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Équations aux dérivées partielles/*Partial Differential Equations*

On the Caffarelli–Kohn–Nirenberg inequalities

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Abstract. Consider the following inequalities due to Caffarelli, Kohn and Nirenberg [3]:

$$\left(\int_{\mathbb{R}^N} |x|^{-bp}|u|^p \, \mathrm{d}x\right)^{2/p} \leqslant C_{a,b} \int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2 \, \mathrm{d}x,$$

where for $N \geqslant 3$: $-\infty < a < \frac{N-2}{2}$, $a \leqslant b \leqslant a+1$, and $p = \frac{2N}{N-2+2(b-a)}$. We shall answer some fundamental questions concerning these inequalities such as the best embedding constants, the existence and nonexistence of extremal functions, and their qualitative properties. We also study the bound state solutions of the corresponding Euler equations and construct positive solutions having prescribed symmetry for certain parameter region. © 2000 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Sur les inégalités de Caffarelli–Kohn–Nirenberg

Résumé. *On considère les inégalités suivantes de Caffarelli, Kohn et Nirenberg* [3] :

$$\left(\int_{\mathbb{R}^N} |x|^{-bp}|u|^p \, \mathrm{d}x\right)^{2/p} \leqslant C_{a,b} \int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2 \, \mathrm{d}x,$$

où $N \geqslant 3$: $-\infty < a < \frac{N-2}{2}$, $a \leqslant b \leqslant a+1$ *et* $p = \frac{2N}{N-2+2(b-a)}$. *Nous répondons à quelques questions fondamentales concernant ces inégalités, telles que meilleures constantes d'injection, existence ou non-existence des fonctions extrémales et leur propriétés qualitatives. On étudie aussi les solutions à état borné qui correspondent à l'équation d'Euler et on construit des solutions positives avec une symétrie prescrite dans certaines régions de l'espace des paramètres.* © 2000 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

Dans [3], entre autres choses, Caffarelli, Kohn et Nirenberg ont établi les inégalités suivantes : pour tout $u \in C_0^\infty(\mathbb{R}^N)$,

$$\left(\int_{\mathbb{R}^N} |x|^{-bp}|u|^p \, \mathrm{d}x\right)^{2/p} \leqslant C_{a,b} \int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2 \, \mathrm{d}x, \tag{1}$$

où, pour $N \geqslant 3$:

$$-\infty < a < (N-2)/2, \quad a \leqslant b \leqslant a+1 \quad \text{et} \quad p = 2N/\left[N-2+2(b-a)\right]. \tag{2}$$

Note présentée par Louis NIRENBERG.

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Soit $\mathcal{D}_a^{1,2}(\mathbb{R}^N)$ la complétion de $\mathrm{C}_0^\infty(\mathbb{R}^N)$ pour le produit scalaire : $(u,v)=\int_{\mathbb{R}^N}|x|^{-2a}\nabla u\cdot\nabla v\,\mathrm{d}x$.

On voit que (1) est vérifiée pour $u\in\mathcal{D}_a^{1,2}(\mathbb{R}^N)$. On définit $S(a,b)=\inf_{u\in\mathcal{D}_a^{1,2}(\mathbb{R}^N)\setminus\{0\}}E_{a,b}(u)$, comme étant les meilleures constantes d'inclusion, où

$$E_{a,b}(u)=\left(\int_{\mathbb{R}^N}|x|^{-2a}|\nabla u|^2\,\mathrm{d}x\right)\times\left(\int_{\mathbb{R}^N}|x|^{-bp}|u|^p\,\mathrm{d}x\right)^{-2/p}.$$

Les fonctions extrémales pour $S(a,b)$ sont des solutions minimales des équations d'Euler :

$$-\mathrm{div}\big(|x|^{-2a}\nabla u\big)=|x|^{-bp}u^{p-1},\quad u\geqslant 0,\quad \text{dans } \mathbb{R}^N. \tag{3}$$

Beaucoup de progrès ont été faits dans le cas où $a\geqslant 0$, [1,15,11,9] (*voir* aussi [13,17]). L'objet de cette Note est l'étude du cas $a<0$ qui n'a pas été très étudié.

