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NONLINEAR SCHRÖDINGER EQUATIONS WITH VANISHING AND DECAYING POTENTIALS

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Abstract. We study the existence and the asymptotic behavior of solutions of (1.1), when V can vanish and decay to zero at infinity.

1. Introduction and main results

This paper deals with NLS with potentials like

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u = K(x)u^p, & x \in \mathbb{R}^n, \\ u \in W^{1,2}(\mathbb{R}^n), & u > 0. \end{cases} \tag{1.1}$$

We are motivated by the recent works [1, 3, 5, 6]. The two former papers deal with the case that V is positive and have an appropriate decay to zero at infinity. The two latter ones, with the case in which V vanishes at some set $\mathcal{Z}$ (referred to in [5] as the case of the critical frequency), but $\liminf_{|x|\to\infty} V(x) > 0$. The main purpose of the present note is to show that, using some ideas of [1, 3] jointly with some arguments of [5, 6], it is possible to extend the results proved in the aforementioned papers to potentials V that can both vanish and decay to zero at infinity.

First of all, we will show that for ε small there exists a ground state solution of (1.1) (semiclassical state), so that the fact that V can vanish does not affect the existence results of [1, 3]. By a ground state we mean a solution which is a mountain pass critical point of the energy functional associated to (1.1).

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Next, we study the concentration of these ground states as $\varepsilon \to 0$, a phenomenon which is important for its implications in quantum mechanics. Loosely speaking, we say that a solution v_ε of (1.1) concentrates at a point x^* if v_ε tends to zero uniformly out of x^*. Concerning concentration, we show that the ground states concentrate on points of the zero set of V. Moreover the behavior of the solutions near these points is similar to the case studied in [5], of V being positive away from zero at infinity. It is not affected by the fact that V decays to zero at infinity, but depends only on the local behaviors of V near the points of concentration where V is zero. However, the decay rates of the solutions at infinity do depend on the decay property of V.

We assume that V and K satisfy

(V) $V \in C(\mathbb{R}^n, \mathbb{R})$, and there exist $R_0, k_1, \alpha > 0$ such that

$$V(x) \geq \frac{k_1}{1+|x|^\alpha}, \qquad |x| \geq R_0;$$

(K) $K \in C(\mathbb{R}^n, \mathbb{R})$, and there exist $k_2, \beta > 0$ such that

$$0 < K(x) \leq \frac{k_2}{1+|x|^\beta}, \qquad x \in \mathbb{R}^n.$$

Let

$$\sigma = \begin{cases} \frac{n+2}{n-2} - \frac{4\beta}{\alpha(n-2)}, & \text{if } 0 < \beta < \alpha \\ 1 & \text{otherwise,} \end{cases} \tag{1.2}$$

and set

$$\mathcal{Z} = \{x \in \mathbb{R}^n : V(x) = 0\}.$$

Let us remark that (V) implies that $\mathcal{Z}$ is bounded. We will be interested in the case that $\mathcal{Z} \neq \emptyset$. Our main existence result is the following one.

Theorem 1. *Suppose that (V) and (K) hold and let $0 < \alpha < 2$, $\beta > 0$ and $\sigma < p < \frac{n+2}{n-2}$. Moreover, assume that $\mathcal{Z} \neq \emptyset$. Then for ε sufficiently small, (1.1) has a ground state $v_\varepsilon \in W^{1,2}(\mathbb{R}^n)$, concentrating at some point $x^* \in \mathcal{Z}$, as $\varepsilon \to 0$. Moreover, there holds*

$$\lim_{\epsilon \to 0} ||v_\varepsilon||_\infty = 0, \qquad \textit{and} \qquad \liminf_{\varepsilon \to 0} \varepsilon^{\frac{-2}{p-1}} \|v_\varepsilon\|_\infty > 0. \tag{1.3}$$

Remarks 2. (i) In [1] it is proved that the growth restriction $\sigma < p < \frac{n+2}{n-2}$ is necessary in order to get a ground state.

(ii) If $\mathcal{Z} = \emptyset$, related existence results can be found in [1, 3]. In the former, any $\varepsilon > 0$ is allowed and $\sigma < p < \frac{n+2}{n-2}$ is assumed. In the latter, it has been proved that for $\varepsilon \ll 1$ (1.1) has a solution (possibly not a ground state) for all $1 < p < \frac{n+2}{n-2}$.

(iii) If $\mathcal{Z} \neq \emptyset$ and $\liminf_{|x|\to\infty} V > 0$, a result similar to the preceding theorem is contained in [5, Theorem 2.1].

