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A Liouville theorem, a-priori bounds, and bifurcating branches of positive solutions for a nonlinear elliptic system

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Abstract The paper is concerned with the local and global bifurcation structure of positive solutions $u, v \in H_0^1(\Omega)$ of the system

$$\begin{cases} -\Delta u + u = \mu_1 u^3 + \beta v^2 u & \text{in } \Omega \\ -\Delta v + v = \mu_2 v^3 + \beta u^2 v & \text{in } \Omega \end{cases}$$

of nonlinear Schrödinger (or Gross-Pitaevskii) type equations in $\Omega \subset \mathbb{R}^N$, $N \leq 3$. The system arises in nonlinear optics and in the Hartree–Fock theory for a double condensate. Local and global bifurcations in terms of the nonlinear coupling parameter β of the system are investigated by using spectral analysis and by establishing a new Liouville type theorem for nonlinear elliptic systems which provides a-priori bounds of solution branches. If the domain is radial, possibly unbounded, then we also control the nodal structure of a certain weighted difference of the components of the solutions along the bifurcating branches.

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Dedicated to Paul Rabinowitz on the occasion of his 70th birthday.

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1 Introduction

In this paper we are concerned with the nonlinear elliptic system

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 u^3 + \beta v^2 u & \text{in } \Omega \\ -\Delta v + \lambda_2 v = \mu_2 v^3 + \beta u^2 v & \text{in } \Omega \\ u, v > 0 & \text{in } \Omega,\ u, v \in H_0^1(\Omega) \end{cases} \tag{1.1}$$

on a possibly unbounded domain $\Omega \subset \mathbb{R}^N$, $N \leq 3$. This system has found considerable interest in recent years as it appears in a number of physical problems, for instance in nonlinear optics. There the solution (u, v) denotes components of the beam in Kerr-like photorefractive media ([1]). With $\mu_j > 0$, $j = 1, 2$, we have self-focusing in both components of the beam. The nonlinear coupling constant β is the interaction between the two components of the beam. Problem (1.1) also arises in the Hartree–Fock theory for a double condensate, i.e., a binary mixture of Bose–Einstein condensates in two different hyperfine states ([15]). In recent years many mathematical works on the existence and on qualitative properties of solutions have appeared, revealing interesting features for the system which are quite different from those of semilinear type Schrödinger equations. Following the work [20] by Lin and Wei about the existence of ground state solutions with small couplings a number of papers have been devoted to the existence theory of solutions in various different parameter regimes of nonlinear couplings; see [2,3,5,6,23,24,35] for the existence of ground state or bound state solutions, [21,22,26,30] for semiclassical states or singularly perturbed settings. In [13,38,39] the authors have investigated the competition case $\beta < 0$, assuming $\lambda_1 = \lambda_2 = 1$ and $\mu_1 = \mu_2 = 1$, and established the existence of multiple positive solutions. We also want to mention the paper [29] where the authors investigate the limit of solutions as $\beta \to -\infty$, and the related work [10] on Lotka–Volterra type competition systems.

The current paper is mostly related to the papers [13,38,39]. We shall use a quite different approach, namely bifurcation techniques. Our results are new and improve significantly some of the results from [13,38,39] where $\lambda_1 = \lambda_2 > 0$ and $\mu_1 = \mu_2 > 0$ is being required. When this condition holds the problem is invariant under the symmetry $(u, v) \mapsto (v, u)$. This invariance is essential to the method used in [13,38,39], namely Lusternik–Schnirelman type arguments for symmetric functionals. Our methods using bifurcation techniques require $\lambda_1 = \lambda_2 > 0$ in order to have a "trivial" branch of solutions. But our arguments do not depend on the symmetry condition $\mu_1 = \mu_2$ so we can extend the existence results from the papers mentioned above to a larger range of parameters. Moreover, we can show that the solutions lie on continuous branches in terms of the nonlinear coupling parameter β, and that these branches are bounded as long as β is bounded. These results are new even in the case $\mu_1 = \mu_2$. The boundedness of the branch is a consequence of a new Liouville type theorem for elliptic systems. We also show that a certain nodal property of a weighted difference of the two components of the solutions is preserved along the solution branches.

We deal with the case $\lambda_1 = \lambda_2 > 0$ and may assume $\lambda_1 = \lambda_2 = 1$. Thus we consider

$$\begin{cases} -\Delta u + u = \mu_1 u^3 + \beta v^2 u & \text{in } \Omega \\ -\Delta v + v = \mu_2 v^3 + \beta u^2 v & \text{in } \Omega \\ u, v > 0 & \text{in } \Omega,\ u, v \in H_0^1(\Omega). \end{cases} \tag{1.2}$$

Fixing $\mu_1, \mu_2 > 0$ we may assume without loss of generality that $\mu_1 \leq \mu_2$. In the case $N = 1$, Ω is a bounded domain. If $N = 2$ or $N = 3$ the domains $\Omega \subset \mathbb{R}^N$ we deal with are bounded or radially symmetric (possibly unbounded).

If $w \in H_0^1(\Omega)$ is a solution of

$$-\Delta w + w = w^3, \ w > 0 \quad \text{in } \Omega \tag{1.3}$$

then a direct calculation shows that for $\beta \in (-\sqrt{\mu_1\mu_2}, \mu_1) \cup (\mu_2, \infty)$ the pair

$$u_\beta = \left(\frac{\mu_2 - \beta}{\mu_1\mu_2 - \beta^2}\right)^{1/2} w, \quad v_\beta = \left(\frac{\mu_1 - \beta}{\mu_1\mu_2 - \beta^2}\right)^{1/2} w$$

solves (1.2). If $\mu_1 = \mu_2 =: \mu$ this simplifies to

$$u_\beta = v_\beta = \left(\frac{1}{|\mu + \beta|}\right)^{1/2} w$$

which is defined for $\beta \neq -\mu$. Thus if $\mu_1 < \mu_2$ we have a "trivial" branch

$$\mathcal{T}_w := \left\{(\beta, u_\beta, v_\beta) \in \mathbb{R} \times H_0^1(\Omega) \times H_0^1(\Omega) : \beta \in (-\sqrt{\mu_1\mu_2}, \mu_1) \cup (\mu_2, \infty)\right\}$$

of solutions of (1.2), and similarly for $\mu_1 = \mu_2$. We are interested in proving bifurcation of nontrivial solutions from this branch. In doing this we considerably improve results due to Dancer, Wei and Weth [13,39]. Our results give that there are infinitely many bifurcation points along this trivial branch, that in case $N = 1$ or Ω radially symmetric, the bifurcating branches are global and unbounded to the left in the β-direction, and that solution branches are prescribed by a nodal property of a weighted difference of the two components u and v.

The paper is organized as follows. In Sect. 2 we state the main results of the paper about local and global bifurcations. We also state a Liouville theorem which is used to establish a-priori bounds of solution branches. This result may be of independent interest. In Sect. 3 we determine all bifurcation points along $\mathcal{T}_w$. Finally, in Sect. 4 we prove the Liouville theorem and using this we investigate the global bifurcation branches.

2 Statement of results

Let $E = H_0^1(\Omega)$ when $N = 1$, or when $\Omega \subset \mathbb{R}^N$ is a bounded domain. If $N \geq 2$ and $\Omega \subset \mathbb{R}^N$ is unbounded we require that Ω is radially symmetric, i.e., the exterior of a ball or all of $\mathbb{R}^N$. In this case we set $E = \{u \in H_0^1(\Omega) : u \text{ is radially symmetric}\}$. In the case of a bounded radial domain actually either choice of E is fine.

