Nonlinear Schrödinger equations with steep potential well

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1 Introduction and main results

We are interested in solutions of nonlinear Schrödinger equations

$$-\Delta u + V(x)u = f(x,u) \quad \text{on } \mathbb{R}^N \tag{1.1}$$

which satisfy the boundary condition $u(x) \to 0$ as $|x| \to \infty$. If the Schrödinger operator $H = -\Delta + V$ on $L^2(\mathbb{R}^N)$ has a discrete spectrum then under appropriate hypotheses on f, various methods may be applied to yield existence of solutions. The problem becomes more difficult when H has essential spectrum. In this paper we investigate a situation where H may have only a finite number of eigenvalues below the infimum of the essential spectrum. We consider positive potentials V which have a potential well whose steepness is

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controlled by a parameter. If the steepness of the potential well is being increased, then H has more and more eigenvalues below $\inf \sigma_{\mathrm{ess}}(H)$. A natural question is whether (1.1) has correspondingly more and more solutions which decay at infinity and are localized near the bottom of the potential. More precisely, we consider potentials of the form $V_\lambda(x) = a_0(x) + \lambda a(x)$ satisfying the conditions

(V_1) $a_0 \in L^\infty(\mathbb{R}^N)$ and $\alpha_0 := \operatorname{ess\,inf} a_0 > 0$;

(V_2) $a \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$, $a(x) \geq 0$ and there exists a closed subset $Z \subset \mathbb{R}^N$ with nonempty interior $\Omega = \operatorname{int} Z$ such that $a(x) = 0$ for $x \in Z$ and $a(x) > 0$ a. e. in $\mathbb{R}^N \setminus Z$.

The set Z is the bottom of the potential well and the parameter λ controls the steepness of the potential well.

To formulate the last basic assumption on the potential, let us denote by $\mu_N(-\Delta + V, G)$ and $\mu_D(-\Delta + V, G)$ infima of the spectra of $-\Delta + V$ on an open set $G \subset \mathbb{R}^N$ with Neumann and Dirichlet boundary conditions, respectively. One has

$$\mu_N(-\Delta + V, G) = \inf_{\psi \in H^1(G)\setminus\{0\}} \frac{\int_G (|\nabla\psi|^2 + V\psi^2)dx}{\|\psi\|^2_{L^2(G)}},$$

$$\mu_D(-\Delta + V, G) = \inf_{\psi \in H^1_0(G)\setminus\{0\}} \frac{\int_G (|\nabla\psi|^2 + V\psi^2)dx}{\|\psi\|^2_{L^2(G)}}.$$

Let $K_r(x) = \{y \in \mathbb{R}^N : |y_i - x_i| < r, i = 1, \dots, N\}$ and $K^c_R = \mathbb{R}^N \setminus \overline{K_R(0)}$. We impose one of the following equivalent conditions

($V_{3,1}$) there exists a sequence $R_j \to \infty$ such that

$$\lim_{\lambda\to\infty} \lim_{j\to\infty} \mu_N(-\Delta + V_\lambda, K^c_{R_j}) = \infty;$$

($V_{3,2}$) for some $r > 0$

$$\lim_{\lambda\to\infty} \lim_{x\to\infty} \mu_N(-\Delta + V_\lambda, K_r(x)) = \infty; \tag{1.2}$$

($V_{3,3}$) for every $r > 0$ condition (1.2) holds;

($V_{3,4}$) $\lim_{\lambda\to\infty} \lim_{R\to\infty} \mu_D(-\Delta + V_\lambda, K^c_R) = \infty$;

($V_{3,5}$) for some $r > 0$

$$\lim_{\lambda\to\infty} \lim_{x\to\infty} \mu_D(-\Delta + V_\lambda, K_r(x)) = \infty; \tag{1.3}$$

($V_{3,6}$) for every $r > 0$ condition (1.3) holds.

The equivalence of all ($V_{3,i}$) will be proved later, in Section 2. Remark that here the cubes $K_r(x)$ can be replaced by balls $B_r(x) = \{y \in \mathbb{R}^N : |y-x| < r\}$. In what follows conditions ($V_{3,i}$) will be refered to as (V_3).

Before continuing with the hypotheses on f we present a geometric condition on a which implies (V_3):

(V_4) there exist $M_0 > 0$ and $r > 0$ such that

$$\operatorname{mes}(\{x \in K_r(y) : \ a(x) < M_0\}) \to 0 \quad \text{as } |y| \to \infty$$

where mes stands for the Lebesgue measure.

We shall prove in section 2 that (V_3) is a consequence of (V_4). The following condition which has been used in [3] is even stronger than (V_4):

(V_5) there exists $M_0 > 0$ such that

$$\operatorname{mes}(\{x \in \mathbb{R}^N : \ a(x) < M_0\}) \ < \ \infty.$$

Thus we see that Z may be unbounded. Moreover, it is allowed that

$$\lim_{R\to\infty} \operatorname{ess\,inf} a|_{\mathbb{R}^N \setminus U_R(Z)} = 0,$$

where $U_R(Z)$ is the open R-neighbourhood of Z. So a can be arbitrarily close to 0 on sets of positive measure which are arbitrarily far away from the bottom. The potential $a(x) = \max\{1, \min\{x_1^2x_2^2 - 1, 0\}\}$ on $\mathbb{R}^2$ satisfies (V_4), hence (V_3), but not (V_5). The same is true for the potential $a(x) = \max\{1, \min\{x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2 - 1, 0\}\}$ on $\mathbb{R}^3$.

On the nonlinearity f we impose the following assumptions.

(f_1) f is a Carathéodory function and there exist constants $c > 0$, $p > 2$ with $p < 2^* = 2N/(N-2)$ if $N \geq 3$, such that

$$|f(x,t)| \leq c(1 + |t|^{p-1}) \quad \text{for } t \in \mathbb{R}, \ \text{a. e. } x \in \mathbb{R}^N;$$

(f_2) $f(x, 0) = 0$ for a. e. $x \in \mathbb{R}^N$, and

$$\lim_{t\to 0} \operatorname*{ess\,sup}_{x\in\mathbb{R}^N} \left| \frac{f(x,t)}{t} \right| = 0 \ ;$$

(f_3) there exists $q > 2$ such that

$$0 < qF(x,t) \equiv q \int_0^t f(x,s)ds \leq tf(x,t)$$

for $t \neq 0$, a. e. $x \in \mathbb{R}^N$;

(f_4) $f(x,-t) = -f(x,t)$ for $t \in \mathbb{R}$, a. e. $x \in \mathbb{R}^N$.

We look for solutions of

$$-\Delta u + (a_0(x) + \lambda a(x))u = f(x,u) \qquad \text{on } \mathbb{R}^N \tag{1.4}$$

in the space

$$E = \left\{ u \in W^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} a(x)u^2 dx < \infty \right\}$$

endowed with the norm

$$\begin{aligned} \|u\|_E &= \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + (a_0(x) + a(x))u^2) dx \right)^{1/2} \\ &\sim \left(\|u\|^2_{W^{1,2}} + \int_{\mathbb{R}^N} a(x)u^2 dx \right)^{1/2}. \end{aligned}$$

For $\lambda \geq 0$ we also consider the norm

$$\|u\|_\lambda = \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + (a_0(x) + \lambda a(x))u^2) dx \right)^{1/2}$$

which is equivalent to $\|\cdot\|_E$ if $\lambda > 0$ (but not uniformly with respect to $\lambda > 0$).

Theorem 1.1. *Suppose $(V_1) - (V_3)$ and $(f_1) - (f_4)$ hold. Then for every integer m there exists $\Lambda_m > 0$ such that (1.4) with $\lambda \geq \Lambda_m$ has at least m distinct pairs $\pm v_1^\lambda, \dots, \pm v_m^\lambda$ of weak solutions $v_i^\lambda \in E$. Moreover, there exist constants a_j, b_j, c_j (independent of λ) with $b_j < a_{j+1}$ and such that*

$$a_j \leq \|v_j^\lambda\|_{L^p} \leq b_j$$

and

$$\|v_j^\lambda\|_\lambda \leq c_j$$

hold for all λ.

Next we investigate the behaviour of the solutions as $\lambda \to \infty$. We show that these solutions are well localized.