(iv) The fact that v_ε concentrates at some point $x^* \in \mathcal{Z}$ agrees with the results of [1], where it is proved that concentration arises at a global minimum of the auxiliary potential $\mathcal{A} := V^{\frac{p+1}{p-1}-\frac{n}{2}} K^{-\frac{2}{p-1}}$. Obviously, in the present case each point in $\mathcal{Z}$ is a global minimum of $\mathcal{A}$, because $\mathcal{A}$ vanishes on $\mathcal{Z}$. But here, in contrast with [1], the ground state does not remain bounded away from zero. Actually, the behavior proved in (1.3) is just the behavior of solutions found in [5]. □

As in [5, 6], one can be more precise about the asymptotic profile of the concentrating solutions, provided one makes some further assumption on the behavior of V near $\mathcal{Z}$. However, here we do not consider all the cases discussed in [5] but we shall focus on the one (which is referred to as finite case in [5]) where V has a polynomial decay to zero near a zero point of V. Without loss of generality we assume $V(0) = 0$. $P(x)$ is said to be of homogenous degree $m > 0$ if $P(\lambda x) = |\lambda|^m P(x)$.

The following theorem shows that also the asymptotic profile of the ground states is quite similar to the one established in [5]. Actually one can prove:

Theorem 3. *Suppose that* (V) *and* (K) *hold and let* $0 < \alpha < 2$, $\beta > 0$ *and* $\sigma < p < \frac{n+2}{n-2}$. *Let* $\mathcal{Z} = \{0\}$ *and suppose that for some* $m > 0$, $V(x) = P_m(x) + Q(x)$ *satisfies* $\lim_{|x|\to 0} |x|^{-m} Q(x) = 0$, *where* P_m *is homogeneous of degree* $m > 0$. *Let* v_ε *be a solution of* (1.1), *localized near* 0, *given in Theorem* 1. *Then for any* $\varepsilon_n \to 0$ *there is a subsequence (denoted still by* ε_n*) such that* $\varepsilon_n^{-\frac{2}{p-1}\frac{m}{m+2}} v_{\varepsilon_n}(\varepsilon_n^{\frac{2}{m+2}} x)$ *converges uniformly to a ground state solution of*

$$-\Delta w + P_m(x) w = K(0) w^p, x \in \mathbb{R}^N. \tag{1.4}$$

Remark 4. The behavior of the solution v_ε found above depends on the fact that the concentration point is a zero of V. If there exists a solution concentrating on a critical point of $\mathcal{A}$ with $V > 0$, its behavior would be like a usual spike that one finds in problems where $\inf_{\mathbb{R}^n} V > 0$. This has been proved in [4] dealing with the radial problem

$$-\varepsilon^2 \Delta u + V(|x|) u = u^p, \quad u \in W^{1,2}(\mathbb{R}^n), \quad u > 0, \tag{1.5}$$

where $p > 1$ and V is radial and satisfies (V). If the weighted potential $M(r) = r^{n-1} V^\ell(r)$, $\ell = \frac{p+1}{p-1} - \frac{1}{2}$, has a minimum or maximum at some

$r^* > R_0$, then it is proved that (1.5) has, for $\varepsilon \ll 1$, a radial solution v_ε which concentrates on the sphere of radius r^*. In such a case, $v_\varepsilon \sim U(\frac{r-r^*}{\varepsilon})$, where U is the positive, radial solution of $-U''+U = U^p$ such that $U'(0) = 0$. This is related to the fact that $V(r^*) > 0$. $\square$

The proofs of Theorems 1 and 3 are carried out in Section 2 and 3, respectively.

2. Proof of Theorem 1

The proof of Theorem 1 is divided into several steps.

A. The functional setting. Let

$$\|u\|_\varepsilon = \int_{\mathbb{R}^n} \left[|\nabla u|^2 + V(\varepsilon x)u^2\right] dx,$$

and let E_ε denote the closure of $C_0^\infty(\mathbb{R}^n)$ with respect to $\|\cdot\|_\varepsilon$. According to the results of [7], E_ε is embedded (respectively, compactly embedded) into the weighted Lebesgue space

$$L_K^{q+1}(\mathbb{R}^n) := \{u \in L^{q+1}(\mathbb{R}^n) : \int_{\mathbb{R}^n} K(\varepsilon x)|u|^{q+1}dx\},$$

provided $0 < \alpha \le 2$, $\beta > 0$ and $\sigma \le q \le \frac{n+2}{n-2}$, respectively $\sigma < q < \frac{n+2}{n-2}$. To be more precise, the results in [7] are proved under the further assumption that $V(x) > 0$ on $\mathbb{R}^n$. These results have been also proved in [1], see in particular Remark 10, and it is easy to check that the arguments carried out in [1] rely only on the behavior of V and K for $|x| \gg 1$, namely on the assumptions (V) and (K).

