ON THE MORSE INDICES OF SIGN CHANGING SOLUTIONS OF NONLINEAR ELLIPTIC PROBLEMS

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a Lipschitz bounded domain and $f\colon \mathbb{R} \to \mathbb{R}$ be of class $\mathcal{C}^1$ with $f(0) = 0$. The goal of this paper is to obtain information on the Morse index and the sign of solutions of the Dirichlet problem

$$\text{(D)} \qquad \begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

This additional information will also yield new existence results for multiple solutions of (D).

We first recall a classical result of Courant [10] which states that the number $n(e)$ of nodal sets of an eigenfunction e of the Laplacian in $\mathcal{C}_0^\infty(\Omega)$

*Research supported by CNSF.
†Research supported by NATO grant CRG 970179 and DFG grant Gi 30/60-1.

is bounded above by the Morse index $\mu(e)$ of e. The inequality $n(u) \leq \mu(u)$ also holds for solutions u of (D) provided $f(0) = 0$ and $f'(t) > f(t)/t$ for $t \neq 0$; cf. [3].

In Theorem 2.1 we give conditions on f which guarantee the existence of a sign changing solution u_1 of mountain pass type in the sense of [15]. In particular, for this solution we have $n(u_1) \geq 2$ and $\mu(u_1) \leq 1$. This clearly shows that one cannot expect a simple relation as above between the Morse index and the number of nodal sets for solutions of nonlinear elliptic problems.

In Theorem 2.2 we assume $f'(t) > f(t)/t$ for $t \neq 0$ as well as certain growth conditions. Here we obtain a solution which has precisely two nodal sets. The solution may be degenerate, however, we can show that its Morse index is 2 and that it looks homologically like a nondegenerate solution. This last statement means that it has the same critical groups as a nondegenerate solution with Morse index 2. This information is used in Corollary 2.4 in order to obtain the existence of a second sign changing solution for a quite general nonlinearity.

The proofs are based on a variational approach to (D). In order to prove that the solutions change sign the maximum principle is used in an essential way. It is worthwile to mention that we make no assumptions at all on the set of positive or negative solutions of (D). There may be infinitely many, they may be degenerate, they need not be isolated and they may even accumulate at 0. We also want to mention that our results hold for more general elliptic operators than the Laplacian as long as they are in divergence form and the strong maximum principle applies. Moreover, the nonlinearity f may also depend on x. An extension to non-autonomous elliptic systems will be sketched in Remark 2.5

The existence of sign changing solutions under stronger assumptions than those considered here has been proved in an earlier paper [2]. Related existence results are due to Castro, Cossio and Neuberger [5], [6], and Dancer and Du [11], [12]. The information on Morse indices and critical groups which we prove here is new. In [2] we obtained certain order relations between the sign changing solution u_1 and other solutions of (D). Simple examples show that the existence arguments used in [2] may not yield the same solutions as the ones we obtain here.

The results on solutions of (D) are stated in section 2. In sections 3 and 4 we develop some abstract critical point theorems for functionals on partially ordered Hilbert spaces. Finally, in section 5 we prove the results about (D).

2 Statement of results

Let $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \dots$ be the eigenvalues of

$$\text{(L)} \qquad \begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We first assume the following hypotheses on f.

(f_1) $f \in \mathcal{C}^1(\mathbb{R}), \quad f(0) = 0.$

(f_2) $\limsup\limits_{|t|\to\infty} f(t)/t < \lambda_1$

(f_3) $f'(t) \to \omega \in \mathbb{R}$ as $|t| \to \infty$. If $\omega = \lambda_k$ with eigenspace $V = \{u \in \mathcal{C}_0^\infty(\Omega) : -\Delta u = \omega u\}$ then

$$\int\limits_\Omega \Big(F(u) - \frac{\omega}{2}u^2\Big)\, dx \to \infty \quad \text{for } u \in V,\ \|u\| \to \infty$$

or

$$\int\limits_\Omega \Big(F(u) - \frac{\omega}{2}u^2\Big)\, dx \to -\infty \quad \text{for } u \in V,\ \|u\| \to \infty.$$

Theorem 2.1. *Suppose (f_1), (f_2) or (f_1), (f_3) with $\omega < \lambda_2$ hold, and that $f'(0) > \lambda_2$. Then (D) has a sign changing solution. If all sign changing solutions with negative energy are isolated then (D) has a solution which changes sign, has Morse index at most 1 and is of mountain pass type.*

The solution will be obtained as a critical point of the energy functional

$$\Phi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - \int_\Omega F(u)\, dx$$

where $F(t) = \int_0^t f(s)\, ds$ is the primitive of f. The notion "mountain pass type" refers to the functional Φ. It will be recalled in section 3 below and is due to Hofer [15]. Observe that we allow a degeneracy at 0 and at infinity: $f'(0) = \lambda_k$ for some $k \ge 3$ is as well possible as $f'(t) \to \lambda_1$ as $|t| \to \infty$.

In our next theorem we treat a complementary case. Here we assume:

(f_4) There exist $p < (N+2)/(N-2)$ and $a_1 > 0$ such that

$$|f(t)| \le a_1(1 + |t|^p) \qquad \text{for all } t \in \mathbb{R}.$$

(f_5) There exist constants $R > 0$, $\theta > 2$ such that

$$0 < \theta F(t) \leq t\ f(t) \quad \text{for } |t| \geq R.$$

(f_6) $f'(t) > f(t)/t$ for all $t \neq 0$.

Theorem 2.2. *Suppose (f_1) and (f_3) with $\omega > \lambda_2$ hold or (f_1), (f_4) and (f_5) hold. Suppose moreover that $f'(0) < \lambda_2$ and that the sign changing solutions are isolated. Then (D) has a sign changing solution u_1 which is neither a local minimum nor of mountain pass type. If in addition (f_6) holds then u_1 has precisely two nodal sets, it has Morse index 2 and its critical groups are like those of a nondegenerate critical point of Morse index 2.*

Recall that the critical groups of an isolated critical point $u_0 \in X$ of Φ with $\Phi(u_0) = c$ are defined by

$$C_k(\Phi, u_0) := H_k(\Phi^c, \Phi^c \setminus \{u_0\}) \qquad \text{for } k \in \mathbb{Z}.$$

Here H denotes singular homology with coefficients in a field $\mathbb{F}$. The coefficients do not play a role in our arguments and $\mathbb{F}$ could be replaced by a module over a ring.

Theorem 2.3. *Suppose (f_1) holds and there exists $k \in \mathbb{N}_0$ with $\lambda_k < f'(0) < \lambda_{k+1}$; here $\lambda_0 := -\infty$. Suppose in addition that (f_3), or (f_4), (f_5) holds. In the case of (f_3) we also assume that there exists $\lambda_l \neq \lambda_k$, $l \geq 1$, with $\lambda_l < \omega < \lambda_{l+1}$. Finally we assume that the sign changing solutions are isolated.*

a) If $k \geq 1$ then there exists a sign changing solution u_1 of (D) satisfying either $\Phi(u_1) > 0$ and $C_{k+1}(\Phi, u_1) \neq 0$ or $\Phi(u_1) < 0$ and $C_{k-1}(\Phi, u_1) \neq 0$.

b) If (f_3) applies with $l \geq 2$ then (D) has a sign changing solution u_1 with $C_l(\Phi, u_1) \neq 0$.

The nondegeneracy assumptions in the Theorem 2.3 can be removed using arguments as in [7] or [1]. Information on Morse indices of sign changing solutions can be used to derive new multiplicity results. Clearly one can combine theorems 2.1 or 2.2 with 2.3 in order to obtain the existence of two sign changing solutions in certain situations. We state one such result.

Corollary 2.4. *Suppose (f_1), (f_3) and (f_6) hold with $\omega > \lambda_3$. Then (D) has two sign changing solutions provided $f'(0) < \lambda_2$. If $f'(0) < \lambda_1$ then (D) has in addition a positive and a negative solution.*

The corollary follows immediately from Theorems 2.2 and 2.3 if $f'(0) \neq \lambda_1$. Using in addition degree theory one can obtain more refined results. In the resonant case at 0 the proof of Theorem 2.3b) works also; cf. Theorem 4.4. A similar result for an ordinary differential equation can be found in [19]. Further multiplicity results can be found in [8], [9]; see also Remark 5.1.

We conclude this section by sketching an extension of our results to elliptic systems. In fact, even for equations the results sketched below are in parts more general than those stated so far.

