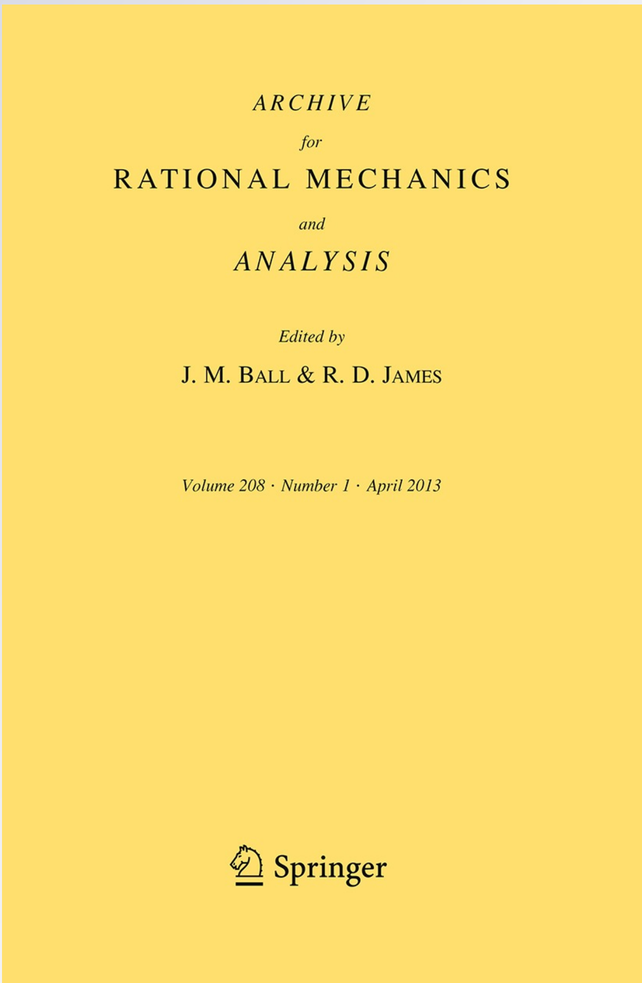
Segregated and Synchronized Vector Solutions for Nonlinear Schrödinger Systems

Shuangjie Peng & Zhi-qiang Wang

Archive for Rational Mechanics and Analysis

ISSN 0003-9527
Volume 208
Number 1

Arch Rational Mech Anal (2013)
208:305-339
DOI 10.1007/s00205-012-0598-0



Arch. Rational Mech. Anal. 208 (2013) 305–339
Digital Object Identifier (DOI) 10.1007/s00205-012-0598-0

Segregated and Synchronized Vector Solutions for Nonlinear Schrödinger Systems

Shuangjie Peng & Zhi-qiang Wang

Communicated by P. Rabinowitz

Abstract

We consider the following nonlinear Schrödinger system in $\mathbb{R}^3$

$$\begin{cases} -\Delta u + P(|x|)u = \mu u^3 + \beta v^2 u, & x \in \mathbb{R}^3, \\ -\Delta v + Q(|x|)v = \nu v^3 + \beta u^2 v, & x \in \mathbb{R}^3, \end{cases}$$

where $P(r)$ and $Q(r)$ are positive radial potentials, $\mu > 0$, $\nu > 0$ and $\beta \in \mathbb{R}$ is a coupling constant. This type of system arises, in particular, in models in Bose–Einstein condensates theory.

We examine the effect of nonlinear coupling on the solution structure. In the repulsive case, we construct an unbounded sequence of non-radial positive vector solutions of segregated type, and in the attractive case we construct an unbounded sequence of non-radial positive vector solutions of synchronized type. Depending upon the system being repulsive or attractive, our results exhibit distinct characteristic features of vector solutions.

1. Introduction

We consider the following nonlinear Schrödinger system

$$\begin{cases} -\Delta u + P(|x|)u = \mu u^3 + \beta v^2 u, & x \in \mathbb{R}^3, \\ -\Delta v + Q(|x|)v = \nu v^3 + \beta u^2 v, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where we assume that $P(x) = P(|x|)$ and $Q(x) = Q(|x|)$ are continuous positive radial functions, $\mu > 0$, $\nu > 0$ and $\beta \in \mathbb{R}$ is a coupling constant.