Pour $a<0$, on montrera de nouveaux phénomènes qui sont en contraste avec ceux correspondant à $a\geqslant 0$. Pour énoncer les résultats, soient $S_p(\mathbb{R}^N)$ les meilleures constantes d'injection de $\mathrm{H}^1(\mathbb{R}^N)$ dans $\mathrm{L}^p(\mathbb{R}^N)$, i.e. $S_p(\mathbb{R}^N)=\inf_{u\in\mathrm{H}^1(\mathbb{R}^N)\setminus\{0\}}\big(\int_{\mathbb{R}^N}|\nabla u|^2+u^2\,\mathrm{d}x\big)\times\big(\int_{\mathbb{R}^N}|u|^p\,\mathrm{d}x\big)^{-2/p}$.

THÉORÈME 1 (Meilleures constantes et non-existence des fonctions extrémales). – (i) $S(a,b)$ *est continue dans le domaine entier des paramètres* (2). (ii) *Pour* $b=a+1$*, on a* $S(a,a+1)=[(N-2-2a)/2]^2$*, et* $S(a,a+1)$ *n'est pas atteint.* (iii) *Pour* $a<0$ *et* $b=a$*, on a* $S(a,a)=S(0,0)$ (*la meilleure constante de Sobolev*) *et* $S(a,a)$ *n'est pas atteint.*

THÉORÈME 2 (Meilleures constantes et existence des fonctions extrémales). – (i) *Pour* $a<b<a+1$, $S(a,b)$ *est toujours atteint.* (ii) *Pour* $b-a\in(0,1)$ *fixe, quand* $a\to-\infty$, $S(a,b)$ *est strictement croissante, et on a* :

$$S(a,b)=\big[(N-2-2a)/2\big]^{2(b-a)}\big[S_p(\mathbb{R}^N)+\mathrm{o}(1)\big].$$

THÉORÈME 3 (Brisure de symétrie). – *Il existe une fonction* $h(a)$ *définie pour* $a\leqslant 0$*, vérifiant* $h(0)=0$, $a<h(a)<a+1$ *pour* $a<0$*, et* $a+1-h(a)\to 0$ *quand* $-a\to\infty$*, telle que pour tout* (a,b), $a<0$ *et* $a<b<h(a)$*, le minimum* $S(a,b)$ *est atteint par une fonction qui n'est pas invariante par rotations.*

THÉORÈME 4 (Proprieté de symétrie). – *Pour* $a<b<a+1$*, le minimum* $S(a,b)$ *est atteint par une fonction* $u(x)$*, qui après une éventuelle dilatation* $u(x)\to\tau^{(N-2-2a)/2}u(\tau x)$*, satisfait la symétrie qu'on appellera* inversion modifiée : $u\big(|x|^{-2}x\big)=|x|^{N-2-2a}u(x)$.

THÉORÈME 5 (Multiplicité asymptotique). – *Soit* $b-a\in(0,1)$ *fixé, alors pour tout entier positif* k*, il existe* $a_k<0$ *tel que* $a<a_k$ *implique* (3) *a une solution à* k*-bosses.*

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1. Introduction

In [3], among other things, Caffarelli, Kohn, and Nirenberg established the following inequalities: for all $u\in\mathrm{C}_0^\infty(\mathbb{R}^N)$,

$$\left(\int_{\mathbb{R}^N}|x|^{-bp}|u|^p\,\mathrm{d}x\right)^{2/p}\leqslant C_{a,b}\int_{\mathbb{R}^N}|x|^{-2a}|\nabla u|^2\,\mathrm{d}x, \tag{1}$$

where, for $N\geqslant 3$:

$$-\infty<a<(N-2)/2,\quad a\leqslant b\leqslant a+1,\quad \text{and}\quad p=2N/[N-2+2(b-a)]. \tag{2}$$

Let $\mathcal{D}_a^{1,2}(\mathbb{R}^N)$ be the completion of $\mathrm{C}_0^\infty(\mathbb{R}^N)$, with respect to the inner product:

$$(u,v)=\int_{\mathbb{R}^N}|x|^{-2a}\nabla u\cdot\nabla v\,\mathrm{d}x.$$

Then we see that (1) holds for $u \in \mathcal{D}_a^{1,2}(\mathbb{R}^N)$. We define:

$$S(a,b) = \inf_{u \in \mathcal{D}_a^{1,2}(\mathbb{R}^N) \setminus \{0\}} E_{a,b}(u), \tag{3}$$

to be the best embedding constants, where

$$E_{a,b}(u) = \left(\int_{\mathbb{R}^N} |x|^{-2a} |\nabla u|^2 \, \mathrm{d}x \right) \times \left(\int_{\mathbb{R}^N} |x|^{-bp} |u|^p \, \mathrm{d}x \right)^{-2/p}.$$