We fix a nondegenerate solution $w \in E$ of (1.3) so that $\mathcal{T}_w \subset \mathbb{R} \times E \times E$. A parameter value β is said to be a parameter of bifurcation from $\mathcal{T}_w$, or simply a bifurcation parameter, if there exists a sequence $(\beta_j, u_j, v_j) \in \mathbb{R} \times E \times E \setminus \mathcal{T}_w$ of solutions of (1.2) such that $(\beta_j, u_j, v_j) \to (\beta, u_\beta, v_\beta)$ as $j \to \infty$. We call β a global bifurcation parameter if a connected set of solutions of (1.2) bifurcates from $\mathcal{T}_w$ at $(\beta, u_\beta, v_\beta)$ in the sense of Rabinowitz. More precisely, setting

$$\mathcal{S} := \{(\beta, u, v) \in \mathbb{R} \times E \times E \setminus \mathcal{T}_w : (\beta, u, v) \ \text{solves (1.2)}\}$$

then β is a global bifurcation parameter if the connected component $\mathcal{S}_\beta$ of $(\beta, u_\beta, v_\beta)$ in $\mathcal{S} \cup \{(\beta, u_\beta, v_\beta)\}$ is unbounded or $\overline{\mathcal{S}_\beta} \cap \mathcal{T}_w \setminus \{(\beta, u_\beta, v_\beta)\} \neq \emptyset$.

The bifurcation parameters depend on the eigenvalues of

$$-\Delta\phi + \phi = \lambda w^2 \phi. \tag{2.1}$$

The eigenvalue problem (2.1) has a sequence of eigenvalues $\lambda_1 = 1 < \lambda_2 < \lambda_3 < \dots$ with $\lambda_k \to \infty$ and multiplicity $n_k = \dim \ker(-\Delta + 1 - \lambda_k w^2)$ where the kernel has to be taken in E. In particular, in the radial setting we only consider radial eigenfunctions here. The first eigenvalue $\lambda_1 = 1$ is simple ($n_1 = 1$) with eigenfunction $w > 0$. The condition that w is non-degenerate means that $\lambda = 3$ is not an eigenvalue of (2.1), so $\lambda_k \neq 3$ for all k. Moreover, if w is a mountain pass solution of (1.3) then $\lambda_2 > 3$. More generally, the Morse index $m(w)$ of w is given by

$$m(w) = n_1 + \cdots + n_{k_0} \quad \text{with } k_0 = \max\{k \in \mathbb{N} : \lambda_k < 3\}. \tag{2.2}$$

Our first result deals with the existence of bifurcation points.

Theorem 2.1 *Assume w is a non-degenerate solution of* (1.3). *There exists a sequence*

$$\mu_1 > \beta_2 > \beta_3 > \cdots > \beta_{k_0} > 0 > \beta_{k_0+1} > \beta_{k_0+2} > \cdots > -\sqrt{\mu_1 \mu_2}$$

of bifurcation parameters of (1.2) *such that $\beta_k \to -\sqrt{\mu_1 \mu_2}$ as $k \to \infty$; here k_0 is as defined in* (2.2). *If the multiplicity n_k of λ_k is odd then β_k is a global bifurcation parameter. If $\mu_1 \neq \mu_2$ then there are no other bifurcation points along $\mathcal{T}_w$ except $(\beta_k, u_{\beta_k}, v_{\beta_k})$, $k \geq 2$. If $\mu_1 = \mu_2 = \mu$ then also $(\beta_1, u_{\beta_1}, v_{\beta_1})$ with $\beta_1 = \mu$ is a bifurcation point.*

Remark 2.2 (a) In the proof of Theorem 2.1 we explicitly determine the bifurcation parameters β_k as a function of λ_k. We also determine explicitly the kernel V_k of the linearization of (1.2) with respect to (u, v) at the trivial solution $(\beta_k, u_{\beta_k}, v_{\beta_k})$. It turns out that its dimension is the same as the multiplicity n_k of λ_k as eigenvalue of (2.1). In fact, the relation between V_k and the k-th eigenspace will be made explicit (see (3.7)). In particular, if $N = 1$ or Ω is radially symmetric and $E = \{u \in H_0^1(\Omega) : u \text{ is radially symmetric}\}$ then $n_k = 1$ for all $k \in \mathbb{N}$.

(b) If $\mu_1 < \mu_2$ then at the end point $\beta_1 = \mu_1$, the trivial branch $\mathcal{T}_w$ intersects the solution branch $\mathcal{T}_1 = \{(\beta, w_1, 0) : \beta \in \mathbb{R}\}$ where $w_1 = \mu_1^{-1/2} w$. So here we have the bifurcation of semitrivial solutions of (1.2) from $\mathcal{T}_w$. Looking at it differently, at $(\mu_1, w_1, 0)$ the branch $\mathcal{T}_w$ bifurcates from the branch $\mathcal{T}_1$ of semitrivial solutions, and the bifurcation points $(\beta_k, u_{\beta_k}, v_{\beta_k})$ are secondary bifurcation points. Theorem 2.1 also shows that there is no secondary bifurcations on the other half of $\mathcal{T}_w$ with $\beta \geq \mu_2$ which meets at $\beta = \mu_2$ the solution branch $\mathcal{T}_2 = \{(\beta, 0, w_2) : \beta \in \mathbb{R}\}$ where $w_2 = \mu_2^{-1/2} w$.

(c) If $\mu_1 = \mu_2 =: \mu$ then at the point $\beta_1 = \mu$ the bifurcating solutions are explicitly given by

$$\left(\mu, u_{\mu,\theta}, v_{\mu,\theta}\right) := \left(\mu, \frac{\cos\theta}{\sqrt{2\mu}} w, \frac{\sin\theta}{\sqrt{2\mu}} w\right) \quad \text{for } 0 < \theta < \frac{\pi}{2}.$$

For other values of θ one obtains non-positive solutions of the elliptic system. The bifurcating set $\mathcal{S}_1^+ := \left\{\left(\mu, u_{\mu,\theta}, v_{\mu,\theta}\right) : 0 < \theta < \frac{\pi}{4}\right\}$ connects $\mathcal{T}_w$ with $\mathcal{T}_1$, and the bifurcating set $\mathcal{S}_1^- := \left\{\left(\mu, u_{\mu,\theta}, v_{\mu,\theta}\right) : \frac{\pi}{4} < \theta < \frac{\pi}{2}\right\}$ connects $\mathcal{T}_w$ with $\mathcal{T}_2$. By [6] at the intersection $\overline{\mathcal{S}_1^+} \cap \mathcal{T}_1 = \{(\mu, u_\mu, 0)\}$ we have bifurcation from a simple eigenvalue in the sense of [11], so there are no further solutions of (1.2) near $(\mu, u_\mu, 0)$ except those contained in $\mathcal{S}_1^+$. The analogous statement holds near $\overline{\mathcal{S}_1^-} \cap \mathcal{T}_2$.

(d) If λ_k is a simple eigenvalue of (2.1) then the bifurcating connected set $\mathcal{S}_k$ is in fact a one-dimensional C^1-curve in a neighborhood of $(\beta_k, u_{\beta_k}, v_{\beta_k})$. As stated in a) this applies if $N = 1$ or in the radial setting.

(e) In the case $\mu_1 = \mu_2 = 1$ and Ω a bounded smooth domain, [13, Theorem 1.2] of Dancer, Wei and Weth states the existence of $\tilde{\beta}_k$ such that (1.2) has at least k solutions for $-1 < \beta < \tilde{\beta}_k$ and infinitely many solutions for $\beta \leq -1$. It seems most likely that this holds with $\tilde{\beta}_k = \beta_{k+1}$. (The index shift occurs because at β_1 there is no bifurcation to the left.) However, if the multiplicity n_k is even then we just obtain local bifurcation from $(\beta_k, u_{\beta_k}, v_{\beta_k})$. And if n_k is odd we do not know whether the bifurcating global connected branch $\mathcal{S}_k$ is unbounded in the β-direction. If so, then as a consequence of [5, Theorem 1.5] the projection $pr_1 : \mathbb{R} \times E \times E \to \mathbb{R}$ satisfies $pr_1(\mathcal{S}_k) \subset (-\infty, \mu_1)$, hence $pr_1(\mathcal{S}_k) \supset (-\infty, \beta_k)$. $\mathcal{S}_k$ may however be bounded in the β-component and unbounded in the (u, v)-component, or it may return to $\mathcal{T}_w$. Comparing Theorem 2.1 with [13, Theorem 1.2] suggests that there should exist infinitely many global solution branches $\mathcal{S}_k$ bifurcating from $\mathcal{T}_w$ and satisfying $pr_1(\mathcal{S}_k) \supset (-\infty, \beta_k)$.