These types of systems arise when one considers standing wave solutions of time-dependent N-coupled Schrödinger systems with $N = 2$ of the form

$$\begin{cases} -i\frac{\partial}{\partial t}\Phi_j = \Delta\Phi_j - V_j(x)\Phi_j + \mu_j|\Phi_j|^2\Phi_j + \Phi_j \sum_{l=1,k\neq j}^{N} \beta_{jk}|\Phi_l|^2, \text{ in } \mathbb{R}^3 \\ \Phi_j = \Phi_j(x,t) \in \mathbb{C}, t > 0, \ j = 1,\dots,N, \end{cases} \tag{1.2}$$

where μ_j and $\beta_{jl} = \beta_{lj}$ are constants. These systems of equations, also known as Gross–Pitaevskii equations, have applications in many physical problems such as in nonlinear optics and in Bose–Einstein condensates theory for multispecies Bose–Einstein condensates. For example, (1.2) with $N = 2$ arises in the Hartree–Fock theory for a double condensate, that is, a binary mixture of a Bose–Einstein condensate in two different hyperfine states $|1\rangle$ and $|2\rangle$ (see [11,27]). Physically, Φ_1 and Φ_2 are the wave functions of the corresponding condensates, μ and ν, and β are the intraspecies and interspecies scattering lengths, respectively. The sign of the scattering length β determines whether the interactions of states are repulsive or attractive. In the attractive case the components of a vector solution tend to go along with each other, leading to synchronization. In the repulsive case, the components tend to segregate from each other, leading to phase separations. These phenomena have been documented in experiments as well as in numeric simulations (for example, [7,19,23] and references therein).

Systems of nonlinear Schrödinger equations have been the subject of extensive mathematical studies in recent years, for example, [2–6,8–10,14,15,17,18,20,21,24,25,29,30,34] and references therein. Phase separation has been proved in several cases with constant potentials, such as in [4,8,9,21,25,29,30] as the coupling constant β tends to negative infinity in the repulsive case. For the totally symmetric case ($\mu_j = \mu > 0$ for all j, and $\beta_{kj} = \beta$ for all $k \neq j$), radial solutions with domain separations are constructed in [25] using variational methods and perturbation methods for N-systems. In [9,30] the minimax method is used to give infinitely many radial positive solutions for 2-systems (see also [26] for generalizations to the N-systems). These examples constitute segregated radial solutions. Segregated radial solutions were obtained in repulsive cases in [4] by global bifurcation methods for the general systems (1.1), establishing the existence of infinite branches of radial solutions with the property that $\sqrt{\mu - \beta}u - \sqrt{\nu - \beta}v$ has exactly k nodal domains for solutions along the kth branch. However, non-radial solutions of the segregated type with an arbitrarily large number of nodal domains are not well known. The work of [16] gives solutions with one component peaking at the origin and the other having a finite number of peaks on a k-polygon. In the symmetric case ($\mu = \nu$ and $P = Q = 1$), [30] gives infinitely many non-radial positive solutions for $\beta \leqq -1$, which are potentially of the segregated type. One of the goals of the current paper is to demonstrate the existence of infinitely many segregated solutions with a potentially large number of nodal domains. On the other hand, for the attractive case (that is, $\beta > 0$) when $P = Q = 1$, it is known (for example, [5,6]) that there are special positive solutions with the two components being positive constant multiples of the unique positive solution of the scalar cubic Schrödinger equation $-\Delta w + w = w^3$. Thus, the two components are in synchronization.

This observation prompts the question of whether there are non-radial synchronized vector solutions. Another goal of our paper is to construct infinitely many non-radial synchronized solutions. Under some assumptions for $P(r)$ and $Q(r)$ near infinity we construct infinitely many non-radial positive solutions for (1.1), for both segregated types and synchronized types.