Theorem 1.2. *Suppose* $(V_1)-(V_3)$ *and* $(f_1)-(f_3)$ *hold. Let* $u_n \in E$, $n \in \mathbb{N}$, *be a weak solution of* (1.4) *for some sequence* $\lambda_n \to \infty$ *and suppose that*

$$\sup_{n\in\mathbb{N}} \|u_n\|_{\lambda_n} < \infty \tag{1.5}$$

and

$$\inf_{n\in\mathbb{N}} \|u_n\|_{L^p} > 0 \tag{1.6}$$

hold. Then there exists $\overline{u} \in W^{1,2}(\mathbb{R}^N)$ *such that* $u_n \to \overline{u}$ *along a subsequence strongly in* $W^{1,2}(\mathbb{R}^N)$. *Moreover,* $\overline{u}$ *is a weak solution of the equation*

$$-\Delta\overline{u} + a_0(x)\overline{u} = f(x,\overline{u}) \qquad \text{for } x \in \Omega = \operatorname{int} Z \tag{1.7}$$

and $\overline{u} = 0$ *a. e. in* $\mathbb{R}^N \setminus Z$.

A natural question is whether $\overline{u} \in W_0^{1,2}(\Omega)$. Without further conditions on a or Z nothing is known. Clearly, if $Z = \overline{\Omega}$ and if the boundary of Ω is locally Lipschitz, then we have the implication

$$u \in W^{1,2}(\mathbb{R}^N),\ u = 0 \text{ a. e. on } \mathbb{R}^N \setminus Z \qquad \Longrightarrow \qquad u \in W_0^{1,2}(\Omega) \tag{1.8}$$

Thus if (1.8) holds, then the solution $\overline{u}$ of (1.7) from Theorem 1.2 lies in $W_0^{1,2}(\Omega)$. However, (1.8) does not always hold. In [18] Polking constructs a compact set $K \subset \mathbb{R}^2$ with $\operatorname{mes}(K) > 0$, $\operatorname{int}(K) = \emptyset$ and such that there exists a nonzero $u \in W^{1,2}(\mathbb{R}^2)$ supported by K. In fact, K is the closed unit disk $B_1(0) \subset \mathbb{R}^2$ with a countable number of small disjoint open disks removed. Setting $Z = K \cup \{x \in \mathbb{R}^2 : 1 \le |x| \le 2\}$ one can find $a \in C^\infty(\mathbb{R}^2, \mathbb{R})$ with $a \ge 0$, $Z = a^{-1}(0)$, and $a(x) \to \infty$ as $|x| \to \infty$. Then a satisfies (V_2) and (V_5), hence (V_3), but (1.8) does not hold. Using the notion of capacity (cf. [14]) it follows from [1] and [9] that (1.8) is a consequence of the condition

(1.9) $\operatorname{cap}\big(G \setminus (Z \cap B_R(0))\big) = \operatorname{cap}\big(G \setminus \operatorname{int}(Z \cap B_R(0))\big)$ for any $G \subset \mathbb{R}^N$ open, any $R > 0$.

In fact, for bounded Z the two properties (1.8) and (1.9) are equivalent.

Finally we examine the decay of the solutions at infinity. Here we need to replace (V_3) by the following assumption which is even stronger than (V_5):

(V_6) there exists $M_0 > 0$, $R_0 > 0$ such that $\{x \in \mathbb{R}^N : a(x) \le M_0\} \subset B_{R_0}(0)$ up to a set of measure zero.

Theorem 1.3. *Suppose* (V_1)*,* (V_2)*,* (V_6) *and* $(f_1)-(f_3)$ *hold. Let* $u_\lambda \in E$ *be a solution of* (1.4) *for* $\lambda \geq \Lambda$ *such that* $\|u_\lambda\|_\lambda$ *is bounded uniformly in* λ*. Then there exist* $\alpha > 0$*,* $\beta > 0$ *independent of* λ *such that*

$$|u_\lambda(x)| \leq \beta\lambda^{-1/2} \exp\big(-\alpha\lambda^{1/2}(|x| - R_0)\big) \quad \textit{for } |x| \geq R_0 .$$

We conclude this section with a discussion of related results. In [19] Rabinowitz obtained the existence of a positive and a negative solution of (1.1) in $E = \{u \in W^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2dx < \infty\}$ provided $\inf V > 0$ and $V(x) \to \infty$ as $|x| \to \infty$. The nonlinearity f has to satisfy $(f_1)-(f_3)$. Moreover, V and f have to be continuous but this is not essential. It is known though not used in [19] that $V(x) \to \infty$ as $|x| \to \infty$ implies

(V_7) the embedding $E \hookrightarrow L^2(\mathbb{R}^N)$ is compact.

In fact, as a consequence of Molchanov's results [15] (see also [11], Corollary 6.2) the following condition (V_8) implies (V_7):

(V_8) for any $M > 0$ and any $r > 0$ it holds:

$$\mathrm{mes}(\{x \in K_r(y) : V(x) \leq M\}) \to 0 \quad \text{as } |y| \to \infty.$$

In [3] and [4] the stronger condition

(V_9) $\mathrm{mes}(\{x \in \mathbb{R}^N : V(x) \leq M\}) < \infty$ for every $M > 0$

has been used. In [3] it is proved that (V_9) implies (V_7) which is of course a consequence of the result of [15] and [11]. In [11] Kondrat'ev and Shubin also show that "λ-free" version of each condition $(V_{3,i})$ is equivalent to (V_7) if V is bounded below. For example, one such condition is

$$\lim_{x\to\infty} \mu_N(-\Delta + V, K_r(x)) = \infty$$

for every $r > 0$ (and $K_r(x)$ can be replaced by $B_r(x)$). Moreover, there is a necessary and sufficient condition expressed in terms of capacity [15] and [11]:

(V_{10}) there exists $c > 0$ such that for every sequence x_k with $|x_k| \to \infty$, every $r > 0$, every compact subset $F_k \subset B_r(x_k)$ with $\mathrm{cap}(F_k) \leq cr^{n-2}$ if $n \geq 3$ ($\mathrm{cap}(F_k) \leq c(\log 1/r)^{-1}$ if $n = 2$), it holds

$$\int_{B_r(x_k)\setminus F_k} V(x)\,dx \quad \to \infty \quad \text{as } k \to \infty.$$

From (V_7) it follows that for $2 \le p < 2^*$

(V_{11}) the embedding $E \hookrightarrow L^p(\mathbb{R}^N)$ is compact.

If f satisfies $(f_1) - (f_3)$, and if V satisfies $\inf V > 0$ and (V_{11}) with p from (f_1), then a simple mountain pass argument yields a positive and a negative solution of (1.1). If f satisfies in addition (f_4) then (1.1) has infinitely many solutions. These results can essentially be found in [3]. There we required (V_9) instead of (V_{11}) but the proof in [3] works under the weaker hypothesis. In [3] it is also proved that (1.4) has a positive and a negative solution for λ large provided (V_1), (V_2), (V_5) and $(f_1) - (f_3)$ hold. In fact, only the case $a_0 \equiv 1$ was considered in [3] which is however not a serious restriction. In [22] these results are obtained under weaker hypotheses on the potential and on the nonlinearity. If V satisfies (V_1), (V_2) and (V_5) then Theorem 2.4 of [3] states the existence of an increasing number of solutions of (1.4) if $\lambda \to \infty$. This result is now a consequence of Theorem 1.1. The proof in [3] implicitely assumes (V_7) instead of (V_5) and therefore has a gap. We would also like to mention the paper [4] where the existence of a sign-changing solution of (1.1) and (1.4) has been proved under somewhat stronger hypotheses on V, a and f (but f need not be odd). The results of [2] and [4] suggest that the solutions obtained in Theorem 1.1 should also change sign, except v_1^λ.