In particular, one has that

$$\int_{\mathbb{R}^n} K(\varepsilon x)|u|^{p+1}dx < +\infty.$$

If $A \subset \mathbb{R}^n$ we set $A_\varepsilon = \{x \in \mathbb{R}^n : \varepsilon x \in A\}$ and denote by A^δ the δ-neighborhood of A. For simplicity we denote $(A^\delta)_\varepsilon$ as A^δ_ε. Fixed $\delta > 0$ small enough, let us consider the following constrained minimization problem

$$m_\varepsilon = \inf\{\|u\|_\varepsilon^2 : \int_{\mathbb{R}^n} K(\varepsilon x)|u|^{p+1}dx = 1, \quad \int_{\mathbb{R}^n\setminus\mathcal{Z}_\varepsilon^\delta} K(\varepsilon x)|u|^{p+1}dx \le \varepsilon^{\frac{3(p+1)}{p-1}}\}.$$

Remark 5. Above, the choice of the exponent $3(p+1)/(p-1)$ has been made to keep our notation as close as possible to that in [5, 6], where localized solutions concentrating on an isolated component of $\mathcal{Z}$ are given for the case the potential has a positive lower bound at infinity. Although we are dealing here with ground state solutions, with minor changes our arguments can be adapted to obtain these type of localized solutions, see also Remark 8. For ground state solutions, some notations can be simplified, for example, in the double constraints problem, the exponent $3(p+1)/(p-1)$ can be replaced by any $a>0$. □

Since, as pointed out before, the embedding of E_ε into L_K^{p+1} is compact provided $\sigma<q<\frac{n+2}{n-2}$, it follows that m_ε is achieved at some $u_\varepsilon \in E_\varepsilon$. Hence $m_\varepsilon>0$ and there exist $\lambda_\varepsilon,\,\mu_\varepsilon\in\mathbb{R}$ such that

$$-\Delta u_\varepsilon + V(\varepsilon x)u_\varepsilon = \lambda_\varepsilon K(\varepsilon x)u_\varepsilon^p + \mu_\varepsilon \chi_{\mathbb{R}^n\setminus \mathcal{Z}_\varepsilon^{4\delta}} K(\varepsilon x) u_\varepsilon^p, \quad u_\varepsilon>0. \tag{2.1}$$

We want to show that $u_\varepsilon\in W^{1,2}(\mathbb{R}^n)$ and there holds

$$\int_{\mathbb{R}^n\setminus \mathcal{Z}_\varepsilon^{\delta}} K(\varepsilon x)|u|^{p+1}dx < \varepsilon^{\frac{3(p+1)}{p-1}}. \tag{2.2}$$

If this is the case, then $\widetilde{u}_\varepsilon = m_\varepsilon^{\frac{1}{p-1}} u_\varepsilon$ is a solution of

$$-\Delta \widetilde{u}_\varepsilon + V(\varepsilon x)\widetilde{u}_\varepsilon = K(\varepsilon x)\widetilde{u}_\varepsilon^p, \qquad u_\varepsilon>0, \tag{2.3}$$

and $v_\varepsilon(x) := \widetilde{u}_\varepsilon(\varepsilon^{-1}x) = m_\varepsilon^{\frac{1}{p-1}} u_\varepsilon(\varepsilon^{-1}x)$ solves (1.1).

B. Some estimates. We first show

Lemma 6. *There holds: $m_\varepsilon = o(1)$ as $\varepsilon\to 0$.*

Proof. Let $x_0\in\mathcal{Z}$. For all $a>0$ there exists $b>0$ such that for all $|x-x_0|\le b$ one has that $V(x)\le a$. Then

$$\begin{aligned} m_\varepsilon &\le \inf_{u\in C_0^\infty(B_{b/\varepsilon}(x_0))} \frac{\int[|\nabla u|^2 + V(\varepsilon x)u^2]dx}{\left(\int_{\mathbb{R}^n} K(\varepsilon x)|u|^{p+1}dx\right)^{2/(p+1)}} \\ &\le \inf_{u\in C_0^\infty(B_{b/\varepsilon}(x_0))} \frac{\int[|\nabla u|^2 + a u^2]dx}{\left(\int_{\mathbb{R}^n} K(\varepsilon x)|u|^{p+1}dx\right)^{2/(p+1)}} \\ &\le \max_{B_b(x_0)} (K(x))^{-2/(p+1)} \inf_{u\in C_0^\infty(B_{b/\varepsilon}(x_0))} \frac{\int[|\nabla u|^2 + a u^2]dx}{\left(\int_{\mathbb{R}^n} |u|^{p+1}dx\right)^{2/(p+1)}}. \end{aligned}$$

Since a can be taken arbitrarily small, the last infimum tends to zero as $\varepsilon\to 0$ and the lemma follows. □

Next, we turn our attention to (2.1). By arguments similar to [5] one finds that $\mu_\varepsilon\le 0\le\lambda_\varepsilon$. We claim that there is a constant Λ such that

$$\limsup_{\varepsilon\to 0}\lambda_\varepsilon \le \Lambda. \tag{2.4}$$

If not, for $\varepsilon_n \to 0$ one has $\lim_{n\to\infty} \lambda_{\varepsilon_n} = +\infty$. Take a cut-off function φ_n such that

$$\varphi_n(x) \begin{cases} 0 & \text{if } x \notin \mathcal{Z}^{\delta}_{\varepsilon_n} \\ 1 & \text{if } x \in \mathcal{Z}^{\delta/2}_{\varepsilon_n}, \end{cases}$$

