at the level c is the topological sum of K_c^G and $A_c = K_c - K_c^G$. Moreover, for each $x \in K_c^G$ there exists a closed invariant neighborhood N_x such that for any $\varepsilon > 0$ small the pair $(N_x \cap f^{c+\varepsilon}, \partial N_x \cap f^{c+\varepsilon})$ is G-homotopy equivalent to the pair $(D^{\mu(x)}, S^{\mu(x)-1})$ where $\mu(x)$ is the Morse index of f at x. This implies in particular that

$$h^*(N_x \cap f^{c+\varepsilon}) \to h^*(\partial N_x \cap f^{c+\varepsilon}) \quad \text{is onto.} \tag{2.7}$$

Let $N := \bigcup_{x \in K_c^G} N_x$ be the disjoint union of these neighborhoods. Now suppose that $\omega^j \cdot 1_{A_c} = 0$. Then by the continuity of the Čech cohomology there exists a closed invariant neighborhood $V \subset M - N$ of A_c such that $\omega^j \cdot 1_V = 0$. Then $U := N \cup V$ is an invariant neighborhood of K_c. The deformation lemma says that for $\varepsilon > 0$ small enough $f^{c+\varepsilon} - \operatorname{int} U$ may be deformed into $f^{c-\varepsilon}$. We fix such an $\varepsilon > 0$. Since $\omega^j \cdot 1_V = 0$ the long exact sequence of the pair $(f^{c+\varepsilon}, V)$ yields that

$$\text{there exists an element } \alpha \in h^j(f^{c+\varepsilon}, V) \text{ with } \alpha|_{f^{c+\varepsilon}} = \omega^j \cdot 1_{f^{c+\varepsilon}}. \tag{2.8}$$

Next we look at the Mayer-Vietoris sequence of the triad $(f^{c+\varepsilon} - \operatorname{int} V; f^{c+\varepsilon} - \operatorname{int} U, N \cap f^{c+\varepsilon})$:

$$\begin{aligned} h^{i+k-1}(f^{c+\varepsilon} - \operatorname{int} U) \oplus h^{i+k-1}(N \cap f^{c+\varepsilon}) &\longrightarrow h^{i+k-1}(\partial N \cap f^{c+\varepsilon}) \\ \xrightarrow{\;\delta\;} h^{i+k}(f^{c+\varepsilon} - \operatorname{int} V) &\longrightarrow h^{i+k}(f^{c+\varepsilon} - \operatorname{int} U) \oplus h^{i+k}(N \cap f^{c+\varepsilon}) \end{aligned}$$

We claim that

$$\omega^i p^*(\eta)|_{f^{c+\varepsilon} - \operatorname{int} V} \in h^{i+k}(f^{c+\varepsilon} - \operatorname{int} V) \quad \text{is zero.} \tag{2.9}$$

To see this observe first that $\delta = 0$ because the homomorphism

$$h^*(N \cap f^{c+\varepsilon}) \longrightarrow h^*(\partial N \cap f^{c+\varepsilon})$$

is onto by (2.7). Moreover, $\omega^i p^*(\eta)|_{f^{c+\varepsilon}-\mathrm{int}\,U}$ is zero because $f^{c+\varepsilon} - \mathrm{int}\,U$ can be equivariantly deformed into $f^{c-\varepsilon}$ and $\omega^i p^*(\eta)|_{f^{c-\varepsilon}} = 0$ by the definition of $c = c_i$. In addition, $\omega^i p^*(\eta)|_{N\cap f^{c+\varepsilon}}$ is zero because $\eta \in h^*(T^k)$ comes from $\xi \in h^*(T^k, T^k - 0)$, hence $\omega^i p^*(\eta)$ comes from $\omega^i p^*(\xi) \in h^*(M, M - p^{-1}(0))$, but $N \cap f^{c+\varepsilon} \subset M - p^{-1}(0)$. By exactness of the above Mayer-Vietoris sequence $\omega^i p^*(\eta)|_{f^{c+\varepsilon}-\mathrm{int}\,V}$ lies in the image of δ which is zero. This proves (2.9).