(f) The first part of the result in Theorem 2.1 about local bifurcations holds also for unbounded domains Ω without radial symmetry. This will be clear from the proof as the Krasnoselski's type bifurcation result is applied (see [18,32,33]).

We now turn to the two cases $N = 1$ or Ω is radial where we can prove a result as suggested in Remark 2.2 e). It is well known that (1.3) has a unique positive (radial if $N \geq 2$) least energy solution w which is nondegenerate (in the class of radial functions if $N \geq 2$) and of mountain pass type; see e.g., [27,28,37,16] for the various domains. Consequently $m(w) = 1$ and $\beta_k \in (-\sqrt{\mu_1\mu_2}, 0)$ for every $k \geq 2$. Moreover, $n_k = 1$ for every $k \in \mathbb{N}$, so each β_k is a global bifurcation point. The next theorem contains some information about the global bifurcating branch (Fig. 1). Recall that we set $E = \{u \in H_0^1(\Omega) : u \text{ is radially symmetric}\}$ in Theorem 2.3 if the domain is radial.

Theorem 2.3 *Suppose $N = 1$ or Ω is radial and let $w \in E$ be the unique positive (radial) solution of (1.3). Then for each integer $k \geq 2$ there exists a connected set $\mathcal{S}_k \subset \mathcal{S} \subset \mathbb{R} \times E \times E$ of solutions (β, u, v) of (1.2) such that $\overline{\mathcal{S}_k} \cap \mathcal{T}_w = \{(\beta_k, u_{\beta_k}, v_{\beta_k})\}$. The projection $pr_1 : \mathcal{S}_k \to \mathbb{R}$ onto the parameter space satisfies $pr_1(\mathcal{S}_k) \supset (-\infty, \beta_k)$. For any $(\beta, u, v) \in \mathcal{S}_k$ the difference $(\mu_1 - \beta)^{1/2}u - (\mu_2 - \beta)^{1/2}v$ has precisely $k - 1$ simple zeroes.*

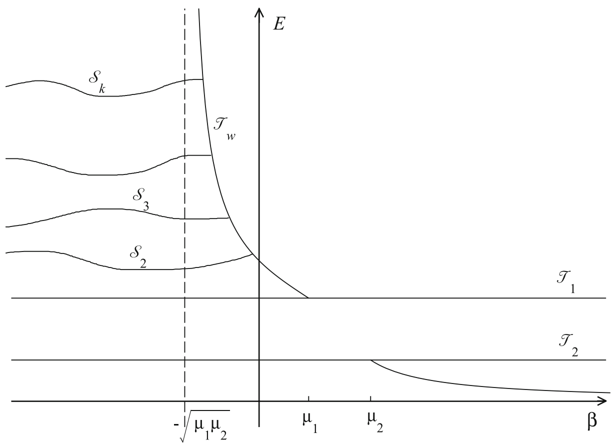

Fig. 1 Schematic diagram of the bifurcation scenario if $\mu_1 < \mu_2$

Thus in the one-dimensional or radial setting we recover and improve [13, Theorem 1.2]. If $\mu_1 = \mu_2$ and $\beta \le -1$ the existence of radial solutions (β, u, v) such that $u - v$ has precisely $k - 1$ zeroes has been obtained by Wei and Weth in [39, Theorem 1.1] for $k \ge 2$ using variational methods which are based on the symmetry $(u, v) \mapsto (v, u)$ of (1.2) in the case $\mu_1 = \mu_2$. Theorem 2.3 improves their result considerably by, firstly, extending it to a larger range of parameters μ_1, μ_2, β, in particular to the case without symmetry $\mu_1 \ne \mu_2$, and, secondly, obtaining the additional information that these solutions lie in fact on connected branches.

Remark 2.4 (a) If $\Omega \subset \mathbb{R}^2$ is a ball or an annulus one can prove that there is global bifurcation at a parameter value β_k corresponding to an eigenvalue λ_k of (2.1), even if λ_k has no radial eigenfunction, so that Theorem 2.3 does not apply. Since (1.2) and (2.1) have an $SO(2)$-symmetry and are variational one can work with the S^1-orthogonal degree from [34]. One can also work with the Leray–Schauder degree in a certain subspace $E \subset H_0^1(\Omega)$. For the latter approach one chooses $m \in \mathbb{N}$ maximal so that there is an eigenfunction of (2.1) of the form $R(r) \cos m\theta$; here (r, θ) are polar coordinates. Then one takes E to be the set of all functions that are even in θ and invariant under rotations of $2\pi/m$ in θ. The bifurcating branches are global in the sense stated above but we do not know whether they are unbounded or return to $\mathcal{T}_w$. Even if they are unbounded we do not know whether they are unbounded in the β-direction.

(b) Equivariant degree theory can also be used for a bounded symmetric domain $\Omega \subset \mathbb{R}^3$. If $\Omega \subset \mathbb{R}^3$ is a ball or an annulus, (1.2) and (2.1) have an $SO(3)$-symmetry. Here one can apply the orthogonal $SO(3)$-equivariant degree. More generally, if Ω is symmetric with respect to a subgroup $G \subset SO(3)$ the orthogonal G-equivariant degree can be used to prove global bifurcation of non-radial solutions. Details are left to the reader and we just refer to the recent monograph [4] on G-equivariant degree theory.

The proof of Theorem 2.3 requires the proof of a-priori bounds for solutions (β, u, v) with a bound on β and a bound on the number of nodal domains of $(\mu_1 - \beta)^{1/2}u - (\mu_2 - \beta)^{1/2}v$.

Theorem 2.5 *Suppose $N = 1$ or Ω is radial. Then, given a compact set $B \subset \mathbb{R}$ and $k \in \mathbb{N}$, the set*

$$\{(\beta, u, v) \in \mathbb{R} \times E \times E : (\beta, u, v) \text{ solves (1.2), } \beta \in B, \text{ and } \\ (\mu_1 - \beta)^{1/2}u - (\mu_2 - \beta)^{1/2}v \text{ has at most } k \text{ zeroes}\}$$

is bounded.

These a-priori bounds are a consequence of a Liouville type theorem for solutions $(u(r), v(r))$ of the system

$$\begin{cases} -u'' - \dfrac{N-1}{c+r}u' = \mu_1 u^3 + \beta v^2 u & \text{in } (-c, \infty), \\ -v'' - \dfrac{N-1}{c+r}v' = \mu_2 v^3 + \beta u^2 v & \text{in } (-c, \infty), \\ u, v \ge 0 \end{cases} \tag{2.3}$$

with $c \in [0, \infty]$ fixed. When $c = \infty$ we understand the terms with u' and v' disappear.

Theorem 2.6 *Let* (u, v) *be a solution of* (2.3)*. Then* $(\mu_1 - \beta)^{1/2}u - (\mu_2 - \beta)^{1/2}v$ *has infinitely many zeroes.*

In [13, Theorem 2.1] it has been proved that the system

$$\begin{cases} -\Delta u = \mu_1 u^3 + \beta v^2 u & \text{in } \mathbb{R}^N \\ -\Delta v = \mu_2 v^3 + \beta u^2 v & \text{in } \mathbb{R}^N \\ u, v \geq 0 & \text{in } \mathbb{R}^N \end{cases} \tag{2.4}$$

has no classical solutions provided $\beta > -\sqrt{\mu_1\mu_2}$. This is not true anymore if $\beta \leq -\sqrt{\mu_1\mu_2}$. For radial solutions, (2.4) reduces to (2.3) with $c = 0$. Our Theorem 2.6 implies that, even if $\beta \leq -\sqrt{\mu_1\mu_2}$, (2.4) does not have nontrivial radial solutions such that $(\mu_1 - \beta)^{1/2}u - (\mu_2 - \beta)^{1/2}v$ has only finitely many zeroes.