Somewhat related concentration phenomena as in Theorem 1.2 and Theorem 1.3 have been observed for positive bound states of the singularly perturbed equation

$$-\epsilon^2 \Delta u + V(x)u = f(x,u) \qquad \text{on } \mathbb{R}^N \tag{1.10}$$

as $\epsilon \to 0$. This kind of results has been initiated by the work [6] of Floer and Weinstein. They considered the homogeneous nonlinearity $f(x,u) = |u|^{p-2}u$ and proved the existence of a positive solution u_ϵ of (1.10) which has the shape of a spike and concentrates near a nondegenerate critical point of V as $\epsilon \to 0$. More recent results of this type can be found in [8], [12], [17], [23], [24] and the references therein. In particular, in [17] a concentration for solutions of (1.10) is proved, if there exists a bounded subset $A \subset \mathbb{R}^N$ such that $\inf V|_A < \inf V|_{\partial A}$. Then (1.10) has a solution u_ϵ which concentrates as $\epsilon \to 0$ at a point $x_\epsilon \in A$ and $V(x_\epsilon) \to \inf V|_A$. Moreover, u_ϵ decays exponentially away from x_ϵ. This point concentration which holds for (1.10) as $\epsilon \to 0$ is quite different from the concentration which we prove for equation (1.4) as $\lambda \to \infty$.

The paper is organized as follows. In section 2 we investigate the spectrum of the Schrödinger operator $H_\lambda = -\Delta + V_\lambda$ as $\lambda \to \infty$. Our main result there is that the infimum of the essential spectrum of $-\Delta + V_\lambda$ converges to ∞

as $\lambda \to \infty$ and that more and more eigenvalues appear below the infimum of the essential spectrum as $\lambda \to \infty$. We also prove in section 2 that (V_4) implies (V_3). Theorem 1.i will be proved in section i+2, $i = 1, 2, 3$.

2 Spectral properties of the Schrödinger operator

The operator $H_\lambda := -\Delta + V_\lambda = -\Delta + \lambda a + a_0$ is a self-adjoint operator on $L^2(\mathbb{R}^N)$, bounded below by α_0, with the form domain $Q(H_\lambda) = E$; see [20], for instance. Thus the quadratic form $\langle H_\lambda u, u\rangle$ which is defined for $u \in \mathcal{D}(H_\lambda)$ extends to a quadratic form on E. Here we write $\langle \cdot\,, \cdot \rangle$ and $\|\cdot\|$ for the scalar product and the norm in $L^2(\mathbb{R}^N)$, respectively. Other norms will be distinguished by an index. We shall occasionally use the notation $\langle H_\lambda u, u\rangle = \int_{\mathbb{R}^N}(|\nabla u|^2 + V_\lambda u^2)dx$ also for $u \in E$.

In order to prove the existence of eigenvalues of H_λ below $\sigma_{\text{ess}}(H_\lambda)$, for given elements $\psi_1, \dots \psi_m$ of E we set

$$\begin{aligned} &U_\lambda(\psi_1, \dots, \psi_m) \\ &\qquad = \inf\{\langle H_\lambda \psi, \psi\rangle : \psi \in E,\ \|\psi\| = 1,\ \langle \psi, \psi_j\rangle = 0 \text{ for } j = 1, \dots, m\}. \end{aligned}$$

For $k \in \mathbb{N}$ we define spectral values of H_λ by the k-th Rayleigh quotient

$$\mu_k(H_\lambda) = \sup_{\psi_1, \dots, \psi_{k-1} \in E} U_\lambda(\psi_1, \dots, \psi_{k-1}).$$

We also use the notation $\mu_{k,N}(H_\lambda, G)$ $(\mu_{k,D}(H_\lambda, G))$ for the $k-$th Rayleigh quotient of the operator H_λ on a domain $G \subset \mathbb{R}^N$ with Neumann (Dirichlet) boundary conditions. In particular,

$$\mu_{1,N}(H_\lambda, G) = \mu_N(H_\lambda, G)$$

and

$$\mu_{1,D}(H_\lambda, G) = \mu_D(H_\lambda, G)$$

.

Obviously, $\mu_k(H_\lambda)$ is nondecreasing as a function of k and as a function of λ. By Theorems XIII.1 and XIII.2 of [20], either $\mu_k(H_\lambda)$ is an eigenvalue of H_λ or $\mu_k(H_\lambda) = \mu_{k+1}(H_\lambda) = \ldots = \inf \sigma_{\text{ess}}(H_\lambda)$. Now we choose a domain $\Omega_0 \subset \Omega$ with $\text{mes}(\Omega_0) < \infty$ and compare H_λ with the operator $L = -\Delta + a_0$ in $L^2(\Omega_0)$. This operator is selfadjoint and positive, with domain $W_0^{1,2}(\Omega_0) \cap W^{2,2}(\Omega_0)$ and form domain $W_0^{1,2}(\Omega_0)$. Since $\text{mes}(\Omega_0) < \infty$ the spectrum

$\sigma(L)$ is discrete and consists of eigenvalues $\mu_k(L)$ of finite multiplicity, $0 < \mu_1(L) < \mu_2(L) \leq \mu_3(L) \leq \dots$, and $\mu_k(L) \to \infty$ as $k \to \infty$.

Considering $W_0^{1,2}(\Omega_0)$ as a subspace of E we see that $0 \leq H_\lambda \leq L$ because $\langle \psi, H_\lambda \psi\rangle = \langle \psi, L\psi\rangle$ for $\psi \in W_0^{1,2}(\Omega_0)$. As a simple consequence of the Courant minimax description of the eigenvalues (see [20], Section XIII.1, for instance), this implies $\mu_k(H_\lambda) \leq \mu_k(L)$ for every k. Since $\mu_k(H_\lambda)$ is nondecreasing in λ we may define $\mu_k = \lim_{\lambda\to\infty} \mu_k(H_\lambda)$. Clearly we have that $\mu_1 \leq \mu_2 \leq \dots$ is a nondecreasing sequence because $\mu_k(H_\lambda)$ is nondecreasing in k for every λ, and $\mu_k \leq \mu_k(L)$.

Proposition 2.1. $\lim_{k\to\infty} \mu_k = \infty$ *if and only if one of equivalent conditions* $(V_{3,i})$, $i = 1, \dots, 6$, *holds.*

Proof. Assume that $(V_{3,1})$ holds, but

$$\lim_{k\to\infty} \mu_k = \sup_k \mu_k \leq C\ .$$

By $(V_{3,1})$ there exists $\lambda_0 > 0$ and $R_j > 0$ such that

$$\mu_N(H_{\lambda_0}, K^c_{R_j}) \geq 2C. \tag{2.1}$$

Since $\lim_{k\to\infty} \mu_{k,N}(H_0, K_{R_j}(0)) = \infty$ there exists $k_0 \in \mathbb{N}$ with

$$\mu_{k_0,N}(H_0, K_{R_j}(0)) \geq 2C$$

. Using the notation (H^N_λ, G) for the operator H_λ on $L^2(G)$ with Neumann boundary conditions, we have for $\lambda \geq \lambda_0$

$$\begin{aligned}(H_\lambda, \mathbb{R}^N) &\geq (H_{\lambda_0}, \mathbb{R}^N)\\ &\geq (H^N_{\lambda_0}, K_{R_j}(0) \cup K^c_{R_j})\\ &= (H^N_{\lambda_0}, K_{R_j}(0)) \oplus (H^N_{\lambda_0}, K^c_{R_j})\\ &\geq (H^N_0, K_{R_j}(0)) \oplus (H^N_{\lambda_0}, K^c_{R_j})\end{aligned}$$

(see [20], XIII.15, Propositions 3 and 4). Now (2.1) implies

$$\begin{aligned}\mu_{k_0}(H_\lambda, \mathbb{R}^N) &\geq \mu_{k_0}((H^N_0, K_{R_j}(0)) \oplus (H^N_{\lambda_0}, K^c_{R_j}))\\ &= \mu_{k_0,N}(H_0, K_{R_j}(0))\\ &\geq 2C\end{aligned}$$

which contradicts the inequality $\mu_{k_0}(H_\lambda, \mathbb{R}^N) \leq \mu_{k_0} \leq C$.

Assume that $\lim_{k\to\infty} \mu_k = \infty$. We want to show that $(V_{3,4})$ holds. If not, then there exists a constant $C > 0$ such that

$$\mu_D(H_\lambda, K^c_R) \leq C$$

for all $\lambda \geq \lambda_0$ and $R \geq R_0$. By the Persson theorem [16] (see also [10]), we have

$$\inf \sigma_{ess}(H_\lambda) = \lim_{R\to\infty} \mu_D(H_\lambda, K_R^c) \leq C. \tag{2.2}$$

Since $\mu_k \to \infty$, there exists k_0 such that $C < \mu_{k_0} < \mu_{k_0+1}$. Then $\mu_{k_0+1}(H_\lambda) > \mu_{k_0} \geq \mu_{k_0}(H_\lambda)$ provided λ is large enough. This implies that $\inf \sigma_{ess}(H_\lambda) \geq \mu_{k_0} > C$ which contradicts (2.2).