The extremal functions for $S(a,b)$ are ground state solutions of the Euler equation:

$$-\mathrm{div}\big(|x|^{-2a} \nabla u\big) = |x|^{-bp} u^{p-1}, \quad u \geqslant 0, \quad \text{in } \mathbb{R}^N. \tag{4}$$

Note that the Caffarelli–Kohn–Nirenberg inequalities (1) (*see* more general versions of the inequalities in [3] and [12]) contain the classical Sobolev inequality ($a = b = 0$) and the Hardy inequality ($a = 0,\ b = 1$) as special cases. These played important roles in many applications by virtue of the complete knowledge of the best constants, extremal functions and their qualitative properties. Thus it is a fundamental task to study the best constants, existence (and nonexistence) of extremal functions, as well as their qualitative properties in inequality (1) for parameters a and b in the full parameter domain (2). Much progress has been done for the parameter region: $0 \leqslant a < (N-2)/2$, $a \leqslant b \leqslant a+1$. In [1,15], the best constant and the minimizers for the Sobolev inequality ($a = b = 0$) were given by Aubin, and Talenti. In [11], Lieb considered the case $a = 0$, $0 < b < 1$ and gave the best constants and explicit minimizers. In [9], Chou and Chu considered the full a-nonnegative region and gave the best constants and explicit minimizers. Also for this a-nonnegative region, Lions in [13,14], and Wang and Willem in [17], have established the compactness of all minimizing sequences up to dilations. The symmetry of the minimizers has also been studied in [11] and [9]. In fact, *all* nonnegative solutions in $\mathcal{D}_a^{1,2}(\mathbb{R}^N)$ for the corresponding Euler equation (4) are *radial solutions* (in the case $a = b = 0$, they are radial with respect to some point) and explicitly given [1,15,11,9]. This was established in [9], using a generalization of the moving plane method (e.g., [10,2,8]).

On the other hand, it seems that little is known for parameters in the a-negative region: $-\infty < a < 0$, $a \leqslant b \leqslant a+1$. The goal of this paper is to settle some of the questions concerning inequalities (1) with parameters in the a-negative region, such as the best constants, the existence (and nonexistence) of minimizers, and the symmetry properties of minimizers. For the a-negative region we shall reveal new phenomena that are strikingly different from those for the a-nonnegative region. We also study the positive bound state solutions and for $-a$ large we construct solutions having prescribed subsymmetry of $\mathbf{O}(N)$.

2. Results on the extremal functions (ground state solutions)

To state the results, let $S_p(\mathbb{R}^N)$ be the best embedding constant from $\mathrm{H}^1(\mathbb{R}^N)$ into $\mathrm{L}^p(\mathbb{R}^N)$, i.e.,

$$S_p(\mathbb{R}^N) = \inf_{u \in \mathrm{H}^1(\mathbb{R}^N) \setminus \{0\}} \left(\int_{\mathbb{R}^N} |\nabla u|^2 + u^2 \, \mathrm{d}x \right) \times \left(\int_{\mathbb{R}^N} |u|^p \, \mathrm{d}x \right)^{-2/p}.$$

THEOREM 1 (Best constants and nonexistence of extremal functions). – (i) $S(a,b)$ *is continuous in the full parameter domain* (2). (ii) *For* $b = a+1$, *we have* $S(a, a+1) = [(N-2-2a)/2]^2$, *and* $S(a, a+1)$ *is not achieved.* (iii) *For* $a < 0$ *and* $b = a$, *we have* $S(a,a) = S(0,0)$ (*the best Sobolev constant*), *and* $S(a,a)$ *is not achieved.*