3 Proof of Theorem 2.1

We first determine explicitly all bifurcation parameters. In order to do this we consider the function

$$f : (-\sqrt{\mu_1\mu_2}, \mu_1) \to (1, \infty), \quad f(\beta) = \frac{3\mu_1\mu_2 - 2\beta(\mu_1 + \mu_2) + \beta^2}{\mu_1\mu_2 - \beta^2}.$$

It is straightforward to check that f is a strictly decreasing diffeomorphism mapping $(-\sqrt{\mu_1\mu_2}, 0]$ to $[3, \infty)$ and $[0, \mu_1)$ to $(1, 3]$. Recall the nondegenerate solution $w > 0$ of (1.3) and the eigenvalues λ_k of the eigenvalue problem (2.1).

Lemma 3.1 *The only possible bifurcation parameters are* $\beta_k := f^{-1}(\lambda_k)$*,* $k \geq 2$ *(*$k \geq 1$ *if* $\mu_1 = \mu_2$*). The dimension of the kernel of the linearization of* (1.2) *with respect to* (u, v) *at the trivial solution* $(\beta_k, u_{\beta_k}, v_{\beta_k})$ *is equal to the multiplicity* n_k *of* λ_k *as eigenvalue of* (2.1)*.*

Proof Linearizing (1.2) at $(\beta, u_\beta, v_\beta)$ yields the system

$$\begin{cases} -\Delta\phi + \phi = 3\mu_1 u_\beta^2 \phi + \beta v_\beta^2 \phi + 2\beta u_\beta v_\beta \psi \\ -\Delta\psi + \psi = 2\beta u_\beta v_\beta \phi + 3\mu_2 v_\beta^2 \psi + \beta u_\beta^2 \psi \end{cases} \tag{3.1}$$

or equivalently

$$\begin{cases} -\Delta\phi + \phi = w^2(a\phi + b\psi) \\ -\Delta\psi + \psi = w^2(b\phi + c\psi) \end{cases} \tag{3.2}$$

with

$$a = a(\beta) = 3\mu_1 \frac{\mu_2 - \beta}{\mu_1\mu_2 - \beta^2} + \beta \frac{\mu_1 - \beta}{\mu_1\mu_2 - \beta^2} = \frac{3\mu_1\mu_2 - 2\mu_1\beta - \beta^2}{\mu_1\mu_2 - \beta^2} \tag{3.3}$$

and

$$b = b(\beta) = 2\beta \frac{\sqrt{(\mu_1 - \beta)(\mu_2 - \beta)}}{\mu_1\mu_2 - \beta^2} \tag{3.4}$$

and

$$c = c(\beta) = 3\mu_2 \frac{\mu_1 - \beta}{\mu_1\mu_2 - \beta^2} + \beta \frac{\mu_2 - \beta}{\mu_1\mu_2 - \beta^2} = \frac{3\mu_1\mu_2 - 2\mu_2\beta - \beta^2}{\mu_1\mu_2 - \beta^2}. \tag{3.5}$$

Let $\gamma_\pm$ be the solutions of $c\gamma - b = a\gamma - b\gamma^2$, that is,

$$\gamma_\pm = \frac{a-c}{2b} \pm \frac{1}{2b}\sqrt{(a-c)^2 + 4b^2}. \tag{3.6}$$

If (ϕ, ψ) is a solution of (3.2) then a simple calculation shows that $\phi - \gamma_\pm\psi$ solves

$$-\Delta(\phi - \gamma_\pm\psi) + (\phi - \gamma_\pm\psi) = (a - b\gamma_\pm)w^2(\phi - \gamma_\pm\psi),$$

and that $a - b\gamma_- = 3$. Consequently, $\phi - \gamma_-\psi$ solves

$$-\Delta(\phi - \gamma_-\psi) + (\phi - \gamma_-\psi) = 3w^2(\phi - \gamma_-\psi).$$

Since w is a nondegenerate solution of (1.3) we obtain that $\phi = \gamma_-\psi$. Plugging this into (3.2) it follows that ψ solves the equation

$$-\Delta\psi + \psi = (b\gamma_- + c)w^2\psi.$$

Next one easily checks that $b\gamma_- + c = f(\beta)$. It follows that the linearization (3.1) has a nontrivial kernel if, and only if, $f(\beta) = \lambda_k$ for some $k \in \mathbb{N}$. Moreover, in that case the kernel is given by

$$V_k = \{(\gamma_-\psi, \psi) : \psi \text{ is an eigenfunction of (2.1) associated to } \lambda_k\}. \tag{3.7}$$

The case $f(\beta) = \lambda_1 = 1$ corresponds to $\beta = \mu_1$. If $\mu_1 < \mu_2$ then we recall from Remark 2.2b that $\overline{\mathcal{T}_w} \cap \mathcal{T}_1 = \{(\mu_1, w_1, 0)\}$, i.e. $\mathcal{T}_w$ bifurcates from $\mathcal{T}_1$ at that point. This is a bifurcation from a simple eigenvalue, hence there can be no further bifurcation of solutions of (1.2), where both components have to be positive, at that point. □

It remains to show that β_k is in fact a bifurcation parameter. By Remark 2.2c this is trivially the case for $\mu_1 = \mu_2$ and $\beta = \beta_1$. Therefore in the sequel we only need to consider the case $k \geq 2$. An important role plays the variational nature of the problem. Solutions of (1.2) are critical points of the functional $J_\beta : E \times E \to \mathbb{R}$ given by

$$J_\beta(u,v) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + |\nabla v|^2 + u^2 + v^2) - \frac{1}{4}\int_\Omega (\mu_1 u^4 + \mu_2 v^4) - \frac{\beta}{2}\int_\Omega u^2 v^2.$$

It is standard to show that J_β is of class C^2. Observe that E embeds compactly into $L^4(\Omega)$; in the case of an unbounded radial domain this is a well known consequence of a lemma of Strauss; see [36] or [40, Corollary 1.26]. It follows easily that ∇J_β is a compact perturbation of $\mathrm{id}_{E\times E}$ and that J_β satisfies the Palais–Smale condition. Let $m(\beta) \in \mathbb{N}_0$ be the Morse index of (u_β, v_β) as critical point of J_β.

Lemma 3.2 *The change of Morse indices at β_k, $k \geq 2$, is given by:*

$$i_k := \lim_{\varepsilon \searrow 0} (m(\beta_k - \varepsilon) - m(\beta_k + \varepsilon)) = n_k.$$

The lemma also holds for $\mu_1 = \mu_2 = \mu$ at $\beta_1 = \mu$. We do not prove this here because the proof is similar to the one we give below and because we do not need the result by Remark 2.2c.

Proof Lemma 3.1 implies $|i_k| \leq n_k$. In order to prove $i_k = n_k$ we introduce some notation. Let

$$\langle (u_1, v_1), (u_2, v_2)\rangle = \int_\Omega (\nabla u_1 \cdot \nabla u_2 + u_1 u_2 + \nabla v_1 \cdot \nabla v_2 + v_1 v_2)$$

be the standard scalar product on $E \times E$ and let $\|\,.\,\|$ be the associated norm. With respect to this product we have

$$\nabla J_\beta(u, v) = (u, v) - (K(\mu_1 u^3 + \beta v^2 u), K(\mu_2 v^3 + \beta u^2 v))$$

where $K = (-\Delta + 1)^{-1}$. Now the Hessian $H_\beta : (E \times E)^2 \to \mathbb{R}$ of J_β at (u_β, v_β), is given by

$$\begin{aligned} H_\beta[(\phi, \psi)^2] &= \|(\phi, \psi)\|^2 - \int_\Omega \left(a(\beta)w^2\phi^2 + 2b(\beta)w^2\phi\psi + c(\beta)w^2\psi^2\right) \\ &= \int_\Omega \left(|\nabla\phi|^2 + \phi^2 + |\nabla\psi|^2 + \psi^2\right) - \int_\Omega \left(a(\beta)\phi^2 + 2b(\beta)\phi\psi + c(\beta)\psi^2\right) w^2 \end{aligned} \tag{3.8}$$

with a, b, c as defined in (3.3)–(3.5). Let $V_\beta^\pm$ denote the positive (resp. negative) eigenspace associated to H_β, and recall the kernel V_k of H_{β_k} given in (3.7). For $0 < \beta_k < \mu_1$ the lemma follows from the following two claims.