Implications $(V_{3,4}) \Longrightarrow (V_{3,5}) \Longrightarrow (V_{3,6})$ and $(V_{3,2}) \Longrightarrow (V_{3,3})$ are obvious.

By Lemma 2.9 [11], there exist $C_1, C_2 > 0$ such that

$$\mu_N(H_\lambda, K_r(x)) \leq \mu_D(H_\lambda, K_r(x)) \leq C_1 \mu_N(H_\lambda, K_r(x)) + C_2 r^{-2}$$

for all $\lambda > 0$, $x \in \mathbb{R}^N$ and $r > 0$. From this equivalences $(V_{3,2}) \Longleftrightarrow (V_{3,5})$ and $(V_{3,3}) \Longleftrightarrow (V_{3,6})$ follow. (In [11] Lemma 2.9 is stated for balls, but the proof given there works for cubes as well).

Now let us prove that $(V_{3,3})$ implies $(V_{3,1})$. Take $R_j = jr$ and decompose $\mathbb{R}^N$ into a countable family of pairwise disjoint cubes $K_r(x_m)$ such that $\overline{K_{R_j}^c}$ is the union of all $\overline{K_r(x_m)}$ with $m \in J(j) = \{m : x_m \in K_{R_j}^c\}$. Let

$$\beta_j(\lambda) = \inf_{m\in J(j)} \mu_N(H_\lambda, K_r(x_m)).$$

Clearly,

$$\lim_{\lambda\to\infty} \lim_{j\to\infty} \beta_j(\lambda) = \infty.$$

Now for $u \in H^1(K_{R_j}^c)$ we have

$$\int_{K_{R_j}^c} (|\nabla u|^2 + V_\lambda u^2) dx = \sum_{m\in J(j)} \int_{K_r(x_m)} (|\nabla u|^2 + V_\lambda u^2) dx \geq$$

$$\geq \beta_j(\lambda) \sum_{m\in J(j)} \|u\|^2_{L^2(K_r(x_m))} = \beta_j(\lambda) \|u\|^2_{L^2(K_{R_j}^c)}.$$

This implies obviously that

$$\lim_{\lambda\to\infty} \lim_{j\to\infty} \mu_N(H_\lambda, K_{R_j}^c) = \infty.$$

$\square$

As a corollary of Proposition 2.1 we obtain:

Corollary 2.2. a) $\inf \sigma_{\mathrm{ess}}(H_\lambda) \to \infty$ *as* $\lambda \to \infty$.

b) *For any k there exists $\lambda_k > 0$ such that H_λ has at least k eigenvalues below the essential spectrum provided $\lambda \geq \lambda_k$.*

For some quantitative estimates it is useful to know more precisely how fast $\inf \sigma_{\text{ess}}(H_\lambda)$ grows. We present one result in this direction.

Proposition 2.3. *Under assumptions* (V_1), (V_2) *and* (V_5) *the essential spectrum* $\sigma_{\text{ess}}(H_\lambda)$ *of* H_λ *is contained in* $[\lambda M_0 + \alpha_0, \infty)$.

It is possible that $\sigma_{\text{ess}}(H_\lambda) = [\lambda M_0 + \alpha_0, \infty)$ for every $\lambda > 0$. This is the case, for instance, if $a_0(x) \equiv \alpha_0$ and $a(x) \to M_0$ as $|x| \to \infty$. Hypothesis (V_5) is satisfied if $a(x) > M_0$ for all x except on a set of finite measure. As it follows from the Lieb-Cwikel-Rozenblum bound [20], under the assumptions of Proposition 2.3 H_λ has only a finite number of eigenvalues below $\lambda M_0 + \alpha_0$ provided $N \geq 3$.

Proof. We set $W_\lambda := V_\lambda - \lambda M_0 - a_0 = \lambda(a - M_0)$ and write $W_\lambda^+ = \max(W_\lambda, 0)$, $W_\lambda^- = \min(W_\lambda, 0)$. Clearly we have for $\lambda \geq 0$

$$\sigma(-\Delta + W_\lambda^+ + \lambda M_0 + a_0) \subset [\lambda M_0 + \alpha_0, \infty) \tag{2.3}$$

because $W_\lambda^+ \geq 0$. Thus Lemma 2.3 follows if we can show that H_λ has the same essential spectrum as $L_\lambda := -\Delta + W_\lambda^+ + \lambda M_0 + a_0$. Since $H_\lambda = L_\lambda + W_\lambda^-$ we only need to prove that the operator of multiplication by W_λ^- is a relative form compact perturbation of L_λ, see [20], Problem 39. The form domain of L_λ is the same as the form domain E of H_λ since W_λ^- is bounded. Thus we have to show that

$$E \to E^*, \; u \mapsto W_\lambda^- \cdot u, \quad \text{is compact .} \tag{2.4}$$

In fact, we shall show that

$$W^{1,2}(\mathbb{R}^N) \to L^2(\mathbb{R}^N), \; u \mapsto W_\lambda^- \cdot u, \quad \text{is compact.} \tag{2.5}$$

Since $E \hookrightarrow W^{1,2}(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N) \hookrightarrow E^*$ are bounded this implies (2.4). Setting

$$S := \{W_\lambda^- \cdot u : u \in W^{1,2}(\mathbb{R}^N), \; \|u\|_{W^{1,2}} \leq 1\} \tag{2.6}$$

we want to prove that S is precompact in $L^2(\mathbb{R}^N)$ which is equivalent to (2.5). Fix $\epsilon > 0$ and set $A := \{x \in \mathbb{R}^N \; : \; a(x) \leq M_0\}$. By (V_5) there exists $R = R(\epsilon) > 0$ such that

$$\text{mes}(A \cap B_R^c) < \epsilon \tag{2.7}$$

where $B_R^c := \mathbb{R}^N \setminus B_R(0)$. Let χ be the characteristic function of $B_R(0)$ and $\chi^c = 1 - \chi$ the characteristic function of B_R^c. Choose $s \in (1, N/(N-2))$ and let $s' = s/(s-1)$ be the dual exponent. For $u \in W^{1,2}(\mathbb{R}^N)$ with $\|u\|_{W^{1,2}} \leq 1$ we have

$$\begin{aligned}\|\chi^c \cdot W_\lambda^- \cdot u\|_{L^2} &= \int_{A\cap B_R^c} (W_\lambda^-)^2 u^2 dx \\ &\leq \left(\int_{A\cap B_R^c} |u|^{2s} dx\right)^{1/s} \left(\int_{A\cap B_R^c} |W_\lambda^-|^{2s'} dx\right)^{1/s'} \\ &\leq C\|u\|_{W^{1,2}}^2 \cdot \epsilon^{1/s'} \\ &\leq C\epsilon^{1/s'}.\end{aligned}$$

Here we used that W_λ^- is bounded and that $W^{1,2}(\mathbb{R}^N)$ embeds into $L^{2s}(\mathbb{R}^N)$ since $2 < 2s < 2N/(N-2)$. On the other hand, the set

$$S_\epsilon = \{\chi \cdot W_\lambda^- \cdot u : \; u \in W^{1,2}(\mathbb{R}^N), \; \|u\|_{W^{1,2}} \leq 1\}$$

is precompact in $L^2(\mathbb{R}^N)$ because $W^{1,2}(\mathbb{R}^N) \hookrightarrow L^2_{\text{loc}}(\mathbb{R}^N)$ is compact. Therefore the set S lies in the $C\epsilon^{1/s'}$-neighbourhood of a precompact set S_ϵ, for every $\epsilon > 0$. It follows that S is precompact itself. □

Proposition 2.4. (V_4) *implies* (V_3).