THEOREM 2 (Best constants and existence of extremal functions). – (i) *For* $a < b < a+1$, $S(a,b)$ *is always achieved.* (ii) *For* $b - a \in (0,1)$ *fixed, as* $a \to -\infty$, $S(a,b)$ *is strictly increasing, and*

$$S(a,b) = \big[(N-2-2a)/2\big]^{2(b-a)} \big[S_p(\mathbb{R}^N) + \mathrm{o}(1)\big].$$

THEOREM 3 (Symmetry breaking). – *There is a function $h(a)$ defined for $a \leqslant 0$, satisfying $h(0) = 0$, $a < h(a) < a+1$ for $a < 0$, and $a + 1 - h(a) \to 0$ as $-a \to \infty$, such that for any (a, b) satisfying $a < 0$ and $a < b < h(a)$, the minimizer for $S(a, b)$ is nonradial.*

THEOREM 4 (Symmetry property). – *For $a < b < a+1$, the minimizer of $S(a,b)$, possibly after a dilation $u(x) \to \tau^{(N-2-2a)/2} u(\tau x)$, satisfies the "modified inversion" symmetry: $u(|x|^{-2}x) = |x|^{N-2-2a} u(x)$.*

Remark 1. – By Theorems 2 and 3, it is likely that there are no closed form minimizers, so it seems to be very difficult to examine the best constants in the interior of the region.

Remark 2. – For a special case: $b = 0$, $-1 < a < 0$, the existence of a minimizer was given in [4] using a quite different method.

Remark 3. – In the case $b = a$, we have $p = 2^*$, the critical Sobolev exponent. The situation is quite delicate since for $a \geqslant 0$, $S(a, a)$ is strictly decreasing in a and is solvable as we mentioned above [9,17], and for $a < 0$, we have $S(a, a) = S(0, 0)$ and the nonexistence result in Theorem 1.

Remark 4. – For Theorem 3 we have an explicit estimate from below of $h(a)$.

Remark 5. – The result in Theorem 4 holds for all nonnegative bound state solutions of (4).

Our approach to the problem in this paper is quite different from that used in the quoted papers [1, 4,9,11,13–15,17] in which the problem was worked directly in $\mathcal{D}_a^{1,2}(\mathbb{R}^N)$. We shall take a detour to convert the problem to an equivalent one defined on $\mathrm{H}^1(\mathbb{R} \times \mathbf{S}^{N-1})$. While taking advantages from the two formulations we shall mainly work with the equivalent one on $\mathrm{H}^1(\mathbb{R} \times \mathbf{S}^{N-1})$. We use the notation $\mathcal{C} = \mathbb{R} \times \mathbf{S}^{N-1}$. In the following we sketch the ideas for proving our main results, we refer the details and more general results to [6] and [7].

To u a smooth function with compact support in $\mathbb{R}^N \setminus \{0\}$, we associate v a smooth function on $\mathcal{C}$ with compact support, by the transformation

$$x \longmapsto u(x) = |x|^{-(N-2-2a)/2}\, v\big(-\ln|x|, x/|x|\big). \tag{5}$$

Using this transformation, we proved in [6]:

PROPOSITION 1. – *The mapping* (5) *is a Hilbert space isomorphism from $\mathcal{D}_a^{1,2}(\mathbb{R}^N)$ to $\mathrm{H}^1(\mathcal{C})$, where the inner product on $\mathrm{H}^1(\mathcal{C})$ is $(v, w) = \int_{\mathcal{C}} \nabla v \cdot \nabla w + \big((N-2-2a)/2\big)^2 v w \,\mathrm{d}\mu$.*

Now we define an energy functional on $\mathrm{H}^1(\mathcal{C})$:

$$F_{a,b}(v) = \left(\int_{\mathcal{C}} |\nabla v|^2 + \left(\frac{N-2-2a}{2}\right)^2 v^2 \,\mathrm{d}\mu\right) \times \left(\int_{\mathcal{C}} v^p \,\mathrm{d}\mu\right)^{-2/p}. \tag{6}$$

If $u \in \mathcal{D}_a^{1,2}(\mathbb{R}^N)$ and $v \in \mathrm{H}^1(\mathcal{C})$ are related through (5), then $E_{a,b}(u) = F_{a,b}(v)$. Moreover, u is solution of (4) if and only if v satisfies

$$-\Delta v + \big[(N-2-2a)/2\big]^2 v = v^{p-1}, \quad v > 0 \quad \text{on } \mathcal{C}. \tag{7}$$

These observations allow us to draw conclusions from the equivalent problem (7). The reformulation enables us to make use of a combination of analytical tools such as compactness argument, rescaling, concentration compactness principle, bifurcation analysis, moving plane method, etc.