$0 \le \varphi_n \le 1$, $|\nabla \varphi_n(x)| \le 2\varepsilon_n/\delta$. Set $u_n = u_{\varepsilon_n}$ and $\lambda_n = \lambda_{\varepsilon_n}$. Multiplying (2.1) by $\varphi_n u_n$ and integrating by parts, we get

$$\begin{aligned} \lambda_n \int_{\mathcal{Z}^{\delta/2}_{\varepsilon_n}} K(\varepsilon_n x) u_n^{p+1} dx &\le \int_{\mathbb{R}^n} \left[|\nabla u_n \cdot \nabla (u_n \varphi_n)| + V(\varepsilon_n x) u_n^2 \varphi_n \right] dx \\ &\le c_1 \int_{\mathbb{R}^n} \left[|\nabla u_n|^2 + |\nabla \varphi_n|^2 |u_n|^2 + V(\varepsilon_n x) u_n^2 \right] dx. \end{aligned}$$

Since $\inf\{V(x) : x \in \mathcal{Z}^{\delta} \setminus \mathcal{Z}^{\delta/2}\} > 0$ and $|\nabla \varphi_n(x)| \le 2\varepsilon_n/\delta$, the above inequality implies, for $n \gg 1$,

$$\lambda_n \int_{\mathcal{Z}^{\delta/2}_{\varepsilon_n}} K(\varepsilon_n x) u_n^{p+1} dx \le c_2 \int_{\mathbb{R}^n} \left[|\nabla u_n|^2 + V(\varepsilon_n x) u_n^2 \right] dx = c_2 m_{\varepsilon_n}.$$

Since $m_{\varepsilon_n} \to 0$ and $\lambda_n \to \infty$, it follows that

$$\int_{\mathcal{Z}^{\delta/2}_{\varepsilon_n}} K(\varepsilon_n x) u_n^{p+1} dx \to 0, \qquad n \to \infty. \tag{2.5}$$

Choose another cut-off function

$$\psi_n(x) \begin{cases} 0 & \text{if } x \in \mathcal{Z}^{\delta/2}_{\varepsilon_n}, \text{ or } x \notin \mathcal{Z}^{5\delta/4}_{\varepsilon_n} \\ 1 & \text{if } x \in \mathcal{Z}^{\delta}_{\varepsilon_n} \setminus \mathcal{Z}^{3\delta/4}_{\varepsilon_n}, \end{cases}$$

such that $0 \le \psi_n \le 1$, $|\nabla \psi_n(x)| \le 4\varepsilon_n/\delta$. Taking $n \gg 1$ such that $\lambda_n \ge 1$, and using arguments similar to the previous ones, we get

$$\begin{aligned} \int_{\mathbb{R}^n} K(\varepsilon_n x) u_n^{p+1} \psi_n dx &\le \lambda_n \int_{\mathbb{R}^n} K(\varepsilon_n x) u_n^{p+1} \psi_n dx \\ &= \int_{\mathbb{R}^n} \left[\nabla u_n \cdot \nabla (u_n \psi_n) + |u_n \psi_n|^2 \right] dx \\ &\le c_1 \int_{\mathbb{R}^n} \left[|\nabla u_n|^2 |\psi_n|^2 + |u_n|^2 |\nabla \psi_n|^2 + |u_n \psi_n|^2 \right] dx \\ &\le c_2 \int_{\mathbb{R}^n} \left[|\nabla u_n|^2 + V(\varepsilon_n x) u_n^2 \right] dx \to 0, \end{aligned}$$

where we have used again that $\inf\{V(x) : x \in \mathcal{Z}^{5\delta/4} \setminus \mathcal{Z}^{\delta/2}\} > 0$. But,

$$\int_{\mathbb{R}^n} K(\varepsilon_n x) \psi_n u_n^{p+1} dx \ge \int_{\mathcal{Z}^{\delta}_{\varepsilon_n} \setminus \mathcal{Z}^{3\delta/4}_{\varepsilon_n}} K(\varepsilon_n x) |u_n|^{p+1} dx.$$

Using (2.11) it follows that

$$\int_{\mathcal{Z}_{\varepsilon_n}^{\delta}\setminus\mathcal{Z}_{\varepsilon_n}^{3\delta/4}} K(\varepsilon_n x)|u_n|^{p+1}dx \to 1, \quad n\to\infty.$$

Hence,

$$\liminf_{n\to\infty}\int_{\mathbb{R}^n} K(\varepsilon_n x)|\psi_n u_n|^{p+1}dx \geq \text{Const.} > 0,$$

a contradiction. This proves that (2.4) holds.