Proof. It is sufficient to prove that

$$\lim_{\lambda\to\infty} \lim_{y\to\infty} \mu_N(-\Delta + V_\lambda, K_r(y)) = \infty$$

follows from (V_4). Let $A_y = \{x \in K_r(y) : a(x) \geq M_0\}$ and $D_y = \{x \in K_r(y) : a(x) < M_0\}$. We first observe that

$$\int_{A_y} \left(|\nabla u|^2 + (\lambda a + a_0)u^2\right) dx \geq (\lambda M_0 + \alpha_0) \int_{A_y} u^2 dx. \tag{2.8}$$

Now we fix $s > 1$ with $s < N/(N-2)$ and let $s' = s/(s-1)$ be the dual exponent. The Sobolev inequality yields a constant $C > 0$ with

$$\|u\|_{L^{2s}(K_r(y))} \leq C\|u\|_{W^{1,2}(K_r(y))} \tag{2.9}$$

for all $u \in W^{1,2}(K_r(y))$. Setting

$$\epsilon(y) = \text{mes}(D_y)$$

we obtain with the help of (2.9) and the Hölder inequality:

$$\begin{aligned}\int_{D_y} u^2 dx &= (\epsilon(y))^{1/s'} (\int_{D_y} u^{2s} dx)^{1/s} \\ &\leq \epsilon(y)^{1/s'} \|u\|_{L^{2s}(K_r(y))}^2 \\ &= \epsilon(y)^{1/s'} C\|u\|_{W^{1,2}(K_r(y))}^2\end{aligned}$$

Condition (V_4) states that $\epsilon(y) \to 0$ as $y \to \infty$. From this and (2.8) it follows easily that the Rayleigh quotient

$$\mu_N(-\Delta + V_\lambda, K_r(y)) = \inf_{u \in W^{1,2}(K_r(y))} \frac{\int_{K_r(y)} (|\nabla u|^2 + V_\lambda u^2)\, dx}{\int_{K_r(y)} u^2 dx}$$

becomes arbitrarily large if $y \to \infty$ and $\lambda \to \infty$. $\square$

3 Proof of Theorem 1.1

Solutions of (1.4) will be obtained as critical points of the functional

$$\Phi_\lambda : E \to \mathbb{R}, \ \Phi_\lambda(u) = \frac{1}{2}\langle H_\lambda u, u\rangle - \int_{\mathbb{R}^N} F(x,u)dx.$$

It is well known that assumption (f_1) implies $\Phi_\lambda \in \mathcal{C}^1(E)$. Clearly, Φ_λ is an even functional by (f_4). As a consequence of the results of section 2, for any $k \in \mathbb{N}$ there exists $\lambda_k > 0$ such that $\mu_k(H_\lambda) < \inf \sigma_{\text{ess}}(H_\lambda)$ and $\mu_j(H_\lambda) > \mu_j/2$ for $j = 1, \dots, k$, all $\lambda \geq \lambda_k$. Let $e_1^\lambda, \dots, e_k^\lambda \in E$ be L^2-orthonormal eigenfunctions of H_λ associated to the eigenvalues $\mu_1(H_\lambda), \dots, \mu_k(H_\lambda)$ and set

$$E_k^\lambda := \text{span}\{e_1^\lambda, \dots, e_k^\lambda\} \subset E \qquad \text{for } \lambda \geq \lambda_k.$$

Next we choose a domain $\Omega_0 \subset \Omega$ with $\text{mes}(\Omega_0) < \infty$ and set $F = W_0^{1,2}(\Omega_0) \subset E$. Since $\text{mes}(\Omega_0) < \infty$, F has an orthonormal basis of eigenfunctions $(f_j)_{j\in\mathbb{N}}$ of the operator $L = -\Delta + a_0$. Setting $F_k = \text{span}\{f_1, \dots, f_k\}$ there exists $R_k > 0$ such that

$$\Phi_\lambda(u) = \Phi_0(u) \leq 0 \quad \text{for any } u \in F_k \text{ with } \|u\|_E \geq R_k. \tag{3.1}$$

Now we define almost critical values c_k^λ as follows. With $B_k = \{u \in F_k : \|u\|_E \leq R_k\}$ and

$$\Gamma_k = \{\gamma : B_k \to E |\ \gamma \text{ is odd, continuous, and } \gamma(u) = u \text{ if } \|u\| = R_k\}$$

we set

$$c_k^\lambda := \inf_{\gamma\in\Gamma_k} \max_{u \in B_k} \Phi_\lambda(\gamma(u)).$$

Clearly we have

$$c_k^\lambda \leq \bar{c}_k := \max_{u\in B_k} \Phi_0(u) \quad \text{for all } k \in \mathbb{N}, \ \lambda \geq 0. \tag{3.2}$$

Moreover, $c_k^\lambda \ge 0$ because $\gamma(0) = 0$ for every $\gamma \in \Gamma_k$. Therefore $c_k^\lambda \in \mathbb{R}$ and standard deformation arguments show that there exist sequences $(u_{k,n}^\lambda)_{n\in\mathbb{N}}$ with

$$\Phi_\lambda(u_{k,n}^\lambda) \to c_k^\lambda \quad \text{and} \quad \Phi'_\lambda(u_{k,n}^\lambda) \to 0 \quad \text{as } n \to \infty \text{ uniformly in } k, \lambda. \tag{3.3}$$

Using (f_3) it follows that $\|u_{k,n}^\lambda\|_\lambda$ is bounded, uniformly in λ and n. This implies $u_{k,n}^\lambda \rightharpoonup u_k^\lambda$ in E and $u_{k,n}^\lambda \to u_k^\lambda$ in $L^q_{\text{loc}}(\mathbb{R}^N)$, $2 \le q < 2N/(N-2)$. It follows that u_k^λ is a weak solution of (1.4).

Next we provide a lower bound for c_k^λ:

$$c_k^\lambda \ge \underline{c}_k^\lambda := \sup_{\rho>0} \inf_{u\in(E_{k-1}^\lambda)^\perp, \|u\|_\lambda=\rho} \Phi_\lambda(u) \quad \text{for all } k \in \mathbb{N},\ \lambda \ge \lambda_k, \tag{3.4}$$

where $(E_{k-1}^\lambda)^\perp = \{u \in E : \langle u, e_j^\lambda \rangle = 0 \text{ for } j = 1, \dots, k-1\}$. In order to see this inequality observe that if ρ satisfies

$$\{u \in (E_{k-1}^\lambda)^\perp \cap F_k : \|u\|_\lambda = \rho, \|u\| \ge R_k\} \ne \emptyset$$

then

$$\inf_{u\in(E_{k-1}^\lambda)^\perp, \|u\|_\lambda=\rho} \Phi_\lambda(u) \le 0$$

by (3.1). Therefore we only need to consider those ρ with

$$\{u \in (E_{k-1}^\lambda)^\perp \cap F_k : \|u\|_\lambda = \rho\} \subset \operatorname{int}(B_k). \tag{3.5}$$

Given $\gamma \in \Gamma_k$ and ρ such that (3.5) holds we set

$$\mathcal{O} = \{u \in B_k : \|\gamma(u)\|_\lambda < \rho\}.$$

This is an open neighbourhood of 0 in B_k and satisfies $\overline{\mathcal{O}} \subset \operatorname{int}(B_k)$ because of (3.5). With $P_k : E \to E_{k-1}$ denoting the L^2-orthogonal projection we consider the map $P_k \circ \gamma : \overline{\mathcal{O}} \to E_{k-1}$. The Borsuk-Ulam theorem yields $u \in \partial\mathcal{O}$ with $P_k(\gamma(u)) = 0$, hence $\gamma(u) \in (E_{k-1}^\lambda)^\perp$ and $\|\gamma(u)\|_\lambda = \rho$. This implies

$$\max_{u\in B_k} \Phi_\lambda(\gamma(u)) \ge \inf_{u\in(E_{k-1}^\lambda)^\perp, \|u\|_\lambda=\rho} \Phi_\lambda(u)$$

for every $\gamma \in \Gamma_k$ and every ρ satisfying (3.5), hence for every $\rho > 0$. This proves (3.4).