Remark 2.5. *We fix an integer* $p \geq 1$ *and a map* $F\colon \overline{\Omega} \times \mathbb{R}^p \to \mathbb{R}$ *of class* $\mathcal{C}^2$. *Setting* $f(x,\xi) := \nabla_\xi F(x,\xi)$ *we study the system*

$$\begin{cases} -\Delta U(x) &=& f(x,U(x)) & \text{for } x \in \Omega; \\ U(x) &=& 0 & \text{for } x \in \partial\Omega. \end{cases}$$

Our basic assumption is that the system is cooperative, that is,

$$\partial^2_{\xi_i\xi_j} F(x,\xi) \geq 0 \quad \text{for } i \neq j,\ (x,\xi) \in \overline{\Omega} \times \mathbb{R}^p$$

and

$$\partial^2_{\xi_i\xi_j} F(x(\xi),\xi) > 0 \quad \text{for all } i \neq j,\ \text{all } \xi \in \mathbb{R}^p,\ \text{and some } x(\xi) \in \overline{\Omega}.$$

We also assume that

$$f(x,0) = 0 \quad \text{for all } x \in \Omega.$$

The conditions on $f'(0)$ *in Theorems 2.1 – 2.3 can be replaced by setting* $M_0(x) := D_\xi f(x,0)$ *and by using the positive eigenvalues* $0 < \lambda^0_1 < \lambda^0_2 \leq \dots$ *of the problem*

$$\begin{cases} \Delta U(x) &=& \lambda M_0(x) U(x) & \text{for } x \in \Omega; \\ U(x) &=& 0 & \text{for } x \in \partial\Omega; \end{cases}$$

cf. [8]. Then the condition $f'(0) > \lambda_k$, *or* $f'(0) < \lambda_k$, *corresponds to* $\lambda^0_k < 1$, *or* $\lambda^0_k > 1$, *respectively.*

Next we formulate conditions at infinity. Suppose there exists a matrix function $M_\infty(x)$ *such that* $|f(x,\xi) - M_\infty(x)\xi| = o(|\xi|)$ *for* $|\xi| \to \infty$. *Let* $0 < \lambda^\infty_1 < \lambda^\infty_2 \leq \dots$ *be the positive eigenvalues of*

$$\begin{cases} -\Delta U(x) &=& \lambda M_\infty(x) U(x) & \text{for } x \in \Omega; \\ U(x) &=& 0 & \text{for } x \in \partial\Omega. \end{cases}$$

We need to assume that

- *$M_\infty(x)$ is weakly cooperative, that is, $M_\infty(x) = (m_{ij}(x))_{i,j=1,\dots,p}$ satisfies $m_{ij}(x) \geq 0$ for all $i \neq j$, $x \in \Omega$;*
- *$M_\infty(x)$ is fully coupled, so the index set $\{1, \dots, p\}$ is not the disjoint union of two nonempty subsets I, J such that $m_{ij}(x) = 0$ for $i \in I$, $j \in J$, $x \in \Omega$;*
- *$m_{ii}(x) > 0$ for some $i \in \{1, \dots, p\}$, some $x \in \Omega$.*

The first two of these assumptions are automatically satisfied if $D_\xi f(x, \xi) \to M_\infty(x)$ for $|\xi| \to \infty$, since we assumed that the system is cooperative.

The system is nondegenerate at infinity if $\lambda_k^\infty \neq 1$ for all k. The Morse index at infinity is the largest k such that $\lambda_k^\infty < 1$. In the resonant case $\lambda_k^\infty = 1$ with eigenspace $V = \{U \in \mathcal{C}_0^\infty(\overline{\Omega}, \mathbb{R}^p) : -\Delta U(x) = M_\infty(x)U(x)$ for $x \in \Omega\}$ we need to assume the Landesmann-Lazer condition

$$\int_\Omega \left(F(x, U(x)) - \frac{1}{2}\langle M_\infty(x)U(x), U(x)\rangle \right) dx \to +\infty \text{ for } U \in V, \|U\| \to \infty$$

or

$$\int_\Omega \left(F(x, U(x)) - \frac{1}{2}\langle M_\infty(x)U(x), U(x)\rangle \right) dx \to -\infty \text{ for } U \in V, \|U\| \to \infty.$$

With these modifications the aymptotically linear case goes through as before.

The superlinear behaviour at infinity generalizes in an obvious way. We assume $|f(x, \xi)| \leq a_1(1 + |\xi|^p)$ for some $p < (N+2)/(N-2)$, some $a_1 > 0$, and

$$0 < \theta F(x, \xi) \leq \langle \xi, f(x, \xi)\rangle_{\mathbb{R}^p} \quad \text{for some } \theta > 2 \text{ and all } |\xi| \text{ large.}$$

We leave the details of the generalization to systems to the interested reader. More information on positivity results for cooperative and fully coupled systems can be found in the papers [18] and [8].

3 Abstract critical point theory on partially ordered Hilbert spaces

We fix an infinite-dimensional Hilbert space E and a closed cone $P_E \subset E$: $P_E = \overline{P}_E$ is convex, $\mathbb{R}^+ \cdot P_E \subset P_E$, $P_E \cap (-P_E) = \{0\}$. This induces a partial order on E given by

$$\begin{aligned} u \geq v \quad &:\Leftrightarrow \quad u - v \in P_E, \\ u > v \quad &:\Leftrightarrow \quad u \geq v \quad \text{and} \quad u \neq v. \end{aligned}$$

The elements of P_E are said to be non-negative, those of $-P_E$ non-positive and those of $E \setminus (P_E \cup (-P_E))$ sign changing. A map $f\colon E \to E$ is order preserving if

$$u \geq v \Rightarrow f(u) \geq f(v) \qquad \text{for all } u, v \in E.$$

In our application P_E has empty interior which causes some technical problems. However, there is a densely embedded Banach space $X \subset E$ such that $P := X \cap P_E$ has nonempty interior $\overset{\circ}{P}$ in X. Elements of $\overset{\circ}{P}$ are called positive, those of $-\overset{\circ}{P}$ negative. We assume that there exists an element $e \in \overset{\circ}{P}$ with $\langle u, e \rangle > 0$ for all $u \in P \setminus \{0\}$. We write $u \gg v$ if $u - v \in \overset{\circ}{P}$. A map $f\colon X \to X$ is strongly order preserving if

$$u > v \quad \Rightarrow \quad f(u) \gg f(v) \qquad \text{for all } u, v \in X.$$

Let $\Phi\colon E \to \mathbb{R}$ be a functional satisfying the hypotheses

(Φ_1) $\Phi \in \mathcal{C}^2(E, \mathbb{R})$, $\Phi(0) = 0$, $\Phi'(0) = 0$ and the Palais-Smale condition holds for Φ. Any critical point of Φ lies in X.

(Φ_2) The gradient of Φ is of the form $\nabla\Phi = Id - K_E$ with $K_E\colon E \to E$ a compact nonlinear operator. In addition, $K_E(X) \subset X$ and the restriction $K := K_E|_X\colon X \to X$ is of class $\mathcal{C}^1$ and strongly order preserving.

(Φ_3) For a critical point u_0 of Φ any eigenvector of the Fréchet derivative $DK_E(u_0) \in \mathcal{L}(E)$ lies in X, the largest eigenvalue of $DK_E(u_0)$ is simple and its eigenspace is spanned by a positive eigenvector.

With these assumptions the Morse index μ_0 and the nullity ν_0 of the trivial critical point 0 of Φ are finite. The eigenspace associated to $\sigma(\Phi''(0)) \cap (-\infty, 0) = \sigma(Id - DK_E(0)) \cap (-\infty, 0)$ is a μ_0-dimensional subspace of X and $\ker \Phi''(0) = \ker (Id - DK_E(0))$ is a ν_0-dimensional subspace of X. For our critical point theorems we need one more assumption on the behaviour of Φ at infinity.

(Φ_4) One of the following holds:

   (i) Φ is bounded below.

   (ii) For every $u \in E - \{0\}$ we have $\Phi(tu) \to -\infty$ as $t \to \infty$. There exists $a < 0$ such that $\Phi(u) \leq a$ implies $\Phi'(u)u < 0$.