From (2.9) it follows that

$$\text{there exists } \beta \in h^{i+k}(f^{c+\varepsilon}, f^{c+\varepsilon} - \mathrm{int}\,V) \text{ with } \beta|_{f^{c+\varepsilon}} = \omega^i p^*(\eta)|_{f^{c+\varepsilon}}\,. \tag{2.10}$$

Using (2.8), (2.10) and the naturality of the cup product we obtain that

$$\omega^{i+j} p^*(\eta)|_{f^{c+\varepsilon}} = (\omega^j \cdot 1_{f^{c+\varepsilon}}) \cdot \omega^i p^*(\eta)|_{f^{c+\varepsilon}}$$

comes from

$$\alpha \cdot \beta \in h^*(f^{c+\varepsilon}, V \cup (f^{c+\varepsilon} - \mathrm{int}\,V)) = h^*(f^{c+\varepsilon}, f^{c+\varepsilon}) = 0\,.$$

Thus $\omega^{i+j} p^*(\eta)|_{f^{c+\varepsilon}} = 0$ which contradicts the definition of $c = c_{i+j}$. This proves Lemma 2.4. □

3 Periodic solutions of even Hamiltonian systems on the torus

Consider the Hamiltonian system

$$-J\dot{z} = H_z(t, z) \tag{3.1}$$

where $J = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}$ is the standard symplectic matrix in $\mathbf{R}^{2N}$. We are interested in the existence of 2π-periodic solutions of (3.1).

We need the following hypotheses.

(H_1) $H \in \mathcal{C}^2(\mathbf{R} \times \mathbf{R}^{2N}, \mathbf{R})$ is 2π-periodic in all variables.

(H_2) H is even in z, that is, $H(t, -z) = H(t, z)$ for all $(t, z) \in \mathbf{R} \times \mathbf{R}^{2N}$.

Note that under the assumptions (H_1) and (H_2) system (3.1) has 2^{2N} constant solutions which are again called the trivial solutions. These are given by

$$W = \{(w_1, \cdots, w_{2N}) \in \mathbf{R}^{2N} \mid w_i \in \{0, \pi\},\ i = 1, \cdots, 2N\}.$$

Of course, there may exist other constant solutions depending on H.

To state our result, let us proceed with some preliminaries. A 2π-periodic solution $z_0(t)$ of (3.1) is said to be nondegenerate if $z_0(t)$ has no Floquet multiplier equal to 1, that is, if the linearized system

$$\dot{x} = JH_{zz}(t, z_0)x$$

has no nontrivial 2π-periodic solutions. For each nondegenerate periodic solution $z_0(t)$ of (3.1), one can define the Maslov index $j = j(z_0) \in \mathbf{Z}$, which is related to $H_{zz}(t, z_0(t))$. (See [CZ2], [LZ] and [Ch] for its definition and a discussion.) Here a constant solution z_0 is considered as a 2π-periodic solution. We are now ready to state our main result in this section.

Theorem 3.2. *Suppose (H_1) and (H_2) hold. Let $w_0 \in W$ be a nondegenerate stationary solution of (3.1) such that $|j(w_0)| > N$ and such that the following holds ($\sigma \in \{\pm 1\}$ denotes the sign of $j(w_0)$).*

(H_3) *All trivial solutions $w \in W$ with*

$$\sigma \int_0^{2\pi} H(t, w)dt > \sigma \int_0^{2\pi} H(t, w_0)dt$$

are nondegenerate.

Then (3.1) has at least $|j(w_0)| - N$ pairs of nontrivial 2π-periodic solutions.

The proof of this theorem is based on a saddle point reduction procedure as in [AZ] or [Ch], which has been kind of standard in dealing with asymptotically linear and periodic Hamiltonian systems. We shall follow the notation in [Ch].

Proof of Theorem 3.2. Define $A = -J\frac{d}{dt}$ with

$$D(A) = H^1(S^1, \mathbf{R}^{2N}) \subset L^2([0, 2\pi], \mathbf{R}^{2N}) =: H.$$

Using Fourier series

$$H = L^2([0, 2\pi], \mathbf{R}^{2N}) = \bigoplus_{l\in} M(l)$$

where $M(l) = \text{span}\{\cos(lt)e_k + \sin(lt)Je_k \mid k = 1, 2, ..., 2N\}$, $l \in \mathbf{Z}$. Here $e_1, \ldots, e_n$ is the standard orthonormal basis of $\mathbf{R}^{2N}$. Clearly one has $\dim M(l) = 2N$.

Let $F : H \to H$ be the Nemytski operator defined by $F(z) = H_z(t, z)$. Now (3.1) is equivalent to the following operator equation

$$Az = F(z), \qquad z \in D(A). \tag{3.3}$$

This equation is of potential type because solutions of (3.3) correspond to the critical points of the following $\mathcal{C}^2$-functional defined for $z \in D(A)$:

$$I(z) = \frac{1}{2}(Az, z) - \int_0^{2\pi} H(t, z)\, dt$$

where $(-,-)$ is the inner product in L^2. Since the Morse indices of critical points of I are infinite we are going to perform the saddle point reduction procedure first.