Lemma 3.1. *$\underline{c}_k^\lambda \to \infty$ as $k \to \infty$, $\lambda \ge \lambda_k$.*

Proof. Using (f_1) and (f_2) there exists $C_1 > 0$ such that

$$F(x,t) \le |t|^2 + C_1|t|^p \quad \text{for } t \in \mathbb{R}, \text{ a. e. } x \in \mathbb{R}^N.$$

Thus we have

$$\begin{aligned}\Phi_\lambda(u) &= \frac{1}{2}\langle H_\lambda u, u\rangle - \int_{\mathbb{R}^N} F(x,u)dx \\ &\ge \frac{1}{2}\langle H_\lambda u, u\rangle - \|u\|^2 - C_1\|u\|_{L^p}^p .\end{aligned}$$

By the definition of $\mu_k(H_\lambda)$ it follows that

$$\|u\|^2 \le \langle H_\lambda u, u\rangle / \mu_k(H_\lambda) \quad \text{for } u \in (E_{k-1}^\lambda)^\perp. \tag{3.6}$$

By the Gagliardo-Nirenberg inequality there exists $C_2 > 0$ such that

$$\|u\|_{L^p} \le C_2\|u\|_{W^{1,2}}^\theta \|u\|^{1-\theta},$$

where $\theta = N(p-2)/2p \in (0,1)$. Using (3.6) and $\|u\|_{W^{1,2}} \le \|u\|_\lambda = \langle H_\lambda u, u\rangle^{1/2}$ we obtain

$$\|u\|_{L^p} \le \frac{C_2}{\mu_k(H_\lambda)^{(1-\theta)/2}}\|u\|_\lambda \qquad \text{for } u \in (E_{k-1}^\lambda)^\perp. \tag{3.7}$$

Now (3.6) and (3.7) imply

$$\Phi_\lambda(u) \ge \left(\frac{1}{2} - \frac{1}{\mu_k(H_\lambda)}\right)\|u\|_\lambda^2 - \frac{C_3}{\mu_k(H_\lambda)^{p(1-\theta)/2}}\|u\|_\lambda^p$$

for $u \in (E_{k-1}^\lambda)^\perp$. Lemma 3.1 follows easily from $\mu_k(H_\lambda) \to \infty$ as $k \to \infty$, $\lambda \ge \lambda_k$, which has been proved in Lemma 2.1. □

Lemma 3.2. *There exists $\alpha_1 > 0$ independent of λ such that*

$$\liminf_{n\to\infty} \|u_{k,n}^\lambda\|_{L^p}^p \ge \alpha_1 \underline{c}_k^\lambda \quad \textit{for all } k \in \mathbb{N}, \ \lambda \ge \lambda_k.$$

Proof. For $\epsilon_0 = (q-2)/4q$ there exists $\alpha > 0$ such that

$$\frac{1}{2}f(x,t)t - F(x,t) \le \epsilon_0 t^2 + \alpha|t|^{p+1} \quad \text{for } t \in \mathbb{R}, \text{ a. e. } x \in \mathbb{R}^N.$$

This is a consequence of (f_1) and (f_2). Setting $\alpha_1 = 1/2\alpha$ the lemma follows as in [3], proof of Lemma 5.1. □

Lemma 3.3. *For each k there exists $\beta_k > 0$ independent of λ such that*

$$\limsup_{n\to\infty} \|u_{k,n}^\lambda\|_{L^p}^p \le \beta_k \quad \text{for all } k \in \mathbb{N}, \lambda \ge \lambda_k.$$

Proof. As in [3], proof of (5.4), one sees that (3.3) implies

$$\|u_{k,n}^\lambda\|_\lambda^2 \le \frac{2q}{q-2} c_k^\lambda + o(1) \quad \text{as } n \to \infty, \text{ uniformly in } \lambda \ge \lambda_k.$$

Lemma 3.3 follows from this, (3.2) and the Sobolev inequality. □

As in [3] we now choose a sequence of integers $k \to \infty$ such that

$$\alpha_1 \underline{c}_{k_{j+1}}^\lambda > 4\beta_{k_j} \qquad \text{for } j \in \mathbb{N},\ \lambda \ge \lambda_{k_{j+1}}. \tag{3.8}$$

This is possible because of Lemma 3.1.

Lemma 3.4. *For each $m \in \mathbb{N}$ there exists $\Lambda_m > 0$ such that $\Lambda_m \ge \lambda_{k_j}$ for $j = 1, \dots, m$ and such that the weak limits $v_j^\lambda := u_{k_j}^\lambda$ of $(u_{k_j,n}^\lambda)_n$ satisfy*

$$\|v_j^\lambda\|_{L^p}^p \ge \gamma_j^\lambda := \frac{\alpha_1}{2} \underline{c}_{k_j}^\lambda \quad \text{for } \lambda \ge \Lambda_m,\ j = 1, \dots, m.$$

This is proved as Lemma 5.10 of [3].

Proof of Theorem 1.1: Given $m \in \mathbb{N}$ the solutions $v_1^\lambda, \dots, v_m^\lambda$ of (1.4) exist for $\lambda \ge \Lambda_m$. They have increasing L^p-norms:

$$\gamma_j^\lambda \le \|v_j^\lambda\|_{L^p}^p \le \beta_{k_j} < \gamma_{j+1}^\lambda/2 < \gamma_{j+1}^\lambda \le \|v_{j+1}^\lambda\|_{L^p}^p.$$

Here we used (3.8) and Lemma 3.3. Setting

$$a_j := \inf_{\lambda \ge \Lambda_m} (\gamma_j^\lambda)^{1/p} \quad \text{and} \quad b_j := (\beta_{k_j})^{1/p}$$

we have $a_{j+1} > b_j$ by (3.8) and $a_j \le \|v_j^\lambda\|_{L^p} \le b_j$ by Lemma 3.3 and Lemma 3.4. Finally,

$$c_j := \sup_{\lambda \ge \Lambda_m} \|v_j^\lambda\|_\lambda < \infty$$

follows from the fact that $\|u_{k,n}^\lambda\|_\lambda$ is bounded for each k uniformly in λ and n; see the statement following (3.3). □

4 Proof of Theorem 1.2

We first observe that $(u_n)_n$ is bounded in E and in $W^{1,2}(\mathbb{R}^N)$. Therefore after passing to a subsequence we have $u_n \rightharpoonup \overline{u}$ in $W^{1,2}(\mathbb{R}^N)$ and $u_n \to \overline{u}$ in $L^{\theta}_{\mathrm{loc}}(\mathbb{R}^N)$, $2 \le \theta < 2^*$. It follows that $\overline{u}$ is a weak solution of (1.7).

Lemma 4.1. *$\overline{u} = 0$ a. e. in $\mathbb{R}^N \setminus Z$.*

Proof. Consider the sets $A_k = \{x \in \mathbb{R}^N : |x| \le k,\ a(x) \ge 1/k\}$, $k \in \mathbb{N}$. Then

$$\int_{A_k} u_n^2 dx \le k \int_{A_k} a(x) u_n^2 dx \le \frac{k}{\lambda_n} \|u_n\|_{\lambda_n} \to 0 \quad \text{as } n \to \infty ,$$

hence, $\int_{A_k} \overline{u}^2 dx = 0$ for every k. This implies $\overline{u}|A_k = 0$ a. e., for every $k \in \mathbb{N}$, and therefore $\overline{u}(x) = 0$ for a. e. $x \in \bigcup_{k\in\mathbb{N}} A_k$, hence $\overline{u} = 0$ a. e. in $\mathbb{R}^N \setminus Z$ by (V_2). □

Lemma 4.2. *$u_n \to \overline{u}$ in $L^{\theta}(\mathbb{R}^N)$, $2 < \theta < 2^*$.*

Proof. Suppose the lemma is wrong. Then by Lemma I.1 of [13] there exists $\delta > 0$, $\rho > 0$ and a sequence $(x_n)_n$ in $\mathbb{R}^N$ such that

$$\int_{B_\rho(x_n)} |u_n - \overline{u}|^2 dx \ge \delta \quad \text{for all } n \in \mathbb{N}. \tag{4.1}$$