C. Exponential decay.

Lemma 7. *There exist $R_1>0$, $C=C(p,n)>0$ and $d=d(p,\alpha,\beta,n)>0$, such that for $|x|\geq\frac{2R_1+C}{\varepsilon}$ there holds*

$$|u_\varepsilon(x)|\leq C|x|^d\exp\Big\{-\tfrac{1}{4}\left|\log\tfrac{3}{4}\right|\frac{\left(|x|^{\frac{2-\alpha}{2}}-(\frac{R_1}{\varepsilon})^{\frac{2-\alpha}{2}}\right)}{\varepsilon^{\alpha/2}}\Big\}. \tag{2.6}$$

This is essentially Lemma 22 of [1]. For the reader's convenience, let us outline below the proof, referring to [1] for more details. The main two steps in the proof of [1] are the following (i) and (ii) below.

(i) For all $\delta_1>0$ there exists $\overline{R}>0$ such that, for all $R\geq\overline{R}$ and all $u\in E_\varepsilon$

$$\begin{aligned}\int_{|x|>\frac{R}{\varepsilon}} &K(\varepsilon x)|u|^{p+1}(x)dx \\ &\leq \delta_1\Big(\int_{|x|>\frac{R}{\varepsilon}}\left[|\nabla u(x)|^2+V(\varepsilon x)u^2(x)\right]dx\Big)^{(p+1)/2}.\end{aligned} \tag{2.7}$$

(ii) Let u_ε satisfy (2.1) and set $\Omega_n=\{|x|>n^{2/(2-\alpha)}\}$. Then for $n\gg1$,

$$\int_{\Omega_{n+1}}\left[|\nabla u_\varepsilon|^2+V(\varepsilon x)u_\varepsilon^2\right]dx\leq\frac{3}{4}\int_{\Omega_n}\left[|\nabla u_\varepsilon|^2+V(\varepsilon x)u_\varepsilon^2\right]dx. \tag{2.8}$$

As for (i), this is nothing but the counterpart of [1, Proposition 11] in our setting and it is easy to check that the proof carried out in [1] can be repeated here. Actually the assumption that $V\geq c>0$ does not play any role, because in (2.7) all the integrals are evaluated for $|x|\gg1$, only.

To prove (2.8) we modify the arguments used in [1, Lemma 17] as follows. Let $\phi_n(r)$ be a piecewise affine function such that

$$\phi_n(r)\equiv0,\quad\forall\, r\leq n^{2/(2-\alpha)},\quad \phi_n(r)\equiv1,\quad\forall\, r\geq(n+1)^{2/(2-\alpha)}.$$

Using (2.1), the fact that $\mu_\varepsilon \le 0$ and $\phi_n \le 1$ we find

$$\int_{\Omega_n} \left[\nabla u_\varepsilon \cdot \nabla(u_\varepsilon \phi_n) + V(\varepsilon x) u_\varepsilon^2 \phi_n\right] dx \le \lambda_n \int_{\Omega_n} K(\varepsilon_n x) u_\varepsilon^{p+1} dx.$$

By a calculation similar to that in [1] and using (2.4), we get

$$\begin{aligned}\int_{\Omega_{n+1}} \left[|\nabla u_\varepsilon|^2 + V(\varepsilon x) u_\varepsilon^2\right] dx &\le \int_{\Omega_n} \phi_n \left[|\nabla u_\varepsilon|^2 + V(\varepsilon x) u_\varepsilon^2\right] dx \\ &\le \lambda_n \int_{\Omega_n} K(\varepsilon x) u_\varepsilon^{p+1} dx - \int_{\Omega_n} (\nabla u_\varepsilon \cdot \nabla \phi_n) dx \\ &\le \Lambda \int_{\Omega_n} K(\varepsilon x) u_\varepsilon^{p+1} dx + \tfrac{1}{2} \int_{\Omega_n} \left[|\nabla u_\varepsilon|^2 + |\nabla \phi_n|^2 u_\varepsilon^2\right] dx.\end{aligned}$$

Since $|\nabla \phi_n| \sim n^{-\alpha/(2-\alpha)}$, we infer that $|\nabla \phi_n|^2 \le V(\varepsilon x)$ in Ω_ε for $n \gg 1$, and hence

$$\begin{aligned}&\int_{\Omega_{n+1}} \left[|\nabla u_\varepsilon|^2 + V(\varepsilon x) u_\varepsilon^2\right] dx \\ &\qquad \le \Lambda \int_{\Omega_n} K(\varepsilon x) u_\varepsilon^{p+1} dx + \tfrac{1}{2} \int_{\Omega_n} \left[|\nabla u_\varepsilon|^2 + V(\varepsilon x) u_\varepsilon^2\right] dx.\end{aligned}$$

This and (2.7) yield, for $\varepsilon \ll 1$,

$$\begin{aligned}&\int_{\Omega_{n+1}} \left[|\nabla u_\varepsilon|^2 + V(\varepsilon x) u_\varepsilon^2\right] dx \\ &\le \delta_1 \Lambda \Big(\int_{\Omega_n} \left[|\nabla u_\varepsilon|^2 + V(\varepsilon x) u_\varepsilon^2\right] dx \Big)^{\frac{p+1}{2}} + \tfrac{1}{2} \int_{\Omega_n} \left[|\nabla u_\varepsilon|^2 + V(\varepsilon x) u_\varepsilon^2\right] dx.\end{aligned}$$