The proof of Theorem 1 (i) depends upon a convergence argument and we refer to [6].

To prove Theorem 1 (ii), we note that $F_{a,a+1}(v) \geqslant [(N-2-2a)/2]^2$ for all $v \in \mathrm{H}^1(\mathcal{C})$. On the other hand, one can easily construct a sequence $(v_n) \subset \mathrm{H}^1(\mathcal{C})$ of radial functions such that $F_{a,a+1}(v_n) \to [(N-2-2a)/2]^2$. Therefore, $S(a, a+1) = [(N-2-2a)/2]^2$. For nonexistence of minimizers, one notes that for $\lambda \geqslant 1$, the equation $-\Delta v + \lambda^2 v = v$ has no nonzero solution in $\mathrm{H}^1(\mathcal{C})$. For $0 < \lambda < 1$, i.e., $(N-4)/2 < a < (N-2)/2$, assume that $S(a, a+1)$ is achieved by some nonnegative function $v \in \mathrm{H}^1(\mathcal{C}) \setminus \{0\}$. By the maximum principle $v > 0$ everywhere. Denote by $f(t)$ the average of v on the spheres $t =$ constant. Then f is a positive function in $\mathrm{H}^1(\mathbb{R})$ and satisfies the ODE: $-f_{tt} + \lambda^2 f = f$, which implies $f \equiv 0$, a contradiction.

Next we sketch the proof of Theorem 1 (iii). The case $a = b = 0$ is well known (the Yamabe problem in $\mathbb{R}^N$). In this case, the infimum $S(0,0)$ is achieved only by functions $U_{\mu,y}(x) = C\,\mu^{(N-2)/2}\big(\mu^2 + |x-y|^2\big)^{-(N-2)/2}$, $\mu > 0$, $y \in \mathbb{R}^N$. Note that for μ small, this function concentrates around y. For $y \neq 0$ we get $S(0,0) = \lim_{\mu \to 0} E_{a,a}\ (U_{\mu,y})$. Due to this fact one concludes that for $a < (N-2)/2$, $S(a,a) \leqslant S(0,0)$. For $a < 0$, assume by contradiction that $S(a,a) < S(0,0)$. Then there is a $v \in \mathrm{H}^1(\mathcal{C})$ such that $F_{a,a}(v) < S(0,0)$. But by the expression (6), $F_{a,a}(v) > F_{0,0}(v) \geqslant S(0,0)$, a contradiction. Hence, $S(a,a) = S(0,0)$ for all $a \leqslant 0$. Next, we show $S(a,a)$ is not achieved for $a < 0$. If the conclusion is not true, for some $a < 0$ and $v \in \mathrm{H}^1(\mathcal{C})$ we get $S(a,a) = F_{a,a}(v)$. But $F_{a,a}(v) > F_{0,0}(v) \geqslant S(0,0)$, contradicting with $S(a,a) \leqslant S(0,0)$.

The proof of Theorem 2 makes use of concentration compactness principle [13] and a more detailed version of it, contained in [5,16]. For Theorem 2 (ii), the method is based on rescalings and is related to the study of:

$$-\Delta u + \lambda^2 u = u^{p-1} \quad \text{in } \mathbb{R}^N, \tag{8}$$

for λ large. A bifurcation argument and a comparison argument are used in the proof of symmetry breaking result Theorem 3. The study of the linearized equation yields the construction of test functions with energy $E_{a,b}$ bellow the least energy in the space of radial functions. Finally, the moving plane method is used in proving the modified inversion symmetry of all bound state solutions. The proofs are rather lengthy and we refer to [6].

3. Bound state solutions having prescribed symmetry

Since our results show that for $a < 0$ there is a region where bound state solutions of (4) are not unique, it is an important issue to investigate the solutions structure for this problem. An interesting question that arises naturally is to seek nonradial solutions having prescribed symmetry G, where G is a subgroup of the orthogonal group $\mathbf{O}(N)$. Note that (4) is radially invariant, in the sense that if u is a solution, so is gu for all $g \in \mathbf{O}(N)$, where $gu(x) = u(g^{-1}x)$. It would be interesting to know whether one can classify all solutions of (4) according to their symmetry. We partially answer this question by constructing solutions with prescribed symmetry group. Here, for $u \in \mathcal{D}_a^{1,2}(\mathbb{R}^N)$, the symmetry group of u is defined to be $\Sigma_u = \{g \in \mathbf{O}(N) : gu = u \text{ a.e.}\}$.