This is only possible if $|x_n| \to \infty$ as $n \to \infty$. Setting $B_n := \{x \in B_\rho(x_n) : a(x) \le M_0\}$ and $R_n := |x_n| - \rho$ we have

$$\begin{aligned} \|u_n\|^2_{\lambda_n} &\ge \lambda_n M_0 \int_{B_\rho(x_n)\setminus B_n} u_n^2\, dx \\ &= \lambda_n M_0 \int_{B_\rho(x_n)\setminus B_n} |u_n - \overline{u}|^2 dx \\ &= \lambda_n M_0 \left(\int_{B_\rho(x_n)} |u_n - \overline{u}|^2 dx - \int_{B_n} |u_n - \overline{u}|^2 dx \right) \\ &= \lambda_n M_0 \left(\delta - \int_{B^c_{R_n}} |u_n - \overline{u}|^2 dx \right). \end{aligned} \tag{4.2}$$

Let $C > 0$ be an upper bound for $\|u_n\|^2_{\lambda_n}$. From (V_3) it follows that

$$\|(u_n - \overline{u})|_{B^c_{R_n}}\|^2_{\lambda_n} \ge \frac{8C}{\delta} \|(u_n - \overline{u})|_{B^c_{R_n}}\|^2_2$$

if n is large enough because $R_n \to \infty$. Putting this into (4.2) we obtain

$$\|u_n\|^2_{\lambda_n} \ge \lambda_n M_0 \delta / 2 \to \infty$$

as $n \to \infty$, in contradiction to the assumption $\sup\limits_{n\in\mathbb{N}} \|u_n\|_{\lambda_n} < \infty$. □

Lemma 4.3. *$u_n \to \overline{u}$ in $W^{1,2}(\mathbb{R}^N)$.*

Proof. Recall that the norm $\|u\|_0 = \int_{\mathbb{R}^N}(|\nabla|u^2 + a_0(x)u^2)dx$ is equivalent to $\|u\|_{W^{1,2}}$. Therefore it suffices to show that

$$\|\overline{u}\|_0 \geq \limsup_{n\to\infty}\|u_n\|_0 \,. \tag{4.3}$$

Since u_n is a weak solution of (1.4) we have

$$\|u_n\|_0^2 \leq \int_{\mathbb{R}^N}(|\nabla u_n|^2 + (\lambda_n a + a_0)u_n^2)\,dx = \int_{\mathbb{R}^N} f(x,u_n)u_n\,dx \,. \tag{4.4}$$

Similarly we obtain

$$\begin{aligned}
\|\overline{u}\|_0^2 &= \lim_{n\to\infty}\int_{\mathbb{R}^N}(\nabla u_n\cdot\nabla\overline{u} + a_0(x)u_n\overline{u})\,dx \\
&= \lim_{n\to\infty}\int_{\mathbb{R}^N}(\nabla u_n\cdot\nabla\overline{u} + (\lambda_n a(x) + a_0(x))u_n\overline{u})\,dx \\
&= \lim_{n\to\infty}\int_{\mathbb{R}^N} f(x,u_n)\overline{u}\,dx \,.
\end{aligned} \tag{4.5}$$

Next we show that

$$\int_{\mathbb{R}^N} f(x,u_n)(\overline{u}-u_n)\,dx \to 0 \qquad \text{as } n\to\infty. \tag{4.6}$$

Let C_1 be a bound for $\|\overline{u}\|_{L^2}$ and $\|u_n\|_{L^2}$, $n\in\mathbb{N}$. The assumptions (f_1) and (f_2) imply that for any $\epsilon > 0$ there exists A_ϵ such that

$$|f(x,t)| \leq \epsilon|t| + A_\epsilon|t|^{p-1} \qquad \text{for } t\in\mathbb{R},\ \text{a. e. } x\in\mathbb{R}^N.$$

This implies

$$\begin{aligned}
\left|\int_{\mathbb{R}^N} f(x,u_n)(\overline{u}-u_n)dx\right| &\leq \epsilon\int_{\mathbb{R}^N}|u_n|\cdot|\overline{u}-u_n|dx + A_\epsilon\int_{\mathbb{R}^N}|u_n|^{p-1}|\overline{u}-u_n|dx \\
&\leq 2\epsilon C_1^2 + A_\epsilon\|u_n\|_{L^p}^{p-1}\cdot\|\overline{u}-u_n\|_{L^p} \,.
\end{aligned}$$

Now (4.6) follows because we can make the first summand as small as we want by choosing ϵ small, and then the second will be arbitrarily small for n large enough by Lemma 4.2. Finally (4.3) is a consequence of (4.4), (4.5) and (4.6). $\square$

5 Proof of Theorem 1.3

Let $u_\lambda \in E$ be a solution of (1.4) for $\lambda \geq \Lambda$ such that $\|u_\lambda\|_\lambda \leq C$ is bounded uniformly in λ.

Lemma 5.1. *The solutions u_λ are bounded, continuous, and $u(x) \to 0$ as $|x| \to \infty$. Moreover, there exists $C_0 > 0$ independent of λ such that $\|u_\lambda\|_{L^\infty} \leq C_0$ for all λ.*

Proof. We first prove the uniform bound on the L^∞-norm of u_λ. In the cases $N = 1$ or $N = 2$ this follows immediately from $\|u_\lambda\|_\lambda \leq C$, hence $\|u_\lambda\|_{W^{1,2}} \leq C_1$ for all $\lambda \geq \Lambda$.

From now on we assume $N \geq 3$. Then $(u_\lambda)_{\lambda\geq\Lambda}$ is a bounded family in $L^{2^*}(\mathbb{R}^N)$ where $2^* = 2N/(N-2)$. Assumption (V_6) implies that the potential V_λ belongs to the Kato class K_N; see [21], $A.2$, for a definition of K_N. Due to the Sobolev estimates for Schrödinger operators (see [21], Theorem B.2.1) H_λ^{-1} is a well defined bounded operator from $L^s(\mathbb{R}^N)$ to $L^r(\mathbb{R}^N)$ provided $1 \leq s \leq r \leq \infty$ and $\frac{1}{s} - \frac{1}{r} < \frac{2}{N}$. Moreover, the argument in [21] also implies that

$$\text{(5.1)}\qquad \begin{aligned} &\text{for } 1 \leq s \leq r \leq \infty \text{ with } \frac{1}{s} - \frac{1}{r} < \frac{2}{N} \text{ there exists } C_{r,s} > 0 \text{ with} \\ &\|H_\lambda^{-1} g\|_{L^r} \leq C_{r,s}\|g\|_{L^s} \text{ for all } \lambda \geq 0 \text{ and all } g \in L^s(\mathbb{R}^N). \end{aligned}$$

Let $\chi_{\lambda,0}$ be the characteristic function of the set $\{x \in \mathbb{R}^N : |u_\lambda(x)| \leq 1\}$ and define $v_{\lambda,0} = \chi_{\lambda,0}u_\lambda$, $w_{\lambda,0} = u_\lambda - v_{\lambda,0} = (1 - \chi_{\lambda,0})u_\lambda$. Setting $g_{\lambda,0} = f(\,\cdot\,, v_{\lambda,0})$ and $h_{\lambda,0} = f(\,\cdot\,, w_{\lambda,0})$ we have $f(\,\cdot\,, u_\lambda) = g_{\lambda,0} + h_{\lambda,0}$. Obviously $\|g_{\lambda,0}\|_{L^\infty}$ is bounded uniformly in λ. Moreover, $\|h_{\lambda,0}\|_{L^{p_0}}$ is bounded uniformly in λ for $p_0 = 2^*/(p-1)$ because assumptions (f_1) and (f_2) imply

$$|f(x,t)| \leq c|t|^{p-1} \qquad \text{for } |t| \geq 1, \text{ a. e. } x \in \mathbb{R}^N.$$