(iii) There exists a compact self-adjoint linear operator $A_E \in \mathcal{L}(E)$ such that $\nabla\Phi(u) = u - A_E u + o(\|u\|_E)$ as $\|u\|_E \to \infty$. All eigenvectors of A_E lie in X, the largest eigenvalue is simple and its eigenspace is spanned by a positive eigenvector $v_\infty \in \overset{\circ}{P}$ such that $\langle u, v_\infty\rangle > 0$ for every $u \in P \setminus \{0\}$. Moreover, the restriction $A := A_E|_X$ is a bounded linear operator $A \in \mathcal{L}(X)$.

In case (i) from (Φ_4) the functional is necessarily coercive since we assumed the Palais-Smale condition; cf. [4]. In case (ii) Φ is anti-coercive. This case corresponds to superlinear nonlinearities in (D). In the last case (iii) Φ is asymptotically quadratic. We do allow the resonance ker $(Id - A_E) \neq 0$. A typical condition which then implies the Palais-Smale condition is the Landesmann-Lazer condition:

$$(\mathrm{LL}_\pm) \qquad \Phi(u) \to \pm\infty \quad \text{for} \quad u \in \ker(Id - A_E),\ \|u\| \to \infty.$$

Some other related conditions can be found in [16] and [1].

If N is a neighbourhood of an isolated critical point u_0 with $\Phi(u_0) = c$ then by excision we have for the critical groups of u_0:

$$C_k(\Phi, u_0) \cong H_k(\Phi^c \cap N, \Phi^c \cap N \setminus \{u_0\}).$$

Setting $V :=$ ker $\Phi''(u_0)$ the generalized Morse lemma yields a diffeomorphism $h\colon N_0 \to N$ from a neighbourhood N_0 of 0 in E to a neighbourhood N of u_0 in E and a $\mathcal{C}^1$-function $\Psi\colon N_0 \cap V \to \mathbb{R}$ such that

$$\Phi(h(v + w)) = \Phi(u_0) + \frac{1}{2}\langle\Phi''(u_0)w, w\rangle + \Psi(v) \tag{3.1}$$

for $v \in V$, $w \in V^\perp$ with $v + w \in N_0$. We now collect some results which we need for further reference and whose proofs are based on the generalized Morse lemma.

Proposition 3.1. *Let u_0 be an isolated critical point with Morse index $\mu = \mu(\Phi, u_0)$ and nullity $\nu = \nu(\Phi, u_0)$. Then*

$$C_k(\Phi, u_0) \cong C_{k-\mu}(\Psi, 0) \qquad \text{for } k \in \mathbb{Z}$$

where Ψ is the function from the generalized Morse lemma. In particular, $C_k(\Phi, u_0) = 0$ for $k \notin [\mu, \mu + \nu]$. Moreover, $C_k(\Phi, u_0) \cong \delta_{k\mu}\mathbb{F}$ if $\nu = 0$.

This is the shifting lemma of Gromoll-Meyer [13]; proofs can also be found in [7] and [17]. The hypotheses (Φ_1) – (Φ_4) are not needed in the proof of Proposition 3.1. It suffices that $\Phi \in \mathcal{C}^2(E, \mathbb{R})$ and $\Phi''(u_0)$ is a Fredholm operator of index 0. The next result is true for any $\Phi \in \mathcal{C}^2(E, \mathbb{R})$ satisfying the Palais-Smale condition.

Proposition 3.2. *Let u_0 be an isolated critical point of Φ. The following are equivalent:*

(i) u_0 *is a local minimum.*

(ii) $C_k(\Phi, u_0) \cong \delta_{k0}\mathbb{F}$ *for all* $k \in \mathbb{Z}$.

(iii) $C_0(\Phi, u_0) \neq 0$.

Clearly we have (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). The implication (iii) $\Rightarrow$ (i) is Theorem I.4.6 in [7].

Next we recall that a critical point u_0 is said to be of mountain pass type if for any small open neighbourhood N of u_0 the set $\Phi^c \cap N \setminus \{u_0\},\ c = \Phi(u_0)$, is not path-connected; cf. [15].

Proposition 3.3. *Let u_0 be an isolated critical point of Φ. The following are equivalent:*

(i) u_0 *is of mountain pass type.*

(ii) $C_k(\Phi, u_0) \cong \delta_{k1}\mathbb{F}$ *for all* $k \in \mathbb{Z}$.

(iii) $C_1(\Phi, u_0) \neq 0$.

That (ii) and (iii) are equivalent follows from Theorem II.1.6 of [7], using hypothesis (Φ_3). The equivalence of (i) and (ii), (iii) will be proved in the next section. Similar results have been proved and used in [20]. The proof shows that Proposition 3.3 holds provided $\Phi \in \mathcal{C}^2(E, \mathbb{R})$, $\Phi''(u_0)$ is Fredholm of index 0 and $\dim \ker \Phi''(u_0) = 1$ if $0 \in \sigma(\Phi''(u_0))$.

Now we state our critical point theorems. For these (Φ_1) – (Φ_4) are needed. Let μ_0, ν_0 denote the Morse index and the nullity of 0, respectively. We define the Morse index μ_∞ and the nullity ν_∞ of infinity depending on which case applies in (Φ_4). In case (i) we set $\mu_\infty = 0 = \nu_\infty$, in case (ii) we set $\mu_\infty := \infty,\ \nu_\infty = 0$. In case (iii) μ_∞ is the number of negative eigenvalues of $Id - A_E$ (counted with multiplicities) and $\nu_\infty := \dim \ker(Id - A_E)$.

Theorem 3.4. *If $\mu_0 \geq 2$ and $\mu_\infty + \nu_\infty \leq 1$ then Φ has a critical point u_1 which changes sign. If all sign changing critical points with negative critical values are isolated then there exists a sign changing critical point of mountain pass type with negative critical value, hence $C_k(\Phi, u_1) \cong \delta_{k1}\mathbb{F}$.*

Theorem 2.1 will be a consequence of this.

Theorem 3.5. *If $\mu_0 \geq 2$ and there is a pair of sub- and super-solutions $\underline{u} \ll 0 \ll \overline{u}$ for the equation $d\Phi(u) = 0$, then the conclusion of Theorem 3.4 remains true.*

Theorem 3.5 improves Theorem V.2 in [9]. The next result implies Theorem 2.2.

Theorem 3.6. *If $\mu_\infty \geq 2$ and $\mu_0 + \nu_0 \leq 1$ then Φ has a critical point u_1 which changes sign and $\Phi(u_1) > 0$. If all sign changing solutions are isolated then there exists a sign changing solution u_1 with $\Phi(u_1) > 0$, Morse index $\mu \in \{1, 2\}$ and $C_0(\Phi, u_1) = 0 = C_1(\Phi, u_1)$, $C_2(\Phi, u_1) \neq 0$. In particular, u_1 is neither a local minimum nor of mountain pass type. If $\mu = 2$ or the nullity $\nu \leq 1$ then $C_k(\Phi, u_1) \cong \delta_{k2}\mathbb{F}$ holds for all k and hence u_2 looks homologically like a nondegenerate critical point of Morse index 2.*

Remark 3.7. *a) In the case $\mu = 2$ one can perturb Φ arbitrarily small in the $\mathcal{C}^2$-norm and only in a neigbourhood of u_1 so that the perturbed functional $\widetilde{\Phi}$ has the same critical points as Φ with u_1 being a non-degenerate critical point of $\widetilde{\Phi}$. The sublevel sets of Φ and of $\widetilde{\Phi}$ are diffeomorphic so that u_1 cannot be distinguished from a nondegenerate critical point.*

b) If $\mu = 1$ then $\nu \geq 1$. In the case $\nu = 1$ the same remarks as in a) apply. If $\nu > 1$ then the geometry of the sublevel sets may be more complicated and the rank of $C_2(\Phi, u_1)$ may be bigger than 1. If $\nu = 2$ one can still prove that $C_k(\Phi, u_1) = 0$ for $k \neq 2$. If $\nu \geq 3$ then even this last statement need not hold. However, if rank $C_k(\Phi, u_1) > \delta_{k2}$ for some $k \geq 2$ then we can show that a second sign changing critical point exists.

c) The hypothesis that all sign changing critical points should be isolated can be weakened. In the proof we produce real numbers $\beta > \alpha > 0$ such that $\Phi(u_1) \in [\alpha, \beta]$. It suffices to assume that all sign changing critical points with values in $[\alpha, \beta]$ are isolated.

Finally we state a critical point theorem which complements Theorems 3.4 and 3.6 and which we need in the proof of Theorem 2.3. For simplicity we only treat the nondegenerate case. An extension to a degenerate situation will be stated at the end of section 4 in Theorem 4.4.