One also has

$$\begin{aligned}&\Big(\int_{\Omega_n} \left[|\nabla u_\varepsilon|^2 + V(\varepsilon x) u_\varepsilon^2\right] dx \Big)^{\frac{p+1}{2}} \\ &\qquad = \Big(\int_{\Omega_n} \left[|\nabla u_\varepsilon|^2 + V(\varepsilon x) u_\varepsilon^2\right] dx \Big)^{\frac{p-1}{2}} \int_{\Omega_n} \left[|\nabla u_\varepsilon|^2 + V(\varepsilon x) u_\varepsilon^2\right] dx \\ &\qquad \le m_\varepsilon^{\frac{p-1}{2}} \int_{\Omega_n} \left[|\nabla u_\varepsilon|^2 + V(\varepsilon x) u_\varepsilon^2\right] dx.\end{aligned}$$

Then we find

$$\int_{\Omega_{n+1}} \left[|\nabla u_\varepsilon|^2 + V(\varepsilon x) u_\varepsilon^2\right] dx \le \Big(\delta_1 \Lambda m_\varepsilon^{\frac{p-1}{2}} + \tfrac{1}{2}\Big) \int_{\Omega_n} \left[|\nabla u_\varepsilon|^2 + V(\varepsilon x) u_\varepsilon^2\right] dx,$$

proving (2.8).

Once we know that $(i) - (ii)$ hold, one can repeat the arguments in [1] proving Lemma 7.

D. Proof of Theorem 1 completed. To complete the proof of Theorem 1 it remains to show that $u_\varepsilon \in W^{1,2}(\mathbb{R}^n)$ and that (2.2) holds.

Fix R_1 and C as in Lemma 7 and let $\rho \geq 2R_1 + C$. Then (2.6) implies

$$\int_{\mathbb{R}^n \setminus B_{\rho/\varepsilon}} K(\varepsilon x) u_\varepsilon^{p+1} dx \sim c \exp\big(-\frac{c}{\varepsilon^{\alpha/2}}\big), \quad c > 0. \tag{2.9}$$

Letting $\Omega_\varepsilon := B_{2\rho/\varepsilon} \setminus \mathcal{Z}_\varepsilon^{2\delta}$, it follows that $\inf_{\Omega_\varepsilon} K(\varepsilon x) \geq \text{constant} > 0$. This and the fact that

$$\int_{\Omega_\varepsilon} K(\varepsilon x) u_\varepsilon^{p+1} dx \leq \int_{\mathbb{R}^n \setminus \mathcal{Z}_\varepsilon^{\delta}} K(\varepsilon x) u_\varepsilon^{p+1} dx \leq \varepsilon^{\frac{3(p+1)}{p-1}},$$

imply that

$$\int_{\Omega_\varepsilon} u_\varepsilon^{p+1} dx \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Then by elliptic estimates, $\|u_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} \to 0$. Taking δ possibly smaller, we can assume that $\mathcal{Z}^{2\delta} \subset B_{2\rho}$ and hence there is $\gamma > 0$ such that $\inf_{\Omega_\varepsilon} V(\varepsilon x) \geq 2\gamma$. For ε small, we have

$$\lambda_\varepsilon \Big(\sup_{\Omega_\varepsilon} [K(\varepsilon x) u_\varepsilon(x)]\Big)^{p-1} \leq \gamma, \quad \sup_{\Omega_\varepsilon} u_\varepsilon(x) \leq 1.$$

Then there holds

$$\begin{cases} -\Delta u_\varepsilon + [V(\varepsilon x) - \gamma] u_\varepsilon \leq 0, & \text{in } \Omega_\varepsilon, \\ u_\varepsilon(x) \leq 1 & \text{on } \partial \mathcal{Z}_\varepsilon^{2\delta}, \\ u_\varepsilon(x) \leq c \exp(-c\, \varepsilon^{-\alpha/2}) & \text{in } \partial B_{2\rho/\varepsilon}. \end{cases}$$

Let Ψ_ε denote the solution of

$$\begin{cases} -\Delta \Psi_\varepsilon + \gamma \Psi_\varepsilon = 0, & \text{in } \Omega_\varepsilon, \\ \Psi_\varepsilon(x) = 1 & \text{on } \partial \mathcal{Z}_\varepsilon^{2\delta}, \\ \Psi_\varepsilon(x) = c \exp(-c\, \varepsilon^{-\alpha/2}) & \text{in } \partial B_{2\rho/\varepsilon}. \end{cases}$$

Then by the comparison principle,

$$u_\varepsilon(x) \leq \Psi_\varepsilon(x), \qquad \forall\, x \in \Omega_\varepsilon.$$

Since Ψ_ε decays exponentially to zero at infinity, there exists $C > 0$ such that

$$u_\varepsilon(x) \leq C \exp\big(-\frac{C}{\varepsilon}\big), \qquad \forall\, x \in B_{\rho/\varepsilon} \setminus \mathcal{Z}_\varepsilon^{4\delta}.$$

Hence $u_\varepsilon \in W^{1,2}(\mathbb{R}^n)$. Moreover, the preceding inequality and (2.9) yield, for $\varepsilon \ll 1$,

$$\int_{\mathbb{R}^n \setminus \mathcal{Z}_\varepsilon^\delta} K(\varepsilon x) u_\varepsilon^{p+1} dx \leq C_1 \exp\left(-\frac{C_1}{\varepsilon}\right) < \varepsilon^{\frac{3(p+1)}{p-1}},$$

proving (2.2).