Because of (H_1), there exists $C > 0$ such that

$$||F'(z)||_{\mathcal{L}(H)} \leq C, \qquad z \in H. \tag{3.4}$$

Take $m \in \mathbf{N}$ such that $m > C$. Let P_m be the linear projection from H into $H_m := \bigoplus_{|l| \leq m} M(l)$, and $P_m^\perp = Id - P_m$. Then (3.3) is equivalent to

$$\begin{aligned} Az_0^\perp &= P_m^\perp F(z_0 + z_0^\perp) && (3.5) \\ Az_0 &= P_m F(z_0 + z_0^\perp), && (3.6) \end{aligned}$$

where $z_0 = P_m z$ and $z_0^\perp = P_m^\perp z$.

By the discussion in §2.1 of Chapter IV in [Ch], we can solve (3.5) for $z_0^\perp$ in terms of $z_0 \in H_m$. There exists a unique $\alpha(z_0) \in H_m^\perp$ such that

$$A\alpha(z_0) = P_m^\perp F(z_0 + \alpha(z_0)). \tag{3.7}$$

The map $\alpha : H_m \to H_m^\perp$ is of class $\mathcal{C}^1$. As a consequence of the uniqueness of α_0 and the fact that $M(0) \subset H_m$ we obtain that $\alpha(z_0)$ is 2π-periodic in all variables and odd, because F is 2π-periodic and even by (H_1) and (H_2). Thus we only have to solve (3.6) with $z_0^\perp$ replaced by $\alpha(z_0)$. The problem is reduced to

$$Az_0 = P_m F(z_0 + \alpha(z_0)), \quad z_0 \in H_m, \tag{3.8}$$

which is still of variational type. Namely, by Theorem 2.1 of Chapter IV in [Ch], there exists a functional $f \in \mathcal{C}^2(H_m, \mathbf{R})$ and a one-one correspondence between the critical points of f with the solutions of (3.8), hence with the solutions of the operator equation (3.3). By (H_1), f can be reduced to a functional defined on

$$M_m = \left(\bigoplus_{0 < |l| \leq m} M(l) \right) \times T^{2N} = (H_m^+ \oplus H_m^-) \times T^{2N}$$

with $H_m^+ = \bigoplus_{0<l\leq m} M(l)$ and $H_m^- = \bigoplus_{-m\leq l<0} M(l)$. And f has the form

$$\begin{aligned} f(z_0) &= \frac{1}{2}(A(z_0+\alpha(z_0)), z_0+\alpha(z_0)) - \int_0^{2\pi} H(t, z_0+\alpha(z_0))\,dt \\ &= \frac{1}{2}(Az_0, z_0) + g(z_0) \end{aligned}$$

where $g(z_0) := \frac{1}{2}(A\alpha(z_0), \alpha(z_0)) - \int_0^{2\pi} H(t, z_0+\alpha(z_0))\,dt$. A is a linear operator preserving H_m.

Now we check the assumptions of Theorem 2.1. First of all, (A_1) is obvious because H_m is finite-dimensional. Using (3.7) and the fact that F is bounded, one can easily check that $g(z_0)$ satisfies (A_2). Moreover, (A_3) and (A_4) follow from the above mentioned invariance properties of α. The Palais-Smale condition is verified for f on M_m because T^{2N} is compact and $A|_{H_m^+\oplus H_m^-}$ is invertible.

In order to apply Theorem 2.1 to f, we need to calculate the Morse index of f at w_0 in the space H_m. By the assumption of w_0 being nondegenerate and Lemma 2.3 of Chapter IV in [Ch], we know that w_0 is a nondegenerate critical point of f in H_m (or in M_m) for any $m > C$. The following lemma links the Morse index of f at w_0 in H_m to the Maslov index of w_0 and $\dim H_m$. We believe this observation is interesting and useful in its own right for applications.

Lemma 3.9. *There exists $C_1 \geq C$ such that if $m > C_1$, the Morse index of f at w_0 in H_m (or in M_m) is $(2m+1)N + j(w_0)$.*

Proof. Calculating directly (as in [Ch], §IV.2.1, for instance), one has

$$\begin{aligned} d^2f(w_0) &= P_m \circ (A - F'(w_0)) \circ (id_{H_m} + d\alpha(w_0)) \\ &= A|_{H_m} - P_m F'(w_0)|_{H_m} - P_m F'(w_0) d\alpha(w_0). \end{aligned}$$