Theorem 3.8. *Suppose $\nu_0 = 0 = \nu_\infty$, $\mu_\infty \geq 1$ and $\mu_0 \neq \mu_\infty$. Suppose moreover that all sign changing critical points are isolated.*

a) If $\mu_0 \geq 1$ then Φ has a critical point u_1 which changes sign and satisfies either $\Phi(u_1) > 0$ and $C_{\mu_0+1}(\Phi, u_1) \neq 0$ or $\Phi(u_1) < 0$ and $C_{\mu_0-1}(\Phi, u_1) \neq 0$.

b) If (Φ_4) (iii) applies with $\mu_\infty \geq 2$ then Φ has a sign changing critical point u_1 with $C_{\mu_\infty}(\Phi, u_1) \neq 0$.

4 Proof of the abstract results

Proof of 3.3. We first show that *(i)* implies *(ii)*. Suppose u_0 is an isolated critical point of mountain pass type at level c. We set $V := \ker \Phi''(u_0)$ and $W := V^\perp = W^+ \oplus W^-$ where $W^\pm$ is the generalized eigenspace of $\Phi''(u_0)$ corresponding to $\sigma(\Phi''(u_0)) \cap \mathbb{R}^\pm$, respectively. Let $\mu := \dim W^-$ be the Morse index and $\nu := \dim V$ be the nullity of u_0. We first prove $\mu \le 1$. The generalized Morse lemma yields a diffeomorphism $h\colon N_0 \to N$ from a neighbourhood N_0 of 0 in E to neighbourhood N of u_0 in E and a $\mathcal{C}^1$-function $\Psi\colon N_0 \cap V \to \mathbb{R}$ such that

$$\Phi(h(v+w)) = \Phi(u_0) + \frac{1}{2}\langle \Phi''(u_0)w, w\rangle + \Psi(v)$$

for $v + w \in N_0$, $v \in V$, $w \in W$. Thus

$$\begin{aligned} \Phi^c \cap N &= \{h(v+w) : \langle \Phi''(u_0)w, w\rangle \le -2\Psi(v)\} \\ &\cong \{v + w \in N_0 : \langle \Phi''(u_0)w, w\rangle \le -2\Psi(v)\}. \end{aligned}$$

We choose $\varepsilon, \delta > 0$ with $B_\varepsilon V \times B_\delta W \subset N_0$ and such that

$$(4.1) \qquad |\langle \Phi''(u_0)w, w\rangle| > 2 \sup_{v \in B_\varepsilon V} |\Psi(v)| \quad \text{for} \quad w \in W^-, \|w\| = \delta.$$

Given $v+w \in B_\varepsilon V \times B_\delta W$ with $\Phi(h(v+w)) \le c$ and $v+w \ne 0$ we connect it along the path $p(t) := v + w^- + (1-t)w^+$ to $p(1) = v + w^- \in B_\varepsilon V \times B_\delta W^-$. Observe that $\Phi(h(p(t))) \le c$ for all $t \in [0,1]$ and $p(t) \in N_0 \setminus \{0\}$ for all $t \in [0,1]$. Assume now $\dim W^- > 1$. We shall show that the set $A = \Phi^c \cap h(B_\varepsilon V \times B_\delta W) \setminus \{u_0\}$ is path connected. If $w^- = 0$ we choose $w_1^- \in W^-$, $\|w_1^-\| = \delta$, arbitrarily. If $w^- \ne 0$ we set $w_1^- := \delta w^-/\|w^-\|$. Now $p(1)$ is connected to $p(2) := v + w_1^-$ along the path $p(t) := v + (2-t)w^- + (t-1)w_1^-$ for $t \in [1,2]$. Again $\Phi(h(p(t))) \le c$ and $p(t) \in N_0 \setminus \{0\}$ for all $t \in [1,2]$. Setting $p(t) := (3-t)v + w_1^-$ for $t \in [2,3]$ we connect $p(2)$ to $p(3) = w_1^-$. Clearly $p(t) \in N_0 \setminus \{0\}$ and by (4.1) we have $\Phi(h(p(t))) \le c$ for $t \in [2,3]$. This shows that every $u \in A$ can be connected by a path within A to a point in $h(\{0\} \times \partial B_\delta W^-) \subset A$. Hence, A is path-connected if $\dim W^- > 1$. However, since u_0 is a mountain pass point A is not path-connected. This contradiction implies $\mu = \dim W^- \le 1$.
Now we prove (i) implies (ii) in Proposition 3.3. In case $\mu = 0$ hypothesis (Φ_3) implies $\nu = 1$, hence $V \cong \mathbb{R}$. Since u_0 is isolated Ψ does not change sign on $(-\varepsilon, 0)$ nor on $(0, \varepsilon)$. From $\Phi^c \cap h(B_\varepsilon V \times B_\delta W) \setminus \{u_0\}$ not being connected it follows that $\Psi(v) < 0$ for $0 < |v| < \varepsilon$. This implies $C_k(\Phi, u_0) \cong C_k(\Psi, 0) \cong \delta_{k1}\mathbb{F}$ as required. In case $\mu = 1$ we have $C_k(\Phi, u_0) \cong C_{k-1}(\Psi, 0)$.

If $C_0(\Psi,0)\neq 0$ then Proposition 3.2 yields $C_k(\Phi,u_0)\cong C_{k-1}(\Psi,0)\cong \delta_{k1}\mathbb{F}$, and we are done. Suppose now $C_0(\Psi,0)=0$. Arguing as above we find $\varepsilon,\delta>0$ with $B_\varepsilon V\times B_\delta W\subset N_0$ and such that every point $u\in A=\Phi^c\cap h(B_\varepsilon V\times B_\delta W)\setminus\{u_0\}$ can be connected within A to a point $h(w_1^-)$ with $w_1^-\in W^-$, $\|w_1^-\|=\delta$. Now $C_0(\Psi,0)=0$ implies that there exists a path $p\colon[0,1]\to\Psi^0\cap B_\varepsilon V$ with $p(0)=0$ and $p(1)\neq 0$. Thus the path $h(p(t)+w_1^-)$ deforms $h(w_1^-)$ to $h(p(1)+w_1^-)$ within A. Using the path $h(p(1)+(1-t)w_1^-)$ we connect $h(p(1)+w_1^-)$ to $h(p(1))$ within A because $\Psi(p(1))\leq 0$. Thus every point in A can be connected within A to $h(p(1))$, so A is path connected. This is a contradiction to u_0 being of mountain pass type. We have proved (i) implies (ii) in 3.3.

Next we show (ii) implies (i). Since $C_1(\Phi,u_0)\neq 0$ the shifting lemma implies $\mu\leq 1$. If $\mu=0$ then $\nu=1$ by (Φ_3). Using the generalized Morse Lemma again we first see that $C_1(\Psi,0)=C_1(\Phi,u_0)\neq 0$ implies $\Psi(v)<0$ for $0<|v|\leq\varepsilon$, $\varepsilon>0$ small. It follows immediately that

$$\begin{aligned} h(B_\varepsilon V\times\{0\})\setminus\{u_0\} &\subset \Phi^c\cap h(B_\varepsilon V\times B_\delta W)\setminus\{u_0\}\\ &\subset h((B_\varepsilon V\setminus\{0\})\times B_\delta W)\end{aligned}$$

Hence, if $U\subset h(B_\varepsilon V\times B_\delta W)$ is a neighbourhood of u_0 then $\Phi^c\cap U\setminus\{u_0\}$ cannot be connected.

In case $\mu=1$ it follows from $C_0(\Psi,0)\cong C_1(\Phi,u_0)\neq 0$ that 0 is a strict local minimum of Ψ. The generalized Morse lemma then implies

$$\begin{aligned} h(\{0\}\times B_\delta W^-)\setminus\{u_0\} &\subset \Phi^c\cap h(B_\varepsilon V\times B_\delta W)\setminus\{u_0\}\\ &\subset \Phi^c\cap h(B_\varepsilon V\times(B_\delta W\setminus B_\delta W^+))\end{aligned}$$

and we conclude as above. □

As in [2] we restrict the negative gradient flow of Φ on E to X and obtain a continuous flow φ^t on X. By construction $\Phi(\varphi^t(u))$ is strictly decreasing in t if u is not a critical point of Φ. Setting $D:=P=\cup(-P)$ and $\dot D:=D\setminus\{0\}$ we have by Lemma 3.1 of [BW]

$$\varphi^t(v)\in\mathring{D}=\mathring{P}\cup(-\mathring{P})\qquad\text{for all } v\in\dot D,\ t>0. \tag{4.2}$$

This is the main order related property of φ^t which we need.