Claim 1 $m(\beta) = m(w) + 1$ *for* $\beta < \mu_1$ *and close to* μ_1.

Claim 2 $m(0) = 2m(w)$

Postponing the proofs of these claims we first deduce $i_k = n_k$ in the range $0 < \beta < \mu_1$. By Lemma 3.1 $m(\beta)$ can only change at $\beta = \beta_k$ and the change is at most n_k. Moreover, $0 < \beta_k < \mu_1$ is equivalent to $1 < f(\beta_k) = \lambda_k < 3$, i.e. $2 \le k \le k_0$. From Claim 1 and Claim 2 it follows that for $\beta_2 < \beta < \mu_1$ we have

$$m(w) - 1 = m(0) - m(\beta) = i_2 + \cdots + i_{k_0} \le n_2 + \cdots + n_{k_0} = m(w) - 1$$

and hence, $i_k = n_k$ for $2 \le k \le k_0$.

Proof of Claim 1 Let $W^- \subset E$ be the eigenspace of (2.1) associated to the eigenvalues $1 = \lambda_1 < \lambda_2 < \cdots < \lambda_{k_0} < 3$ and W^+ the eigenspace of (2.1) associated to the eigenvalues $3 < \lambda_{k_0+1} < \lambda_{k_0+2} < \cdots$. Then we have

$$\int_\Omega w^2\phi^2 \le \int_\Omega \left(|\nabla\phi|^2 + \phi^2\right) \le \lambda_{k_0} \int_\Omega w^2\phi^2 < 3 \int_\Omega w^2\phi^2 \quad \text{for } \phi \in W^- \setminus \{0\}, \tag{3.9}$$

and

$$\int_\Omega \left(|\nabla\phi|^2 + \phi^2\right) \ge \lambda_{k_0+1} \int_\Omega w^2\phi^2 > 3 \int_\Omega w^2\phi^2 \quad \text{for } \phi \in W^+ \setminus \{0\}. \tag{3.10}$$

We claim that H_β is negative definite on the space $W^- \times \mathbb{R}w \subset E \times E$ and positive definite on the orthogonal complement $W^+ \times (\mathbb{R}w)^\perp$. Looking at (3.8) and using (3.9), (3.10), this follows easily from $a(\beta) \to 3$, $b(\beta) \to 0$, and $c(\beta) \to 1$ as $\beta \to \mu_1$. □

Proof of Claim 2 The claim follows in the same way using that $a(0) = 3 = c(0)$ and $b(0) = 0$. H_0 is negative definite on $W^- \times W^-$ and positive definite on $W^+ \times W^+$. □

For $-\sqrt{\mu_1\mu_2} < \beta_k < 0$ the equality $i_k = n_k = \dim V_k$ follows immediately from the following two claims.

Claim 3 *For $\beta > \beta_k$ and close to β_k, H_β is positive definite on $V_{\beta_k}^+ \oplus V_k$ and negative definite on $V_{\beta_k}^-$.*

Claim 4 *For $\beta < \beta_k$ and close to β_k, H_β is positive definite on $V_{\beta_k}^+$ and negative definite on $V_{\beta_k}^- \oplus V_k$.*

Both claims follow from $H_\beta = H_{\beta_k} + (\beta - \beta_k)H'_{\beta_k} + o(|\beta - \beta_k|)$ for $\beta \to \beta_k$ if we can show that the derivative $H'_{\beta_k} = \frac{\partial}{\partial\beta} H_\beta|_{\beta=\beta_k}$ is positive definite on the kernel V_k. The derivative is simply given by

$$H'_\beta[(\phi, \psi)^2] = -\int\limits_\Omega \left(a'(\beta)\phi^2 + 2b'(\beta)\phi\psi + c'(\beta)\psi^2\right) w^2.$$

Let $(\phi, \psi) = (\gamma_-(\beta_k)\psi, \psi) \in V_k \setminus \{0\}$ be an arbitrary nontrivial element of the kernel (see (3.7)). So $\psi \in E \setminus \{0\}$ is an eigenfunction of (2.1) associated to λ_k and

$$\gamma_-(\beta) = \frac{a(\beta) - c(\beta)}{2b(\beta)} - \frac{1}{2b(\beta)}\sqrt{(a(\beta) - c(\beta))^2 + 4b^2(\beta)}$$

is as in (3.6). We have to show that

$$\begin{aligned} H'_\beta[(\gamma_-(\beta)\psi, \psi)^2] &= -\int\limits_\Omega \left(a'(\beta)(\gamma_-(\beta)\psi)^2 + 2b'(\beta)\gamma_-(\beta)\psi^2 + c'(\beta)\psi^2\right) w^2 \\ &= -\left(a'(\beta)\gamma_-^2(\beta) + 2b'(\beta)\gamma_-(\beta) + c'(\beta)\right)\int\limits_\Omega w^2\psi^2 \\ &> 0 \end{aligned}$$

for $\beta = \beta_k$. Clearly $\gamma_-(\beta) < 0$ for all β so it is sufficient to prove that $a'(\beta_k) < 0$, $b'(\beta_k) > 0$, and $c'(\beta_k) < 0$. For a we have

$$a'(\beta) = -\frac{2\mu_1(\mu_1\mu_2 - 2\beta\mu_2 + \beta^2)}{(\mu_1\mu_2 - \beta^2)^2} < 0$$

provided $-\sqrt{\mu_1\mu_2} < \beta < 0$, which is the case for the β_k's which we consider here. For b we get

$$b'(\beta) = \frac{2\mu_1^2\mu_2^2 - 4(\mu_1 + \mu_2)\mu_1\mu_2\beta + 4\mu_1\mu_2\beta^2 - 2(\mu_1 + \mu_2)\beta^3 + \beta^4}{(\mu_1\mu_2 - \beta^2)^2(\mu_1 - \beta)^{1/2}(\mu_2 - \beta)^{1/2}} > 0$$

for $-\sqrt{\mu_1\mu_2} < \beta < 0$. And finally, for c we have

$$c'(\beta) = -\frac{2\mu_2(\mu_1\mu_2 - 2\beta\mu_1 + \beta^2)}{(\mu_1\mu_2 - \beta^2)^2} < 0$$

provided $-\sqrt{\mu_1\mu_2} < \beta < 0$. □

In order to prove Theorem 2.1 we shall apply classical bifurcation results going back to Krasnoselski [18] and Rabinowitz [31]. However, we need to guarantee that the bifurcating critical points of J_β are in fact positive. In order to achieve this we modify the problem and consider the functional $J_\beta^+ : E \times E \to \mathbb{R}$ defined by

$$
\begin{aligned}
J_\beta^+(u,v) &= \frac{1}{2}\int_\Omega (|\nabla u|^2 + |\nabla v|^2 + u^2 + v^2) - \frac{1}{4}\int_\Omega (\mu_1 u_+^4 + \mu_2 v_+^4) - \frac{\beta}{2}\int_\Omega (u_+^2 v_+^2)\\
&= \frac{1}{2}\|(u,v)\|^2 - \frac{1}{4}\left(\mu_1 |u_+|_4^4 + \mu_2 |v_+|_4^4\right) - \frac{\beta}{2}\int_\Omega u_+^2 v_+^2.
\end{aligned}
$$

Here u_+ and v_+ are the positive parts of u and v, and $|\,.\,|_p$ denotes the L^p-norm. It is standard to prove that J_β^+ is of class C^{2-0} and satisfies the Palais-Smale condition. The Euler-Lagrange equation associated to J_β is a modification of (1.2):

$$
\begin{cases}
-\Delta u + u = \mu_1 u_+^3 + \beta v_+^2 u_+ & \text{in } \Omega\\
-\Delta v + v = \mu_2 v_+^3 + \beta u_+^2 v_+ & \text{in } \Omega\\
u, v \in H_0^1(\Omega),
\end{cases}
\tag{3.11}
$$

This system has only nonnegative solutions as can be seen by multiplying the first equation with u_-, the second with v_- and integrating. Consequently every solution of (3.11) is a solution of (1.2). And every non-negative solution of (1.2) is also a solution of (3.11). This applies in particular to the elements of $\mathcal{T}_w$.