Now we define $v_{\lambda,1} = H_\lambda^{-1} g_{\lambda,0}$ and $w_{\lambda,1} = H_\lambda^{-1} h_{\lambda,0}$ so that $u_\lambda = v_{\lambda,1} + w_{\lambda,1}$. It follows from (5.1) that there exists $C_2 > 0$ with

$$\text{(5.2)}\qquad \|v_{\lambda,1}\|_{L^\infty} \leq C_2 \text{ and } \|w_{\lambda,1}\|_{L^{p_1}} \leq C_2 \text{ for all } \lambda;$$

here $p_1 = \infty$ if $p_0 > N/2$, and p_1 is arbitrarily close to and less than $p_0N/(N-2p_0)$ if $p_0 \leq N/2$. In the case $p_0 > N/2$ we are done. In the case $p_0 \leq N/2$ let $\chi_{\lambda,1}$ be the characteristic function of the set

$$A_\lambda = \{x \in \mathbb{R}^N : |w_{\lambda,1}(x)| \leq C_2 + 1\}.$$

Setting

$$\overline{v}_{\lambda,1} = \chi_{\lambda,1}u_\lambda = \chi_{\lambda,1}(v_{\lambda,1} + w_{\lambda,1})$$

and

$$\overline{w}_{\lambda,1} = u_\lambda - \overline{v}_{\lambda,1} = (1-\chi_{\lambda,1})(v_{\lambda,1} + w_{\lambda,1})$$

we have that $\|\overline{v}_{\lambda,1}\|_{L^\infty}$ is bounded uniformly in λ. In addition, $\|\overline{w}_{\lambda,1}\|_{L^{p_1}}$ is uniformly bounded because

$$\|(1-\chi_{\lambda,1})v_{\lambda 1}\|_{L^{p_1}} \le \|v_{\lambda 1}\|_{L^\infty} \cdot (\mathrm{mes}(\mathbb{R}^N \setminus A_\lambda))^{1/p_1}$$

and $\|(1-\chi_{\lambda,1})w_{\lambda 1}\|_{L^{p_1}} \le \|w_{\lambda,1}\|_{L^{p_1}}$ are uniformly bounded. Now we repeat the above argument with $g_{\lambda,1} = f(\,\cdot\,,\overline{v}_{\lambda,1})$ and $h_{\lambda,1} = f(\,\cdot\,,\overline{w}_{\lambda,1})$. Thus, for $v_{\lambda,2} = H_\lambda^{-1} g_{\lambda,1}$ and $w_{\lambda,2} = H_\lambda^{-1} h_{\lambda,1}$ we obtain a constant $C_3 > 0$ with

$$\|v_{\lambda,2}\|_{L^\infty} \le C_3 \quad \text{and} \quad \|w_{\lambda,2}\|_{L^{p_2}} \le C_3 \text{ for all } \lambda; \tag{5.3}$$

here $p_2 = \infty$ if $p_1 > N/2$, and p_2 is arbitrarily close to and less than $p_1 N/(N-2p_0)$. After a finite number of such steps we obtain a uniform bound for $\|u_\lambda\|_{L^\infty}$ also in the case $N \ge 3$.

According to [21], Theorem B.3.3, we have $u_\lambda \in \mathcal{C}(\mathbb{R}^N)$ because $u_\lambda \in L^\infty(\mathbb{R}^N)$. It remains to prove that $u_\lambda(x) \to 0$ as $|x| \to \infty$. We set $g_\lambda(x) := f(x, u_\lambda(x))/u_\lambda(x)$ and $W_\lambda(x) := V_\lambda(x) - g_\lambda(x)$ so that u_λ solves the equation

$$-\Delta u_\lambda + W_\lambda u_\lambda = 0 \quad \text{on } \mathbb{R}^N. \tag{5.4}$$

Since $u_\lambda \in L^\infty(\mathbb{R}^N)$ we have $g_\lambda \in L^\infty(\mathbb{R}^N)$ using assumptions (f_1) and (f_2). Therefore $W_\lambda \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ is bounded from below. Now Theorem C.3.1 of [21] implies $u_\lambda(x) \to 0$ as $|x| \to \infty$. □

A simple consequence of the uniform bound $\|u_\lambda\|_\lambda \le C$ is

Lemma 5.2. *For every $M > 0$ there exists a constant $C_M > 0$ independent of λ such that*

$$\|u_\lambda\|_{L^2(\{x \in \mathbb{R}^N : a(x) > M\})} \le C_M/\lambda^{1/2}.$$

The assumption (V_6) was not needed for Lemmas 5.1 and 5.2. Theorem C.3.5 of [21] shows that $u_\lambda(x)$ grows like $C_\lambda \exp(-\alpha\lambda^{1/2}|x|)$ at infinity. For the proof of Theorem 1.3 we shall now establish the dependence of C_λ on λ.

Proof of Theorem 1.3: Set $B^c_{R_0} = \mathbb{R}^N \setminus B_{R_0}(0)$ where R_0 is from (V_6). Clearly we have $a(x) > M_0$ for $x \in B^c_{R_0}$. Since u_λ is a solution of (5.4), the subsolution estimate (see [21], Theorem C.1.2) yields

$$|u_\lambda(x)| \le C_1 \|u_\lambda\|_{L^1(B_1(x))} \quad \text{for all } x \in \mathbb{R}^N. \tag{5.5}$$

As a consequence of Lemma 5.1 the negative part W_λ^- of $W_\lambda = V_\lambda - g_\lambda = V_\lambda - f(\cdot, u_\lambda)/u_\lambda$ is bounded uniformly in λ. It follows from [21], A.2.1, that the norm of W_λ^- in K_N is bounded uniformly in λ. This implies that the constant C_1 in (5.5) is independent of λ. Lemma 5.2 implies

$$\|u_\lambda\|_{L^2(B^c_{R_0})} \le C_2/\lambda^{1/2} \quad \text{for all } \lambda \tag{5.6}$$

where C_2 is also independent of λ. Now (5.5) and (5.6) yield a constant $C_3 > 0$ such that

$$|u_\lambda(x)| \le C_3/\lambda^{1/2} \quad \text{for all } \lambda, \text{ all } x \in B^c_{R_0}. \tag{5.7}$$

Next we need to recall some facts about the fundamental solution w_μ of $-\Delta+\mu^2$. This grows like $|x|^{-(N-1)/2}e^{-\mu|x|}$ for $|x| \to \infty$ and we may normalize it so that

$$w_\mu(x) = |x|^{-(N-1)/2}e^{-\mu|x|}(1+o(1)) \text{ as } |x| \to \infty, \tag{5.8}$$

uniformly with respect to $\mu \ge \mu_0 > 0$. For example, in dimension $N = 1$ the true fundamental solution is $(2\mu)^{-1}\exp(-\mu|x|)$ and we set $w_\mu(x) = \exp(-\mu|x|)$. Moreover, w_μ is of class $\mathcal{C}^\infty$ outside the origin and it is positive because $(-\Delta + \mu^2)^{-1}$ is positive in $L^2(\mathbb{R}^N)$. We refer the reader to [5], Proposition 3.1 of Chapter 3 and Theorem 4.2 of Chapter 2, for these results. As a consequence of (5.4) we obtain for $b \in \mathbb{R}$

$$(-\Delta + 2\lambda b)u_\lambda^2 = -2(W_\lambda - \lambda b)u_\lambda^2 - 2|\nabla u_\lambda|^2 \tag{5.9}$$

in the sense of distributions. For a fixed $b \in (0, M_0)$ there exists $\lambda > 0$ such that $W_\lambda - \lambda b \ge 0$ on $B^c_{R_0}$ for all $\lambda \ge \Lambda$. Here we used again the uniform bound of $\|u_\lambda\|_\infty$ from Lemma 5.1. It follows that

$$(-\Delta + 2\lambda b)u_\lambda^2 \le 0 \quad \text{for } \lambda \ge \Lambda, \quad \text{a. e. } x \in B^c_{R_0}. \tag{5.10}$$

Given $\lambda \ge \Lambda$ we define $v_\lambda := u_\lambda^2 - \theta_\lambda w_\mu$, where $\mu = \sqrt{2\lambda b}$ and θ_λ is chosen such that $v_\lambda(x) < 0$ if $|x| = R_0$. In fact, by (5.7) and (5.8) we may choose $\theta_\lambda = C_4\lambda^{-1}\exp(\mu R_0)$ with C_4 independent of λ. Since $(-\Delta + \mu^2)v_\lambda \le 0$ by (5.10) and since $v_\lambda(x) \to 0$ as $|x| \to \infty$ it follows from the maximum principle ([7]), Theorem 8.1) that $v_\lambda(x) \le 0$ for $x \in B^c_{R_0}$, all $\lambda \ge \Lambda$. Thus we have

$$u_\lambda^2(x) \le C_4\lambda^{-1}\exp(\mu R_0)w_\mu(x) \quad \text{for } x \in B^c_{R_0},\ \lambda \ge \Lambda.$$

Theorem 1.3 follows from this and (5.8). $\square$

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