Now define $v_\varepsilon(x) = m_\varepsilon^{\frac{1}{p-1}} u_\varepsilon(\varepsilon^{-1}x)$. Then v_ε solves equation (1.1). By Lemma 6 we have $||v_\varepsilon||_\infty \to 0$ as $\varepsilon \to 0$.

Finally, we show $\liminf_{\varepsilon\to 0} \varepsilon^{\frac{-2}{p-1}} ||v_\varepsilon||_\infty > 0$. Set $\widehat{v}_\varepsilon = \varepsilon^{\frac{-2}{p-1}} v_\varepsilon$, and note that $\widehat{v}_\varepsilon$ satisfies

$$-\Delta \widehat{v}_\varepsilon + \frac{1}{\varepsilon^2} V(x) \widehat{v}_\varepsilon = K(x) \widehat{v}_\varepsilon^p. \tag{2.10}$$

Choose a cut-off function $\phi \in C_0^\infty(\mathbb{R}^n)$ satisfying $\phi(x) = 1$ for $x \in B_R(0)$ such that $R > \max\{R_0, 2R_1 + C\}$ and $\mathcal{Z}^{4\delta} \subset B_R(0)$, where R_0 is from (V) and $2R_1 + C$ is from Lemma 7 so that (2.6) holds. We have

$$\int_{\mathbb{R}^n} K(x) \widehat{v}_\varepsilon^{p+1} \leq k_2 \int_{\mathbb{R}^n} \widehat{v}_\varepsilon^{p-1} (\phi^2 \widehat{v}_\varepsilon^2 + (1-\phi)^2 \widehat{v}_\varepsilon^2). \tag{2.11}$$

There holds, for some $C > 0$,

$$\begin{aligned} \int_{\mathbb{R}^n} \widehat{v}_\varepsilon^{p-1} \phi^2 \widehat{v}_\varepsilon^2 &\leq C\, ||\widehat{v}_\varepsilon||_\infty^{p-1} \int_{\mathbb{R}^n} |\nabla(\phi \widehat{v}_\varepsilon)|^2 \\ &\leq C\, ||\widehat{v}_\varepsilon||_\infty^{p-1} \int_{\mathbb{R}^n} \left[|\nabla \widehat{v}_\varepsilon|^2 + \frac{1}{\varepsilon^2} V(x) \widehat{v}_\varepsilon^2 \right] \end{aligned} \tag{2.12}$$

Furthermore, choosing $\delta > 0$ such that $p - 1 - \delta > 0$ we use (V) to infer that

$$\begin{aligned} &\int_{\mathbb{R}^n} \widehat{v}_\varepsilon^{p-1} (1-\phi)^2 \widehat{v}_\varepsilon^2 \leq \|\widehat{v}_\varepsilon\|_\infty^{p-1-\delta} \int_{\mathbb{R}^n} \widehat{v}_\varepsilon^\delta (1-\phi)^2 \widehat{v}_\varepsilon^2 \\ &\leq k_1^{-1} \|\widehat{v}_\varepsilon\|_\infty^{p-1-\delta} \int_{\mathbb{R}^n} (1 + |x|^\alpha) \widehat{v}_\varepsilon^\delta V(x) \widehat{v}_\varepsilon^2. \end{aligned}$$

Using Lemma 7, we find that $(1 + |x|^\alpha)\widehat{v}_\varepsilon^\delta \leq 1$, provided $|x| > R$ and ε is sufficiently small. From this and the previous equation we get

$$\begin{aligned} \int_{\mathbb{R}^n} \widehat{v}_\varepsilon^{p-1} (1-\phi)^2 \widehat{v}_\varepsilon^2 &\leq k_1^{-1} \|\widehat{v}_\varepsilon\|_\infty^{p-1-\delta} \int_{\mathbb{R}^n} V(x) \widehat{v}_\varepsilon^2 \\ &\leq k_1^{-1} \|\widehat{v}_\varepsilon\|_\infty^{p-1-\delta} \int_{\mathbb{R}^n} \left[|\nabla \widehat{v}_\varepsilon|^2 + \frac{1}{\varepsilon^2} V(x) \widehat{v}_\varepsilon^2 \right] \end{aligned} \tag{2.13}$$