Proof of 3.4. For $\gamma\in\mathbb{R}$ let $j_\gamma\colon H_1(X,\dot D)\to H_1(X,\Phi^\gamma\cup\dot D)$ be the homomorphism induced by the inclusion $(X,\dot D)\hookrightarrow(X,\Phi^\gamma\cup\dot D)$ on the first homology level. Since X is contractible and $\dot D$ homotopy equivalent to a two point space we get $H_1(X,\dot D)\cong\mathbb{F}$. Now consider $\Gamma:=\{\gamma\in\mathbb{R}:j_\gamma\neq 0\}$ and $c:=\sup\Gamma$.

Lemma 4.1. $\Gamma \neq \emptyset$

Proof of 4.1. Since $\mu_\infty + \nu_\infty \le 1$ we have that (Φ_4) (i) or (iii) applies. In case (i) we set $v_\infty := e \in \overset{\circ}{P}$, in case (iii) we take $v_\infty \in \overset{\circ}{P}$ as the positive eigenvector of $Id - A$ with $\|v_\infty\| = 1$ belonging to the largest eigenvalue of A. With $X_1 := X \cap \langle v_\infty \rangle^\perp$ we see that Φ is bounded below on X_1. For $\gamma < \inf \Phi(X_1)$ we have the inclusions

$$(X, \dot{D}) \overset{j}{\hookrightarrow} (X, \Phi^\gamma \cup \dot{D}) \overset{i}{\hookrightarrow} (X, X \setminus X_1).$$

Clearly, $j \circ i$ is a homotopy equivalence which implies $\gamma \in \Gamma$. □

Lemma 4.2. $c < 0$

Proof of 4.2. Since $\mu_0 \ge 2$ the two smallest eigenvalues $\lambda_1 < \lambda_2$ of $\Phi''(0)$ are negative. Let e_1 and e_2 be normalized eigenvectors of $\Phi''(0)$ belonging to λ_1 and λ_2, respectively. By (Φ_3) we have $e_1 \in \overset{\circ}{D}$. Let S_ρ be the sphere of radius ρ in span $\{e_1, e_2\}$. For $\rho > 0$ small we have $\max \Phi(S_\rho) < 0$. We claim that $\gamma \notin \Gamma$ for every $\gamma \ge \max \Phi(S_\rho)$, every $\rho > 0$. If $\gamma \ge \max \Phi(S_\rho)$ we have $(X, S_\rho \cup \dot{D}) \subset (X, \Phi^\gamma \cup \dot{D})$. Therefore it suffices to show that $H_1(X, S_\rho \cup \dot{D}) \cong 0$. This is an immediate consequence of the long exact sequence of the pair $(X, S_\rho \cup \dot{D})$ and the facts that $H_1(X) = 0$ and $H_0(S_\rho \cup \dot{D}) \to H_0(X)$ is an isomorphism since $\dot{D} \cup S_\rho$ is path connected. □

Now a standard deformation argument yields that c is a critical value. Suppose there are only finitely many sign changing critical points $u_1, \dots, u_r$ of Φ at the level c. Then we choose $\delta > 0$ and neighbourhoods $N_i \subset X \setminus D$ of u_i with the following properties:

(i) $N_i \cap N_j = \emptyset$ for $i \neq j$;

(ii) u_i is the only critical point of Φ in N_i;

(iii) $\Phi^{c-\delta} \cup N_i$ is positively invariant under φ^t;

(iv) there exists $T \ge 0$ with $\varphi^T(\Phi^{c+\delta}) \subset \Phi^{c-\delta} \cup N_1 \cup \dots \cup N_r$.

Since $c + \delta \notin \Gamma$ and $c - \delta \in \Gamma$ the long exact sequence of the triple $(X, \Phi^{c+\delta} \cup \dot{D}, \Phi^{c-\delta} \cup \dot{D})$ yields $H_1(\Phi^{c+\delta} \cup \dot{D}, \Phi^{c-\delta} \cup \dot{D}) \neq 0$:

$$\begin{array}{ccccc} & & H_1(X, \dot{D}) & & \\ & & \downarrow \neq 0 & \searrow 0 & \\ H_1(\Phi^{c+\delta} \cup \dot{D}, \Phi^{c-\delta} \cup \dot{D}) & \longrightarrow & H_1(X, \Phi^{c-\delta} \cup \dot{D}) & \longrightarrow & H_1(X, \Phi^{c+\delta} \cup \dot{D}) \end{array}$$

Setting $N := N_1 \cup \cdots \cup N_r$ properties (iii) and (iv) imply that $\Phi^{c-\delta} \cup N \cup \dot{D}$ is a strong deformation retract of $\Phi^{c+\delta} \cup \dot{D}$, hence

$$H_1(\Phi^{c-\delta} \cup N \cup \dot{D}, \Phi^{c-\delta} \cup \dot{D}) \cong H_1(\Phi^{c+\delta} \cup \dot{D}, \Phi^{c-\delta} \cup \dot{D}).$$

The excision property of homology implies

$$H_1(N, N \cap \Phi^{c-\delta}) \cong H_1(\Phi^{c-\delta} \cup N, \Phi^{c-\delta}) \cong H_1(\Phi^{c-\delta} \cup N \cup \dot{D}, \Phi^{c-\delta} \cup \dot{D}).$$

Now properties (i) and (ii) yield

$$H_1(N, N \cap \Phi^{c-\delta}) \cong \bigoplus_{i=1}^{r} H_1(N_i, N_i \cap \Phi^{c-\delta}) \cong \bigoplus_{i=1}^{r} C_1(\Phi, u_i).$$

It follows that there exists $i \in \{1, \dots, r\}$ with $C_1(\Phi, u_i) \neq 0$. By Proposition 3.3 u_i is of mountain pass type and $C_k(\Phi, u_i) \cong \delta_{k1}\mathbb{F}$. $\square$

Proof of 3.5. The proof proceeds as the one of 3.4. We simply replace X and $\dot{D}$ by the order interval $[\underline{u}, \overline{u}]$ and $[\underline{u}, \overline{u}] \cap \dot{D}$, respectively. $\square$

Proof of 3.6. Since $\mu_\infty \geq 2$ either (Φ_4) (ii) or (iii) applies. In both cases there exists $v_\infty \in \mathring{P}$ and $w_\infty \perp v_\infty$ with $\|v_\infty\|$, $\|w_\infty\| = 1$ and such that $\Phi(u) < 0$ for any $u \in$ span $\{v_\infty, w_\infty\}$ with $\|u\| \geq R$, where $R > 0$ is large. Now we set

$$B := \{sv_\infty + tw_\infty : |s| \leq R, \quad 0 \leq t \leq R\}$$

and

$$\partial B := \{sv_\infty + tw_\infty \in B : |s| = R \text{ or } t \in \{0, R\}\}.$$

By construction we have $\partial B \subset \Phi^0 \cup D$. Let $\beta := \max \Phi(B)$ so that $(B, \partial B) \hookrightarrow (\Phi^\beta \cup D, \Phi^0 \cup D)$. Let $\xi_\beta \in H_2(\Phi^\beta \cup D, \Phi^0 \cup D)$ be the image of $1 \in \mathbb{F} \cong H_2(B, \partial B)$ under the homomorphism

$$\mathbb{F} \cong H_2(B, \partial B) \to H_2(\Phi^\beta \cup D, \Phi^0 \cup D)$$

induced by the inclusion. For $\gamma \leq \beta$ let

$$j_\gamma \colon H_2(\Phi^\gamma \cup D, \Phi^0 \cup D) \to H_2(\Phi^\beta \cup D, \Phi^0 \cup D)$$

be also induced by the inclusion. Now we define

$$\Gamma := \{\gamma \leq \beta : \xi_\beta \in \text{ image } (j_\gamma)\}$$

and $c := \inf \Gamma$.