We need to recall the concept of critical groups (see e.g., [9,25]). For an isolated critical point (u,v) of J_β^+ with $J_\beta^+(u,v) = c$ the critical groups are defined by

$$
C_*(J_\beta^+, (u,v)) := H_*((J_\beta^+)^c, (J_\beta^+)^c \setminus \{(u,v)\}).
$$

Here H_* denotes singular homology with coefficients in a field.

Lemma 3.3 *For $\beta \in (-\sqrt{\mu_1\mu_2}, \mu_1) \setminus \{\beta_k : k \in \mathbb{N}\}$ ($\beta > -\mu$ if $\mu = \mu_1 = \mu_2$) the critical groups of (u_β, v_β) are given by $\dim C_k(J_\beta^+, (u_\beta, v_\beta)) = \delta_{km(\beta)}$, and the local degree by $\deg(\nabla J_\beta^+, (u_\beta, v_\beta)) = (-1)^{m(\beta)}$. Here $m(\beta)$ is the index of the quadratic form $H_\beta = D^2 J_\beta(u_\beta, v_\beta)$ from (3.8).*

Recall that due to the compact embedding of E into $L^4(\Omega)$, the gradient of J_β^+ is a compact perturbation of $\mathrm{id}_{E\times E}$, so the Leray–Schauder degree can be applied. By Lemma 3.3 the critical groups of (u_β, v_β) considered as critical point of J_β or of J_β^+ are identical. The same holds for the local degrees of ∇J_β or of ∇J_β^+ at (u_β, v_β). The computation of the critical groups and the local degree of (u_β, v_β) with J_β^+ replaced by J_β is easy because ∇J_β is of class C^1. The argument for J_β^+ is a bit more complicated because ∇J_β^+ is not differentiable, not even at (u_β, v_β).

Proof Let $V_\beta^\pm$ be the positive (resp. negative) eigenspace of H_β. In particular, $\dim V_\beta^- = m(\beta)$ and $V_\beta^- + V_\beta^+ = E \times E$. Then there exist subspaces $W_\beta^\pm \subset C_0^\infty(\Omega)$ with $\dim W_\beta^- = m(\beta)$, $\mathrm{clos}(W_\beta^- + W_\beta^+) = E \times E$, and such that H_β is negative definite on W_β^- and positive definite on W_β^+. Let $w_n \in W_\beta^+$ be such that $\mathrm{span}\{w_n : n \in \mathbb{N}\} = W_\beta^+$ and set $W_\beta^n := W_\beta^- + \mathrm{span}\{w_k : k = 1\ldots, n\}$. Then J_β^+ coincides with J_β in a neighborhood $U \subset (u_\beta, v_\beta) + W_\beta^n$ of (u_β, v_β) in $(u_\beta, v_\beta) + W_\beta^n$. Consequently, $J_\beta^+|_U$ is of class C^2 and has (u_β, v_β) as a nondegenerate critical point with Morse index $m(\beta)$. Now [7, Theorem I.5.10] yields $\dim C_k(J_\beta^+, (u_\beta, v_\beta)) = \delta_{km(\beta)}$. This in turn implies that the local degree of ∇J_β^+ at (u_β, v_β) is $(-1)^{m(\beta)}$; see [19, Theorem 3.2]. □

Proof of Theorem 2.1 By Lemma 3.2 and Lemma 3.3 the bifurcation theorem for variational maps as formulated in [25, Theorem 8.9] applies and yields that each β_k is in fact a bifurcation parameter for critical points of J_β^+. The maximum principle implies that these critical points must be strictly positive, hence they are solutions of (1.2).

If the multiplicity n_k of λ_k is odd then the crossing number i_k is not zero by Lemma 3.2 and the local degree of (u_β, v_β) as zero of ∇J_β^+ changes. Then we can apply Rabinowitz' global bifurcation theorem; see [31] and [17, Theorem II.3.3]. In fact, a straightforward modification of it yields a connected set $\mathcal{S}_k$ of critical points (β, u, v) of J_β^+ bifurcating from $(\beta_k, u_{\beta_k}, v_{\beta_k})$, and $\mathcal{S}_k$ is either unbounded or returns to $\mathcal{T}_w$. If one of the components u, v is not strictly positive, then by the maximum principle this component would be 0. That means, there would be bifurcation from one of the (semi-)trivial branches

$$\mathcal{T}_0 := \{(\beta, 0, 0) \in \mathbb{R} \times E \times E : \beta \in \mathbb{R}\},$$
$$\mathcal{T}_1 := \{(\beta, \mu_1^{-1/2} w, 0) \in \mathbb{R} \times E \times E : \beta \in \mathbb{R}\},$$

or

$$\mathcal{T}_2 := \{(\beta, 0, \mu_2^{-1/2} w) \in \mathbb{R} \times E \times E : \beta \in \mathbb{R}\}.$$

It is clear that there is no bifurcation from $\mathcal{T}_0$. Due to the results in [6] there is only one bifurcation point on $\mathcal{T}_1$ that produces nonnegative solutions. This is at $\beta = \mu_1$ where bifurcation from a simple eigenvalue takes place; see the proof of [6, Lemma 2.2]. According to the Crandall–Rabinowitz theorem (see [11] or [17, Theorem I.5.1]) there is locally a unique bifurcating branch which, in the case $\mu_1 < \mu_2$, must be our trivial branch $\mathcal{T}_w \cap \big((-\sqrt{\mu_1\mu_2}, \mu_1) \times E \times E\big)$, so $\overline{\mathcal{S}}_k \cap \mathcal{T}_1 = \emptyset$. Similarly, there is only one bifurcation point on $\mathcal{T}_2$ where nonnegative solutions bifurcate, namely at $\beta = \mu_2$. Again we have bifurcation from a simple eigenvalue and the unique bifurcating branch here is $\mathcal{T}_w \cap ((\mu_2, \infty) \times E \times E)$ in the case $\mu_1 < \mu_2$, so $\overline{\mathcal{S}}_k \cap \mathcal{T}_2 = \emptyset$. If $\mu_1 = \mu_2$ then $\overline{\mathcal{S}}_k \cap \mathcal{T}_1 = \emptyset = \overline{\mathcal{S}}_k \cap \mathcal{T}_2$ holds for $k \geq 2$ according to Remark 2.2c). It follows that all solutions on $\mathcal{S}_k$ must be strictly positive, hence they are solutions of (1.2).

Finally, if $\mathcal{S}_k$ is bounded there exists a solution $(\beta, u, v) \in \partial\mathcal{S}_k \setminus \{(\beta_k, u_{\beta_k}, v_{\beta_k})\}$. There are two possibilities: Either $(\beta, u, v) \in \mathcal{T}_w \setminus \{(\beta_k, u_{\beta_k}, v_{\beta_k})\}$, and we are done, or one of the components u, v is not strictly positive. In the latter case, by the maximum principle this component would then be 0 and we would have bifurcation from one of the (semi-)trivial branches $\mathcal{T}_0$, $\mathcal{T}_1$ or $\mathcal{T}_2$, which is not possible as shown above. □

4 Proof of Theorems 2.3, 2.5 and 2.6

We begin with the proof of the Liouville type theorem.

Proof of Theorem 2.6 Let (u, v) be a classical radial solution of the system (2.3) such that $(\mu_1 - \beta)^{1/2} u - (\mu_2 - \beta)^{1/2} v$ has only finitely many zeroes. If $\beta > -\sqrt{\mu_1\mu_2}$ then $u = v = 0$ according to [13, Theorem 2.1]. In fact, for this range of β problem (2.3) has no classical nontrivial solution at all. Thus we only need to consider the case $\beta \leq -1$. The argument below works for $\beta < \mu_1 \leq \mu_2$. We consider the case c is finite, the case $c = \infty$ is similar and simpler.