Using (2.11 - 2.12 - 2.13) and the equation (2.10) we infer that $\|\widehat{v}_\varepsilon\|_\infty \geq C > 0$ and thus $\liminf_{\varepsilon\to 0} \varepsilon^{-\frac{2}{p-1}} ||v_\varepsilon||_\infty > 0$. This completes the proof of Theorem 1

Remark 8. We may also consider localized solutions concentrating near an isolated subset A of the set of zeros of V, $\mathcal{Z}$. I.e., we require $d(A, \mathcal{Z} \setminus A) > 0$. This was done in [6] for the case $\liminf_{|x|\to\infty} V$ is positive. Slightly refined versions of the arguments above give results of this type. We refer to [6] for details. □

3. Proof of Theorem 3

In this section we will give the proof of Theorem 3. By [6], the equation has a least energy solution v_ε which has an exponential decay at infinity.

By a scaling depending on m, we define

$$w_\varepsilon(x) = \varepsilon^{-\frac{2}{p-1}\frac{m}{m+2}} v_\varepsilon(\varepsilon^{\frac{2}{m+2}} x).$$

Then w_ε satisfies

$$-\Delta w_\varepsilon(x) + (P_m(x) + \varepsilon^{-\frac{2m}{m+2}} Q(\varepsilon^{\frac{2}{m+2}} x)) w_\varepsilon = K(\varepsilon^{\frac{2}{m+2}} x) w_\varepsilon^p.$$

By Lemma 7 and the construction of the solution we have that for each $\delta_1 > 0$ there exist $C, c > 0$ such that for $|\varepsilon^{\frac{2}{m+2}} x| \geq \delta_1$,

$$w_\varepsilon(x) \leq C\varepsilon^{-\frac{2}{p-1}\frac{m}{m+2}} \exp(-c\varepsilon^{-\frac{m}{m+2}} |x|). \tag{3.1}$$

By using (V), we first claim that for $|x| \geq \varepsilon^{-\frac{2}{m+2}} R_0$,

$$\varepsilon^{-\frac{2m}{m+2}} V(\varepsilon^{-\frac{2m}{m+2}}) \geq \frac{k_1}{|x|^\alpha}. \tag{3.2}$$

By the property of Q there exists $\delta_2 > 0$ such that for $|\varepsilon^{\frac{2}{m+2}} x| \leq \delta_2$, $P_m(x) + \varepsilon^{-\frac{2m}{m+2}} Q(\varepsilon^{\frac{2}{m+2}} x) \geq \frac{1}{2} P_m(x)$. Thus there is $R_2 > 0$ such that for $|x| \geq R_2$

$$\varepsilon^{-\frac{2m}{m+2}} V(\varepsilon^{-\frac{2m}{m+2}}) \geq \frac{k_1}{|x|^\alpha}. \tag{3.3}$$

Since a ground state solution w to (1.4) is exponentially decaying at infinity, we have

$$\limsup_{\varepsilon\to 0} \int_{\mathbb{R}^n} |\nabla w_\varepsilon|^2 + (P_m(x) + \varepsilon^{-\frac{2m}{m+2}} Q(\varepsilon^{\frac{2}{m+2}} x)) w_\varepsilon^2 \leq \int_{\mathbb{R}^n} |\nabla w|^2 + P_m(x) w^2.$$

From these and the elliptic estimates we get that the L^∞ norm of w_ε is uniformly bounded for ε small. By the fact that $P_m(x) \to \infty$ as $|x| \to \infty$

and an elliptic estimate again, w_ε tends to zero as $|x| \to \infty$ uniformly for ε small. Then by this and (3.1) we have $R_3 \geq R_2$ such that for $|x| \geq R_3$,

$$K(\varepsilon^{\frac{2}{m+2}} x) w_\varepsilon^{p-1}(x) \leq \frac{1}{2} \frac{k_1}{|x|^\alpha}.$$

Thus for $|x| \geq R_3$, $\Delta w_\varepsilon - \frac{1}{2} \frac{k_1}{|x|^\alpha} w_\varepsilon \geq 0$. By Lemma 6 of [3] and the comparison principle, we get for some constant $C, C > 0$,

$$w_\varepsilon(x) \leq C \exp(-c|x|^{\frac{2-\alpha}{2}}).$$

Next we show $\liminf_{\varepsilon \to 0} ||w_\varepsilon||_\infty > 0$. If not, using the above estimate and similar to the end of the last section, we have

$$\int_{\mathbb{R}^n} |\nabla w_\varepsilon|^2 + \varepsilon^{-\frac{2m}{m+2}} V(\varepsilon^{\frac{2}{m+2}} x) w_\varepsilon^2 \leq C \exp(-c\varepsilon^{\frac{(2-\alpha)(p+1)}{m+2}}).$$

Scaling back to v_ε we would have $||v_\varepsilon||_\infty$ tending to zero exponentially as $\varepsilon \to 0$, which is a contradiction with (1.3). The convergence of the solutions to a least energy solution of (1.4) follows from the elliptic estimates and the uniformly exponential decay property.

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