Lemma 4.3. $\xi_\beta \neq 0$

Proof. Let $e_1 \in \overset{\circ}{P}$ be the first eigenvector of $\Phi''(0)$ and set $X_1 := \text{span }\{e_1\}$, $X_2 := X_1^\perp \cap X$. Since $\mu_0 + \nu_0 \leq 1$ we obtain $\inf \Phi(S_\rho X_2) \geq \alpha > 0$ for some $\rho > 0$ small. This implies

$$(B, \partial B) \subset (\Phi^\beta \cup D, \Phi^0 \cup D) \subset (X, X \setminus S_\rho X_2).$$

Therefore it suffices to show that the homomorphism

$$H_2(B, \partial B) \to H_2(X, X \setminus S_\rho X_2)$$

induced by the inclusion is not 0. Choose $w_0 \in X_2$ with $\|w_0\| = 1$ arbitrarily and set

$$B_0 := \{sv_\infty + tw_0 : |s| \leq R, \quad 0 \leq t \leq R\}.$$

Rotating w_∞ into w_0 we may deform $(B, \partial B)$ into $(B_0, \partial B_0)$ within $(X, X \setminus S_\rho X_2)$. Therefore it suffices to show that $H_2(B_0, \partial B_0) \to H_2(X, X \setminus S_\rho X_2)$ is not 0.

Setting

$$B_1 := \{se_1 + tw_0 : |s| \leq R, \quad 0 \leq t \leq R\}$$

the rotation from v_∞ to e_1 inside $\overset{\circ}{P}$ deforms $(B_0, \partial B_0)$ into $(B_1, \partial B_1)$ within $(X, X \setminus S_\rho X_2)$. Thus we have to show that $H_2(B_1, \partial B_1) \to H_2(X, X \setminus S_\rho X_2)$ is not 0. In order to see this we define

$$C_1 := \{se_1 + tw_0 \in B_1 : |s| \leq R, t = 0, R\} \subset \partial B_1$$

and

$$C_2 := \{se_1 + tw_0 \in B_1 : |s| = R, \ 0 \leq t \leq R\} \subset \partial B_1$$

so that $\partial B_1 = C_1 \cup C_2$ and the inclusion $(C_1, C_1 \cap C_2) \hookrightarrow (\partial B_1, C_2)$ is an excision. Similarly, we define

$$A_1 := X_1 \times (X_2 \setminus S_\rho X_2)$$

and

$$A_2 := (X_1 \setminus \{0\}) \times X_2$$

so that $X \setminus S_\rho X_2 = A_1 \cup A_2$ and the inclusion $(A_1, A_1 \cap A_2) \hookrightarrow (X \setminus S_\rho X_2, A_2)$ is an excision.

Next we consider the commutative diagram

$$\begin{array}{ccccc} H_2(B_1, C_2) & \to & H_2(B_1, \partial B_1) & \xrightarrow{\partial} & H_1(\partial B_1, C_2) \\ & & \downarrow i_1 & & \downarrow i_2 \\ & & H_2(X, X \setminus S_\rho X_2) & \to & H_1(X \setminus S_\rho X_2, A_2) \end{array}$$

The top row is exact as a part from the long exact sequence of the triple $(B_1, \partial B_1, C_2)$. Using $H_2(B_1, C_2) \cong H_1(C_2) = 0$ we see that $i_1 \neq 0$ follows from i_2 being injective. Using the excision isomophisms i_2 is injective if, and only if, $i_3\colon H_1(C_1, C_1 \cap C_2) \to H_1(A_1, A_1 \cap A_2)$ is injective. This is an immediate consequence of the 5-lemma because the inclusions $C_1 \hookrightarrow A_1$ and $C_1 \cap C_2 \hookrightarrow A_1 \cap A_2$ are homotopy equivalences, hence they induce isomorphisms on homology level. Therefore, i_3 is even an isomorphism. □

As a consequence of Lemma 4.3 we have $0 \notin \Gamma$ because $j_0 = 0$. Clearly $\beta \in \Gamma$, hence $c \in [0, \beta]$. Morover, since $\mu_0 + \nu_0 \leq 1$ the sign changing solutions cannot accumulate at 0. Assuming that the sign changing solutions are isolated we see that there can only be finitely many sign changing solutions with values in $[0, \beta]$. This implies that $\Phi^0 \cup D$ is a strong deformation retract of $\Phi^\gamma \cup D$ for $\gamma > 0$ small enough, and consequently $c > 0$. Let $u_1, \ldots, u_r$ be all sign changing critical points at the level $c \in (0, \beta]$. We choose $\delta > 0$ and neighbourhoods N_i of u_i, $i = 1, \ldots, r$, as in the proof of Theorem 3.4.

Consider the commutative diagram

$$\begin{array}{ccccc} H_2(\Phi^{c-\delta} \cup D, \Phi^0 \cup D) & \xrightarrow{j} & H_2(\Phi^{c+\delta} \cup D, \Phi^0 \cup D) & \to & H_2(\Phi^{c+\delta} \cup D, \Phi^{c-\delta} \cup D) \\ j_{c-\delta} \searrow & & \downarrow j_{c+\delta} & & \\ & & H_2(\Phi^\beta \cup D, \Phi^0 \cup D) & & \end{array}$$

Since $c + \delta \in \Gamma$ there exists $\xi_{c+\delta} \in H_2(\Phi^{c+\delta} \cup D, \Phi^0 \cup D)$ with $j_{c+\delta}(\xi_{c+\delta}) = \xi_\beta$. Now $\xi_{c+\delta}$ cannot lie in the image of j because $c - \delta \notin \Gamma$, that is, $\xi_\beta \notin$ image $(j_{c-\delta})$. Therefore the exactness of the top row yields $H_2(\Phi^{c+\delta} \cup D, \Phi^{c-\delta} \cup D) \neq 0$. As in the proof of Theorem 3.4 this implies that $C_2(\Phi, u_i) \neq 0$ for some $i \in \{1, \ldots, r\}$. This is the required sign changing solution which is neither a local minimum nor of mountain pass type.

In case $\mu = 2$ the shifting theorem yields

$$C_k(\Phi, u_i) \cong C_{k-2}(\Psi, 0) \quad \text{for all } k$$

where Ψ is defined in a neighbourhood of 0 in ker $D^2\Phi(u_i)$. Thus $C_0(\Psi, 0) \neq 0$, so 0 is a local minimum of Ψ according to Proposition 3.2 and moreover, $C_k(\Psi, 0) \cong \delta_{k0}\mathbb{F}$. Hence we obtain $C_k(\Phi, u_i) \cong \delta_{k2}\mathbb{F}$ for all k.

Next we suppose $\mu \le 1$ and $\nu = \nu(\Phi, u_i) \le 1$, hence $\mu = 1$ and $\nu = 1$ because $C_2(\Phi, u_i) \ne 0$. Then we have

$$C_k(\Phi, u_i) \cong C_{k-1}(\Psi, 0)$$

again by the shifting theorem. This implies $C_1(\Psi, 0) \ne 0$, hence Ψ is a local maximum of Ψ. From this we deduce $C_k(\Psi, 0) \cong \delta_{k1}\mathbb{F}$, hence $C_k(\Phi, u_i) \cong \delta_{k2}\mathbb{F}$, as claimed. $\square$

Proof of 3.8 a). By theorems 3.4 and 3.6 we only need to consider the case $\mu_0 \ge 2$ and $\mu_\infty \ge 2$. Using excision and the fact that 0 is a nondegenerate critical point we obtain for $\varepsilon > 0$ small

$$H_{\mu_0}(\Phi^\varepsilon \cup \dot{D}, \Phi^{-\varepsilon} \cup \dot{D}) \cong H_{\mu_0}(\Phi^\varepsilon, \Phi^{-\varepsilon}) \ne 0 \tag{4.3}$$

It is not difficult to see (cf. [2] and [9]) that for $\alpha > 0$ large

$$H_*(X, \Phi^{-\alpha} \cup \dot{D}) \cong H_*(X, \Phi^{-\alpha}) \cong \delta_{*\mu_\infty} \cdot \mathbb{F} \tag{4.4}$$

This uses $\mu_\infty \ge 1$ and $\nu_\infty = 0$. We fix such a small ε and such a large α. Consider the following diagram:

$$\begin{array}{ccccc}
H_{\mu_0+1}(X, \Phi^\varepsilon \cup \dot{D}) & \stackrel{\partial}{\to} & H_{\mu_0}(\Phi^\varepsilon \cup \dot{D}, \Phi^{-\alpha} \cup \dot{D}) & \to & H_{\mu_o}(X, \Phi^{-\alpha} \cup \dot{D}) \\
 & & \downarrow j & & \\
 & & H_{\mu_0}(\Phi^\varepsilon \cup \dot{D}, \Phi^{-\varepsilon} \cup \dot{D}) & & \\
 & & \downarrow & & \\
 & & H_{\mu_0-1}(\Phi^{-\varepsilon} \cup \dot{D}, \Phi^{-\alpha} \cup \dot{D}) & &
\end{array}$$

Since the vertical maps are part of the long exact sequence of the triple $(\Phi^\varepsilon \cup \dot{D}, \Phi^{-\varepsilon} \cup \dot{D}, \Phi^{-\alpha} \cup \dot{D})$ it follows from (4.3) that $H_{\mu_0}(\Phi^\varepsilon \cup \dot{D}, \Phi^{-\alpha} \cup \dot{D}) \ne 0$ or $H_{\mu_0-1}(\Phi^{-\varepsilon} \cup \dot{D}, \Phi^{-\alpha} \cup \dot{D}) \ne 0$. In the former case we deduce $H_{\mu_0+1}(X, \Phi^\varepsilon \cup \dot{D}) \ne 0$ because $H_{\mu_0}(X, \Phi^{-\alpha} \cup \dot{D}) = 0$ by (4.4) and because the top row in the diagram is part of the exact sequence of the triple $(X, \Phi^\varepsilon \cup \dot{D}, \Phi^{-\alpha} \cup \dot{D})$.