Suppose $(u, v) \neq (0, 0)$. Setting

$$\alpha := \left(\frac{\mu_1 - \beta}{\mu_2 - \beta}\right)^{1/2} \tag{4.1}$$

we claim that the difference $\alpha u - v$ must have infinitely many zeroes. The function $\alpha u - v$ solves the equation

$$\begin{aligned}-(\alpha u - v)'' - \frac{N-1}{c+r}(\alpha u - v)' &= \alpha\mu_1 u^3 - \beta u^2 v + \alpha\beta u v^2 - \mu_2 v^3 \\ &= \left(\mu_1 u^2 + (\mu_1 - \beta)^{1/2}(\mu_2 - \beta)^{1/2} uv + \mu_2 v^2\right)(\alpha u - v)\end{aligned}$$

as a simple calculation shows. Setting $f = \alpha u - v$ and

$$q = \mu_1 u^2 + (\mu_1 - \beta)^{1/2}(\mu_2 - \beta)^{1/2} uv + \mu_2 v^2$$

we obtain the simple equation

$$-f'' - \frac{N-1}{c+r} f' = q(r) f. \tag{4.2}$$

Claim 1 *Given $r_0 > -c$ such that $f(r_0) \ge 0$ and $f'(r_0) > 0$ there exists $s_0 > r_0$ with $f'(r) > 0$ for $r_0 < r < s_0$, $f'(s_0) = 0$.*

Proof Since $f'(r_0) > 0$ we may assume that $c_0 := f(r_0) > 0$. Now we define

$$s_0 := \sup\{s > r_0 : f'(r) > 0 \;\text{ for } r_0 \le r \le s\} \in (r_0, \infty]$$

and observe that f is strongly increasing on the interval (r_0, s_0). Then we have

$$u(r) > \frac{f(r)}{\alpha} \ge \frac{f(r_0)}{\alpha} = \frac{c_0}{\alpha} > 0 \quad \text{for all } r \in (r_0, s_0)$$

and therefore $q(r) \ge \mu_1 u^2(r) \ge \mu_1 c_0^2/\alpha^2$ for $r \in (r_0, s_0)$. This in turn yields

$$f''(r) = -\frac{N-1}{c+r} f'(r) - q(r)f(r) \le -q(r)f(r) \le -\mu_1 c_0^3/\alpha^2 \quad \text{for } r \in (r_0, s_0),$$

hence $s_0 < \infty$. □

Claim 2 *Given $s_0 > -c$ such that $f(s_0) > 0$ and $f'(s_0) \le 0$ there exists $r_1 > s_0$ with $f(r) > 0$ for $s_0 < r < r_1$, $f(r_1) = 0$.*

Proof If $f'(s_0) = 0$ then $f''(s_0) = -\frac{N-1}{c+s_0} f'(s_0) - q(s_0)f(s_0) < 0$, so increasing s_0 we may assume that $f'(s_0) < 0$. Now we define

$$r_1 := \sup\{r > s_0 : f(s) > 0 \;\text{ for } s_0 \le s \le r\} \in (s_0, \infty]$$

and want to show that $r_1 < \infty$. Observe that

$$((c+r)^{N-1} f'(r))' = -(c+r)^{N-1} q(r) f(r) < 0. \tag{4.3}$$

Therefore $(c+r)^{N-1} f'$ is strictly decreasing on the interval (s_0, r_1). For $N = 1$ or $N = 2$ this implies easily $r_1 < \infty$.

It remains to consider the case $N = 3$. Suppose to the contrary that $r_1 = \infty$, hence $f(r) > 0$ for $r > s_0$. Below c_i denotes various positive constants. We first claim that

$$f(r) \to 0 \quad \text{as } r \to \infty. \tag{4.4}$$

(4.3) implies $(c+r)^2 f' < 0$, hence $f' < 0$ in $[s_0, \infty)$, and therefore $f(r) \to c_1 \ge 0$ as $r \to \infty$. If $c_1 > 0$ then f, hence u, q and qf are bounded away from 0 in $[s_0, \infty)$. Now (4.3) implies $((c+r)^2 f'(r))' \le -c_2(c+r)^2$ and thus $(c+r)^2 f'(r) \le -c_3(c+r)^3$ for r large. This implies $f'(r) \to -\infty$ as $r \to \infty$, hence $r_1 < \infty$, a contradiction.

Next we claim that

$$(c+r)^2 f'(r) \to -\infty \quad \text{as } r \to \infty. \tag{4.5}$$

In order to see this, observe that (4.2) implies $((c+r)f)'' < 0$ in $[s_0, \infty)$, and consequently $((c+r)f)' > 0$ because $(c+r)f > 0$ in $[s_0, \infty)$. It follows that $f(r) > c_1/(c+r)$, hence $q(r) > c_2/(c+r)^2$ and

$$(c+r)^2 f'(r) = c_3 + \int_{r_0}^{r} ((c+s)^2 f'(s))'\, ds = c_3 - \int_{r_0}^{r} (c+s)^2 q(s) f(s)\, ds$$

$$< c_3 - \int_{r_0}^{r} \frac{c_4}{c+s}\, ds \to -\infty \quad \text{as } r \to \infty.$$

Next we prove that

$$(c+r)^2 q(r) \to \infty \quad \text{as } r \to \infty. \tag{4.6}$$

By (4.5), for any $C > 0$ there exists $R(C) > 0$ such that $f'(r) < -C/(c+r)^2$ for $r > R(C)$. Using (4.4) it follows that

$$f(r) = -\int_{r}^{\infty} f'(s) ds \geq \int_{r}^{\infty} \frac{C}{(c+s)^2}\, ds = \frac{C}{c+r},$$

hence $(c+r)f(r) > C$ and $(c+r)^2 q(r) > C^2/\alpha^2$ for $r > R(C)$. Since $C > 0$ was arbitrary, (4.6) follows.

Now (4.6) implies that the differential operator $D := -\frac{d}{dt}\left((c+r)^2 \frac{d}{dt}\right) - (c+r)^2 q$ on $L^2((s_0, \infty))$ is unbounded below. Then [14, Theorem XIII.7.40] implies that f being a solution of $Df = 0$ has arbitrarily large zeroes, contradicting the assumption $r_1 = \infty$. This proves Claim 2. □

We have proved that given $r_0 > -c$ with $f(r_0) \geq 0$ and $f(r) > 0$ for $r > r_0$ close to r_0 there exists $r_1 > r_0$ with $f(r_1) = 0$ and $f'(r_1) < 0$. Using analogous arguments one sees that given $r_1 > -c$ with $f(r_1) \leq 0$ and $f(r) < 0$ for $r > r_1$ close to r_1 there exists $r_2 > r_1$ with $f(r_2) = 0$ and $f'(r_2) > 0$. It follows that $f = \alpha u - v$ has infinitely many zeroes. This completes the proof of the Theorem. □

Remark 4.1 Claim 2 in the proof of Theorem 2.6 in the case $N = 2, 3$ can also be derived from [8, Theorem 3.3(iii)] which asserts that $-\Delta u \geq u^q$ has no positive solution in the exterior of a ball if $q \leq \frac{N}{N-2}$. Using the definition of f and $c \geq 0$, if $f' \leq 0$ Eq. (4.2) yields the inequality $-\Delta f \geq \mu_1 \alpha^2 f^3$. It follows readily that f cannot be positive for all r large, so f has to have infinitely many zeroes.