DEFINITION 1. – Let $G \subset \mathbf{O}(N)$ be a closed subgroup, so that G acts on $\mathbf{S}^{N-1}$. We say G has a locally minimal orbit set $\Omega \subset \mathbf{S}^{N-1}$, if there exist $k \in \mathbf{N}$, and $\delta > 0$, such that: (a) Ω is G-invariant; (b) $\#\,Gy = k$ for any $y \in \Omega$; (c) $\#\,Gy > k$ for any $y \in \mathbf{S}^{N-1}$ with $0 < \operatorname{dist}(y, \Omega) < \delta$.

DEFINITION 2. – We say G is maximal with respect to a locally minimal orbit set Ω if for any closed subgroup H, with $G \leqslant H \leqslant \mathbf{O}(N)$, $H \neq G$, we have $\#\,Hy > k$ for any $y \in \Omega$.

THEOREM 5 (Solutions having prescribed symmetry). – *Assume $G \subset \mathbf{O}(N)$ is a closed subgroup having a locally minimal orbit set $\Omega \subset \mathbf{S}^{N-1}$ (corresponding to an integer k). Let $b - a \in (0,1)$ fixed, then for $-a$ sufficiently large,* (4) *has a G-invariant, k-bump solution u, concentrating near Ω as $a \to -\infty$. Moreover, if G is maximal with respect to Ω, then $\Sigma_u = G$.*

The idea of proving Theorem 5 comes from our earlier work in [5,16], where other type of radially invariant elliptic problems have been shown to have nonradial positive solutions with prescribed symmetry. A rather general local minimization scheme has been used in [5] and [16] and overcoming some technical complications we employ this method again. Let us sketch the procedure here. It is more convenient to work with the equivalent equation (7). Denote: $\lambda = (N-2-2a)/2$. It is known that problem (8) has a unique (up to translations) positive solution, and for $\lambda \to \infty$ these solutions concentrate around some point. This concentration phenomenon allows local study of Palais-Smale sequences on $\mathcal{C}$. Under the requirement of symmetry, we show $(P.S.)$ sequences which have most of the mass in certain open sets around a locally minimal orbit Ω, have convergent subsequences. To be more precise, we define an open subset in $\mathrm{H}^1_G(\mathcal{C})$ (the space of functions invariant under the action of G),

$$\mathcal{K}^{\sigma}_{G,\lambda} = \left\{ u \in \mathrm{H}^1_G(\mathcal{C}) : \int_{\mathcal{C}} |u|^p \,\mathrm{d}\mu = 1,\ \int_{\Lambda_\lambda} |u|^p \,\mathrm{d}\mu > 1-\sigma \right\}.$$

Here $\sigma = (k+1)/(k+2)$, and Λ_λ is a suitably chosen open subset of $\mathcal{C}$, which contains Ω. Provided λ is sufficiently large, it can be shown that local minimizing sequences in the set $\mathcal{K}^{\sigma}_{G,\lambda}$ have a subsequence convergent to a solution of (7) (for details, we refer to [7]). The novelty in this approach is that we are able to obtain by one method, both global minimizers in the space of symmetric functions and local minimizers.

4. Final remarks

1. The inequality (1) also holds for $N=1$ and $N=2$ with corresponding conditions on $a,\ b$. The conditions for these cases are: – for $N=2$: $-\infty < a < 0$, $a < b \leqslant a+1$, and $p = 2/(b-a)$; – for $N=1$: $-\infty < a < -1/2$, $a+1/2 < b \leqslant a+1$, and $p = 2/[-1+2(b-a)]$. In [6] we also study these cases, and for $N=1$, we have a complete solution for the problem.

2. An open question related to Theorem 3 is: can one find the exact form of the curve $h(a)$, where the bifurcation of minimizers for $S(a,b)$ from radial to nonradial occurs?

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