Case 1: $H_{\mu_0+1}(X, \Phi^\varepsilon \cup \dot{D}) \ne 0$

We choose a nontrivial element $\xi \in H_{\mu_0+1}(X, \Phi^\varepsilon \cup \dot{D})$. For $\gamma \ge \varepsilon$ let

$$j_\gamma \colon H_{\mu_0+1}(X, \Phi^\varepsilon \cup \dot{D}) \to H_{\mu_0+1}(X, \Phi^\gamma \cup \dot{D})$$

be the homomorphism induced by the corresponding inclusion. Now we consider the set

$$\Gamma := \{\gamma \ge \varepsilon : j_\gamma(\xi) \ne 0\}$$

and define $c := \sup\Gamma$. First we observe that $c < \infty$. Namely, if σ is a singular chain representing ξ then the carrier $|\sigma| \subset X$ of σ is compact and contained in Φ^γ for $\gamma \geq \max\Phi(|\sigma|)$. Hence $j_\gamma(\xi) = 0$ for $\gamma \geq \max\Phi(|\sigma|)$ and $c < \max\Phi(|\sigma|) < \infty$. Thus $c \in [\varepsilon, \infty)$ and we may proceed as before in order to obtain a sign changing critical point u_1 of Φ with $\Phi(u_1) = c$ and $C_{\mu_0+1}(\Phi, u_1) \neq 0$.

Case 2: $H_{\mu_0-1}(\Phi^{-\varepsilon} \cup \dot{D}, \Phi^{-\alpha} \cup \dot{D}) \neq 0$
Here we consider

$$\Gamma := \{\gamma \in [-\alpha,\ -\varepsilon] : H_{\mu_0-1}(\Phi^\gamma \cup \dot{D}, \Phi^{-\alpha} \cup \dot{D}) = 0\}$$

and $c := \sup\Gamma \in [-\alpha, -\varepsilon]$. It is easy to see that Φ has a sign changing critical point u_1 on the level c with $C_{\mu_0-1}(\Phi, u_1) \neq 0$. □

Proof of 3.8 b). Consider $\alpha > 0$ as in (4.4) and define

$$\Gamma := \{\gamma \geq -\alpha : H_{\mu_\infty}(\Phi^\gamma \cup \dot{D}, \Phi^{-\alpha} \cup \dot{D}) \neq 0\}$$

In order to see that $\Gamma \neq \emptyset$ choose a nontrivial element $\xi \in H_{\mu_\infty}(X, \Phi^{-\alpha} \cup \dot{D})$ which exists by (4.4). If σ is a singular chain representing ξ and if $\gamma \geq \max\Phi(|\sigma|)$ then ξ comes from $H_{\mu_\infty}(\Phi^\gamma \cup \dot{D}, \Phi^{-\alpha} \cup \dot{D})$. This implies $[\max\Phi(|\sigma|), \infty) \subset \Gamma$. Thus $c := \inf\Gamma \in [-\alpha, \infty)$ is finite. As before, Φ has a critical point u_1 with $C_{\mu_\infty}(\Phi, u_1) \neq 0$ and $u_1 \in X \setminus \dot{D}$. Since $C_{\mu_\infty}(\Phi, 0) = 0$ we have $u_1 \neq 0$, so u_1 changes sign. □

Theorem 3.8 may be extended to the degenerate situation where the nullities ν_0 and ν_∞ may be nontrivial, provided a Landesmann-Lazer condition holds. Let us state just one result in this direction.

Theorem 4.4. *Suppose Φ satisfies (Φ_4) (iii) and (LL)$_-$. If $2 \leq \mu_\infty + \nu_\infty \notin [\mu_0, \mu_0+\nu_0]$ then Φ has a sign changing critical point u_1 with $C_{\mu_\infty+\nu_\infty}(\Phi, u_1) \neq 0$, provided the sign changing critical points are isolated.*

The proof is left to the interested reader; cf. [1] and [7] for related results (without the sign changing conclusion). From Theorem 4.4 one can also deduce Theorem 3.4, provided the Landesmann-Lazer condition holds in the degenerate case in 3.4

5 Proof of the results for (D)

The natural variational setting is to consider

$$\Phi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega F(u)\,dx \quad \text{for } u \in E = H_0^1(\Omega)$$

and to use the norm $\int_\Omega |\nabla u|^2\, dx$. However, under the conditions of the theorems 2.1 and 2.2, Φ may not satisfy the smoothness condition in (Φ_1) and it also may not satisfy the order preserving condition in (Φ_2). In order to overcome these problems we shall modify Φ in a proper way with respect to different problems. Let us start with the proof of Theorem 2.1.

Proof of 2.1. In [14], Proof of Theorem 8, it is shown that in the case (f_2) there exists a $\mathcal{C}^1$-modification $\tilde f$ of f satisfying

— $\tilde f(0) = 0$,

— $\limsup_{|t|\to\infty} \tilde f(t)/t < \lambda_1$,

— $\tilde f'(0) > \lambda_2$,

and in addition

— $|\tilde f'(t)| < a$ for all $t \in \mathbb{R}$ and some $a > 0$.

Moreover, solutions of (D) are precisely the solutions of the modified Dirichlet problem

$$\begin{cases} -\Delta u &= \tilde f(u) \qquad \text{in } \Omega \\ \quad\; u &= 0 \qquad\quad\; \text{on } \partial\Omega. \end{cases}$$

Thus we may assume that $|f'(t)| < a$ in addition to (f_2). If (f_3) holds then $f'(t)$ is bounded anyway. We fix $a > 0$ with $|f'(t)| < a$ for all $t \in \mathbb{R}$.

Now we use the following equivalent norm on E:

$$\|u\|_E^2 := \int_\Omega (|\nabla u|^2 + au^2)\, dx.$$

The ordering on E is given by the closed cone $P_E := \{u \in E : u \geq 0 \text{ a.e. in } \Omega\}$. Let X be the Banach space $\mathcal{C}_0^1(\Omega)$ with the usual norm. It is well known that X is dense in E and that $P := X \cap P_E = \{u \in X : u \geq 0\}$ has nonempty interior $\overset{\circ}{P}$. The element $e \in \overset{\circ}{P}$ such that $\langle u, e\rangle > 0$ for all $u \in P \setminus \{0\}$ is the unique normalized positive eigenfunction of $-\Delta$ on Ω with homogeneous Dirichlet boundary condition.