Now we turn to the

Proof of Theorem 2.5 This is done by a standard blow-up argument. In dimensions $N = 2$ and $N = 3$ we write the system in the radial variable $r = |x|$ for $r \in (a, b)$ with $0 \leq a < b \leq \infty$. Suppose there exists a sequence (β_n, u_n, v_n) of (radial) solutions of (1.2) with $\beta_n \to \beta$, $\|u_n\|_\infty \to \infty$ and such that the difference $(\mu_1 - \beta_n)^{1/2} u_n - (\mu_2 - \beta_n)^{1/2} v_n$ has at most k zeroes for every $n \in \mathbb{N}$. We may assume that $\|v_n\|_\infty \leq \|u_n\|_\infty$ and choose r_n, such that $u_n(r_n) = \|u_n\|_\infty$. Now we set $\varepsilon_n := \|u_n\|_\infty^{-1}$ and $\tilde{u}_n(r) := \varepsilon_n u_n(r_n + \varepsilon_n r)$, $\tilde{v}_n(r) := \varepsilon_n v_n(r_n + \varepsilon_n r)$. Then clearly $\tilde{u}_n, \tilde{v}_n$ are bounded in L^∞ and satisfy the system

$$\begin{cases} -\tilde{u}_n'' - \dfrac{\varepsilon_n(N-1)}{r_n+\varepsilon_n r}\tilde{u}_n' + \varepsilon_n^2\tilde{u}_n = \mu_1\tilde{u}_n^3 + \beta_n\tilde{v}_n^2\tilde{u}_n \\ -\tilde{v}_n'' - \dfrac{\varepsilon_n(N-1)}{r_n+\varepsilon_n r}\tilde{v}_n' + \varepsilon_n^2\tilde{v}_n = \mu_2\tilde{v}_n^3 + \beta_n\tilde{u}_n^2\tilde{v}_n \end{cases} \tag{4.7}$$

on the scaled domain $\frac{a-r_n}{\varepsilon_n} < r < \frac{b-r_n}{\varepsilon_n}$.

If $N = 1$ let the domain be (a, b) with $-\infty \le a < b \le \infty$. Then, after passing to a subsequence, $\frac{a-r_n}{\varepsilon_n}$ and $\frac{b-r_n}{\varepsilon_n}$ converge in $[-\infty, \infty]$, and $(\tilde{u}_n, \tilde{v}_n)$ converge in C^2_{loc} as $n \to \infty$ towards a solution (u, v) of

$$\begin{cases} -u'' = \mu_1 u^3 + \beta v^2 u, \\ -v'' = \mu_2 v^3 + \beta u^2 v, \\ u, v \ge 0. \end{cases} \tag{4.8}$$

Here u and v are defined on an interval of the following possible forms: $(-\infty, \infty)$, $(-c, \infty)$ with $c \ge 0$, and $(-\infty, c)$ with $c \ge 0$. But for the last possibility $(u(-r), v(-r))$ solves (4.8) on $(-c, \infty)$ reducing to the second possibility. In any case, we obtain a solution (u, v) of (2.3) with $N = 1$ which is nontrivial because $u(0) = \lim_{n\to\infty} \tilde{u}_n(0) = 1$. Observe that $(\mu_1 - \beta)^{1/2}u - (\mu_2 - \beta)^{1/2}v$ can have at most k simple zeroes because this holds true for all $(\mu_1 - \beta)^{1/2}\tilde{u}_n - (\mu_2 - \beta)^{1/2}\tilde{v}_n$. This contradicts the Liouville theorem 2.6.

Now we consider the dimensions $N = 2$ or $N = 3$. Up to a subsequence we may assume $r_n/\varepsilon_n \to c \in [0, \infty]$ as $n \to \infty$. Suppose first $r_n/\varepsilon_n \to \infty$ along a subsequence, so that $\frac{\varepsilon_n(N-1)}{r_n+\varepsilon_n r} \to 0$. Then $(\tilde{u}_n, \tilde{v}_n)$ converge in C^2_{loc} along a subsequence towards a solution (u, v) of (4.8) on domains of three possible forms: $(-\infty, \infty)$, $(-c, \infty)$ with $c \ge 0$, and $(-\infty, c)$ with $c \ge 0$. As above we may reduce the third to the second possibility and obtain a contradiction with Theorem 2.6 because the solution is nontrivial and $(\mu_1 - \beta)^{1/2}u - (\mu_2 - \beta)^{1/2}v$ has at most k simple zeroes.

It remains to consider the case where $r_n/\varepsilon_n \to c \in [0, \infty)$ along a subsequence, so that $\frac{\varepsilon_n(N-1)}{r_n+\varepsilon_n r} \to \frac{N-1}{c+r}$. Then after passing to a subsequence, $(\tilde{u}_n, \tilde{v}_n)$ converge in C^2_{loc} as $n \to \infty$ towards a solution (u, v) of (2.3). Since $\varepsilon_n \to 0$ we must have $r_n \to 0$ and $a = 0$ which implies that (u, v) solves (2.3) on $(0, \infty)$. Again we obtain a contradiction to the Liouville theorem 2.6. □

Finally we give the

Proof of Theorem 2.3 In the one-dimensional and the radial setting all eigenvalues are simple, so each bifurcating branch $\mathcal{S}_k$ must be global. Now for $(\beta, u, v) \in \mathcal{S}_k$ near the bifurcation point $(\beta_k, u_{\beta_k}, v_{\beta_k})$ the proofs of Lemma 3.1 and Lemma 3.2 imply

$$u = u_{\beta_k} + (\beta - \beta_k)\gamma_-(\beta_k)\phi_k + o(\beta - \beta_k)$$

and

$$v = v_{\beta_k} + (\beta - \beta_k)\phi_k + o(\beta - \beta_k)$$

as $\beta \to \beta_k$. Here $\gamma_-(\beta_k)$ is given in (3.6) and ϕ_k is the k-th eigenfunction of (2.1). With α as in (4.1) we claim that

$$\alpha u - v = (\beta - \beta_k)\alpha\phi_k + o(\beta - \beta_k)$$

has precisely $k - 1$ simple zeroes provided β is close to β_k. Here we first note that ϕ_k has precisely $k - 1$ simple zeroes (see Theorem XIII.7.53 and Corollary 7.56. of [14] for a related case, and also [12]). Now $f = \alpha u - v$ solves, in radial coordinates, the equation

$$\begin{aligned} -f'' - \frac{N-1}{r} f' + f &= \alpha\mu_1 u^3 + \alpha\beta v^2 u - \mu_2 v^3 - \beta u^2 v \\ &= \big(\mu_1 u^2 + (\mu_1 - \beta)^{1/2}(\mu_2 - \beta)^{1/2} uv + \mu_2 v^2\big) \cdot f \\ &=: q(r) f. \end{aligned}$$

This implies that f cannot have a double zero because otherwise $f = 0$, hence $\alpha u = v$, which in turn implies $u = u_\beta$, $v = v_\beta$. Now we bootstrap the perturbation term $o(\beta - \beta_k)$ from the H^1-norm to the C^1-norm, so (u, v) converges to $(u_{\beta_k}, v_{\beta_k})$ in the C^1-norm as $\beta \to \beta_k$. If the domain is bounded we easily deduce the claim. If the domain is unbounded and f has more than $k - 1$ zeroes then there have to be zeroes of f moving to infinity as $\beta \to \beta_k$. Then there exist a positive maximum (or a negative minimum) of f moving to infinity as $\beta \to \beta_k$. Using the fact that u and v both go to zero as $r \to \infty$ uniformly for β close to β_k we get $-f'' + f = q(r) f$ with $q(r) < 1$ at a large positive maximum (or negative minimum) r of f, which is not possible. The claim is proved.

It follows from the same argument that $\alpha u - v$ has precisely $k - 1$ simple zeroes for every $(\beta, u, v) \in \overline{\mathcal{S}_k} \setminus \{(\beta_k, u_{\beta_k}, v_{\beta_k})\}$. As a consequence, $\overline{\mathcal{S}_k} \cap \mathcal{T}_w = \{(\beta_k, u_{\beta_k}, v_{\beta_k})\}$, and $\mathcal{S}_k$ must be unbounded. Now Theorem 2.5 implies that $\mathcal{S}_k$ must be unbounded in the β-direction, i.e., $pr_1(\mathcal{S}_k) \subset \mathbb{R}$ is unbounded. Since the branch $\mathcal{S}_k$ cannot approach to $\mathcal{T}_i$ for $\beta \le 0$ with $i = 0, 1, 2$ and since for $\beta = 0$ the only positive solution to (1.2) is (u_0, v_0) it follows that $pr_1(\mathcal{S}_k) \subset (-\infty, 0)$, hence $pr_1(\mathcal{S}_k) \supset (-\infty, \beta_k)$. □

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