As in Theorem 1.2 and its proof in Chapter IV of [Ch], there exist $N_1 > 0$ and $\delta > 0$, such that for $m > N_1$,

$$||(A|_{H_m} - P_m F'(w_0)|_{H_m})z_0|| \geq \delta ||z_0||, \qquad z_0 \in H_m,$$

and that the Morse index for the linear operator $A - P_m F'(w_0)$ restricted to the space H_m is

$$\frac{1}{2}\dim H_m + j(w_0) = (2m+1)N + j(w_0).$$

So it suffices to estimate the last term in the expression of $d^2f(w_0)$ to be as small as we please for m large. In fact, using the fact that $(A|_{H_m})^{-1}$ exists and $||(A|_{H_m})^{-1}|| \leq \frac{1}{m+1}$, and differentiating (3.7) one gets for any $z_0 \in H_m$

$$||d\alpha(z_0)||_{\mathcal{L}(H_m,H_m^\perp)} \leq \frac{C}{m+1} \cdot (1 + ||d\alpha(z_0)||_{\mathcal{L}(H_m,H_m^\perp)})$$

This implies

$$||d\alpha(z_0)||_{\mathcal{L}(H_m,H_m^\perp)} \leq \frac{C}{m+1-C}$$

hence, using (3.4) once more we obtain

$$||P_m F'(w_0) d\alpha(w_0)||_{\mathcal{L}(H_m)} \leq \frac{C^2}{m+1-C}$$

and Lemma 3.9 follows. □

We return to the proof of Theorem 3.2. To finish the proof, let us consider the case $j(w_0) > 0$. By Lemma 3.9, the Morse index for f at w_0 in M_m is $(2m+1)N + j(w_0)$. For $w \in W \subset \mathrm{Ker}\, A = M(0) \subset H_m$ we have $\alpha(w) = 0$ because w is a 2π-periodic solution of (3.1). Then one obtains

$$f(w) = -\int_0^{2\pi} H(t,w)dt.$$

Therefore $f(w) < f(w_0)$ if and only if

$$-\int_0^{2\pi} H(t,w)dt < -\int_0^{2\pi} H(t,w_0)dt\,.$$

Thus we see that (H_3) implies (A_6). Now we can apply Theorem 2.1 and obtain the existence of

$$(2m+1)N + j(w_0) - (2N + \dim H_m^-) = j(w_0) - N$$

pairs of nontrivial critical points of f, that is, $j(w_0) - N$ pairs of nontrivial 2π-periodic solutions for (3.1).

The case where $j(w_0) < 0$ can be treated in a similar way by applying Theorem 2.1 to $-f$ instead of f. Thus the proof of Theorem 3.2 is complete. □

Remark 3.10. The Maslov index $j(w_0)$ can be estimated directly in some special cases. For example, if there exists $m \in \mathbf{N}$ such that

$$mI_{2N} < \pm H_{zz}(t,w_0) < (m+1)I_{2N} \qquad \text{for all } t \in [0,2\pi],$$

then the Maslov index is $\pm 2mN$. If (H_3) holds in addition, then (3.1) has at least $(2m-1)N$ pairs of nontrivial 2π-periodic solutions.

A similar result as in Theorem 3.2 can be proved for second order systems of ordinary differential equations with even, periodic nonlinearity. Consider the existence of 2π-periodic solutions for

$$-\frac{d^2x}{dt^2} = V_x(t,x) \tag{3.11}$$

where V satisfies

(V_1) $V \in \mathcal{C}^2(\mathbf{R}\times\mathbf{R}^N,\mathbf{R})$ is 2π-periodic in all variables.

(V_2) V is even in x, that is, $V(t,-x) = V(t,x)$ for all $(t,x)\in\mathbf{R}\times\mathbf{R}^N$.

Under the assumptions (V_1) and (V_2) system (3.11) has at least 2^N constant solutions, 2^N of which are given precisely by

$$W = \{(w_1,\dots,w_N)\in\mathbf{R}^N \mid w_i\in\{0,\pi\},\ i=1,\dots,N\}\,.$$

Let $x(t)$ be a 2π-periodic solution of (3.11). Then one can define the Morse index $\mu(x)$ of x with respect to (3.11) to be the Morse index of the linear operator $-d^2/dt^2 - V_{xx}(t,x(t))$.

Theorem 3.12. *Suppose (V_1) and (V_2) hold. Let $w_0\in W$ be a solution of (3.11) such that $\mu(w_0) > N$ and such that the following holds.*

(V_3) *All trivial solutions $w\in W$ with*

$$\int_0^{2\pi} V(t,w)dt > \int_0^{2\pi} V(t,w_0)dt$$

are nondegenerate.

Then (3.11) has at least $\mu(w_0) - N$ pairs of 2π-periodic solutions which are not in W.

Observe that we do not require w_0 to be non-degenerate. This is due to the fact that one does not need to perform a saddle point reduction in order to prove Theorem 3.12. The abstract critical point theorem 2.1 does not require w_0 to be non-degenerate. We expect that one can also get rid of the non-degeneracy assumption in Theorem 3.2 . The Maslov index of a degenerate solution is defined in [L].

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