Clearly we have

$$\Phi(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + au^2)\, dx - \int_\Omega G(u)\, dx \quad \text{for } u \in E$$

where $g(t) = f(t) + at$, $G(t) = \int_0^t g(s)\, ds$. Then it is easy to check that $\Phi \in \mathcal{C}^2(E, \mathbb{R})$. The Palais-Smale condition is satisfied by Φ in case of (f_2)

because then Φ is coercive and $\nabla\Phi$ is a compact perturbation of the identity. If (f_3) holds with $\omega \neq \lambda_k$ the Palais-Smale condition can be deduced similarly (cf. [7]). If $\omega = \lambda_k$, we can also refer to [7] for the Palais-Smale condition because the Landesman-Lazer condition is satisfied. It follows from standard regularity theory that critical points of Φ lie in X. Thus (Φ_1) from section 3 is satisfied. Since $g'(t) > 0$ it follows that Φ satisfies (Φ_2) and (Φ_3). Morover, if (f_2) holds we see that (Φ_4)(i) is satisfied, and if (f_3) holds with $\omega \in (\lambda_1, \lambda_2)$ we see that (Φ_4)(iii) is satisfied. Since $f'(0) > \lambda_2$, the Morse index μ_0 of Φ at 0 is at least 2. In case (f_2) holds, we have $\mu_\infty = 0 = \nu_\infty$. If (f_3) holds, (μ_∞, ν_∞) is $(0,0)$, or $(0,1)$, or $(1,0)$. Thus $\mu_\infty + \nu_\infty \leq 1$ is verified. Applying Theorem 3.4 we obtain Theorem 2.1. □

Proof of 2.2. To prove 2.2 in the superlinear case we also need to modify the nonlinearity f such that we can get Φ to satisfy (Φ_1) and (Φ_2). In the asymptotically linear case, no modification is needed because then $f'(t)$ is bounded. In the following we only give the proof in the superlinear case and leave the simpler verification of the assumptions of Theorem 3.6 for the asymptotically linear case to the reader.

We want to show that Φ has a sign changing critical point u such that $C_0(\Phi, u) = C_1(\Phi, u) = 0$ and $C_2(\Phi, u) \neq 0$. If in addition (f_6) holds, we also want to show

$$C_k(\Phi, u) \cong \delta_{k2}\mathbb{F} \quad \text{for } k \in \mathbb{Z}.$$

First it is easy to see that by (f_5) there exists $C > 0$ such that

$$F(u) \geq C|u|^\mu \quad \text{for } |u| \geq R. \tag{5.1}$$

Next we claim that there exists $A > 0$ and $R_n \to \infty$ such that

$$0 \leq \liminf_{n\to\infty} \frac{f'(R_n)}{|R_n|^{p-1}} \leq \limsup_{n\to\infty} \frac{f'(R_n)}{|R_n|^{p-1}} \leq A. \tag{5.2}$$

If the claim is not true, then either

$$\lim_{R\to\infty} \frac{f'(R)}{|R|^{p-1}} = +\infty \tag{5.3}$$

or

$$\limsup_{R\to\infty} \frac{f'(R)}{|R|^{p-1}} < 0 \tag{5.4}$$

If (5.4) holds we get a contradiction with (5.1). And if (5.3) holds we get a contradiction with (f_4). Similarly, there exists a sequence $r_n \to +\infty$, such that

$$\text{(5.5)} \qquad 0 \le \liminf_{n\to\infty} \frac{f'(-r_n)}{|r_n|^{p-1}} \le \limsup_{n\to\infty} \frac{f'(-r_n)}{|r_n|^{p-1}} \le A.$$

Define

$$f_n(t) = \begin{cases} f(t) & -r_n \le t \le R_n \\ f(R_n) + f'(R_n)(t - R_n) + |t - R_n|^{p-1}(t - R_n) & t \ge R_n \\ f(-r_n) + f'(-r_n)(t + r_n) + |t + r_n|^{p-1}(t + r_n) & t \le -r_n \end{cases}$$

where $1 < p < \frac{N+2}{N-2}$ is from (f_4). Observe that $p \ge \mu$ by (5.1). Because of (5.2) and (5.5) we have

$$\text{(5.6)} \qquad |f_n(t)| \le a_0(1 + |t|^p)$$

and

$$\text{(5.7)} \qquad \mu F_n(t) \le f_n(t)t \quad \text{for } |t| \ge R$$

uniformly satisfied for all n. There also exists $a_n > 0$ such that

$$\text{(5.8)} \qquad |f_n'(t)| \le a_n(1 + |t|^{p-1}) \quad \text{for every } t \in \mathbb{R}$$

and

$$\text{(5.9)} \qquad f_n'(t) > -a_n \quad \text{for every } t \in R.$$

Next, for each n we use an equivalent norm on E given by

$$\|u\|_{E_n}^2 = \int_\Omega (|\nabla u|^2 + a_n u^2)\, dx$$

and we write E_n for the space E with this norm. Now we define $g_n(t) = f_n(t) + a_n t$, $G_n(t) = \int_0^t g_n(s)\, ds$ and

$$\Phi_n(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + a_n u^2)\, dx - \int_\Omega G_n(u)\, dx$$

Then $\Phi_n \in \mathcal{C}^2(E_n, \mathbb{R})$ by (5.8). It is easy to check using (5.6) and (5.7) that Φ_n satisfies the Palais-Smale condition on E_n. Thus (Φ_1) is satisfied. Also it is easy to see that (Φ_2) and (Φ_3) are satisfied. With (5.7) uniformly satisfied for all n, we also get that for each $u \in E \setminus \{0\}$ there exists $T(u) > 0$ such that

$$\Phi_n(tu) < 0 \qquad \text{for} \;\; t > T(u)$$

and there exists $a < 0$ such that $\Phi_n(u) \le a$ implies $\Phi_n'(u)u < 0$. These are all independent of n. Therefore we may apply Theorem 3.6 to Φ_n. Moreover, in checking the proof of Theorem 3.6, we see the critical value c_n is bounded above by

$$\beta_n = \max \Phi_n(B_n)$$

where

$$B_n = \{sv_\infty + tw_\infty : |s| \le T_n, \quad 0 \le t \le T_n\}$$

for some $T_n > 0$ such that

$$\Phi_n(u) < 0 \quad \text{for all} \quad u \in \operatorname{span}\{v_\infty, w_\infty\} \quad \text{with} \quad \|u\|_{E_n} \ge T_n$$

where v_∞, w_∞ are normalized eigenfunctions corresponding to the first and the second eigenvalues. From the analysis above we see that we can choose $T_0 > 0$ independent of n such that

$$\Phi_n(u) < 0 \quad \text{for} \quad u \in \operatorname{span}\{v_\infty, w_\infty\} \quad \text{with} \quad \|u\|_{E_n} \ge T_0.$$

Therefore we may choose a fixed B_0 such that

$$c_n \le \beta_0 = \max \Phi_n(B_0).$$

Thus c_n is uniformly bounded. Next we observe that

$$\begin{aligned}
c_n &= \Phi_n(u_n) - \frac{1}{\mu}\Phi_n'(u_n)u_n \\
&= \left(\frac{1}{2} - \frac{1}{\mu}\right)\int_\Omega (|\nabla u_n|^2 + a_n u_n^2)\, dx - \int_\Omega \left(G_n(u_n) - \frac{1}{\mu} g_n(u_n)u_n\right) dx \\
&= \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|_{E_n}^2 - \int_\Omega \left(F_n(u_n) - \frac{1}{\mu} f_n(u_n)u_n\right) dx \\
&\ge \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|^2 - C
\end{aligned}$$

where $C > 0$ is independent of n because of (5.7). Therefore u_n is bounded in E. Then by (5.6), equation (D), and standard elliptic estimates we get

$$\|u_n\|_{\mathcal{C}_0^1(\Omega)} \le C$$

independent of n. Thus for n large, u_n is a solution of the original problem (D), a sign-changing solution. We fix n large enough and write $u_n = u$. We denote the restriction of Φ and Φ_n to $\mathcal{C}_0^1(\overline{\Omega})$ by $\tilde{\Phi}$ and $\tilde{\Phi}_n$, respectively. By a Theorem from [7] we have for $k = 0, 1$:

$$0 = C_k(\Phi_n, u) = C_k(\tilde{\Phi}_n, u) = C_k(\tilde{\Phi}, u) = C_k(\Phi, u)$$

and

$$0 \neq C_2(\Phi_n, u) = C_2(\Phi, u).$$

If in addition (f_6) is satisfied, we also get

$$C_k(\Phi, u) = C_k(\Phi_n, u) = \delta_{q2}\mathbb{F}$$

because now a sign changing solution has Morse index $\mu \geq 2$. Combining this with the fact $\mu \leq 2$ we conclude $\mu = 2$. □

Proof of 2.3. This is a direct consequence of Theorem 3.8. □

Remark 5.1. *In case (f_2) of Theorem 2.1 we can as usual construct a positive and a negative solution which are both local mimimizers of the functional. Some order relation between these solutions and the sign changing solution is also given in our early paper [2] in this case. If $f'(0)$ is not an eigenvalue, one can get a fourth nontrivial solution using either degree theory or Morse theory. Similarly, in Theorem 2.2 with $f'(0) < \lambda_1$ we can obtain as usual a positive and a negative solution of mountain pass type; cf. [2].*

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