We assume that $P(r) > 0 and Q(r) > 0$ satisfy the following conditions:

(P): There are constants $a \in \mathbb{R}$, $m > 1$, and $\theta > 0$, such that as $r \to +\infty$

$$P(r) = 1 + \frac{a}{r^m} + O\left(\frac{1}{r^{m+\theta}}\right). \tag{1.3}$$

(Q): There are constants $b \in \mathbb{R}$, $n > 1$, and $\varepsilon > 0$, such that as $r \to +\infty$

$$Q(r) = 1 + \frac{b}{r^n} + O\left(\frac{1}{r^{n+\varepsilon}}\right). \tag{1.4}$$

Our main results in this paper can be stated as follows:

Theorem 1.1. *Suppose that $P(r)$ satisfies (P) and $Q(r)$ satisfies (Q). Then there exists a decreasing sequence $\{\beta_k\} \subset (-\sqrt{\mu\nu}, 0)$ with $\beta_k \to -\sqrt{\mu\nu}$ as $k \to \infty$ such that for $\beta \in (-\sqrt{\mu\nu}, 0) \cup (0, \min\{\mu, \nu\}) \cup (\max\{\mu, \nu\}, \infty)$ and $\beta \neq \beta_k$ for any k, problem (1.1) has infinitely many non-radial positive synchronized solutions (u_ℓ, v_ℓ), whose energy can be arbitrarily large, provided one of the following two conditions holds:*

(i) *$m < n$, $a > 0$ and $b \in \mathbb{R}$; or $m > n$, $a \in \mathbb{R}$ and $b > 0$;*
(ii) *$m = n$, $aB + bC > 0$, where B and C are defined in Proposition A.2.*

Furthermore, $\lim_{\ell\to\infty} \max u_\ell > 0$, $\lim_{\ell\to\infty} \max v_\ell > 0$, *and as* $\ell \to \infty$

$$||\sqrt{|\mu-\beta|}u_\ell - \sqrt{|\nu-\beta|}v_\ell||_{H^1} + ||\sqrt{|\mu-\beta|}u_\ell - \sqrt{|\nu-\beta|}v_\ell||_{L^\infty} \to 0.$$

Theorem 1.2. *Suppose that $P(r)$ satisfies (P), $Q(r)$ satisfies (Q) and $m = n$, $a > 0$, $b > 0$. Then there exists $\bar{\beta}^* > 0$ such that, for $\beta < \bar{\beta}^*$, problem (1.1) has infinitely many non-radial positive segregated solutions (u_ℓ, v_ℓ), whose energy can be arbitrarily large. Furthermore,* $\lim_{\ell\to\infty} \max u_\ell > 0$, $\lim_{\ell\to\infty} \max v_\ell > 0$, *and as* $\ell \to \infty$

$$||\sqrt{\nu}u_\ell(\cdot) - \sqrt{\mu}v_\ell(T_\ell\cdot)||_{H^1} + ||\sqrt{\nu}u_\ell(\cdot) - \sqrt{\mu}v_\ell(T_\ell\cdot)||_{L^\infty} \to 0.$$

Here $T_\ell \in SO(3)$ is the rotation on the (x_1, x_2) plane of $\frac{\pi}{\ell}$.

Remark 1.3. The segregated or synchronized natures of these solutions are demonstrated from the L^∞ estimates in the theorems; this distinction will be clearer in Theorems 1.7 and 1.8, stated later, after we fix the notation. Roughly speaking, synchronized solutions are small perturbations of (U_r, V_r), where U_r and V_r are sums of positive constant multiples of translated W to the vertices of a large sized l-polygon, where W is the unique positive radial solution of $-\Delta W + W = W^3$. Segregated solutions are small perturbations of $(\bar{U}_r, \bar{V}_\rho)$, where $\bar{U}_r$ ($\bar{V}_\rho$, resp.) are sums of translated W_μ (W_ν, resp.) to the vertices of a large sized l-polygon, with one polygon being a π/l rotation shift of the other, and where W_μ (W_ν, resp.) is the unique positive radial solution of $-\Delta W + W = \mu W^3$ ($-\Delta W + W = \nu W^3$, respectively). In other words, synchronized solutions and segregated solutions both have a large number of bumps near infinity, while the locations of the bumps for u and v are roughly the same for synchronized solutions and the locations of the bumps for u and v have an angular shift for segregated solutions.

Remark 1.4. 1). In Theorem 1.1 the requirement of $\beta \notin (\min\{\mu, \nu\}, \max\{\mu, \nu\})$ is necessary as there is a non-existence result on positive solutions (for example, [5]), and in particular there are no positive solutions when $P = Q$ for such β. 2). Note that for the small coupling constant we can obtain simultaneously infinitely many synchronized and segregated solutions. With the constant potentials, segregation was discussed for the repulsive case with β negatively large (for example, [4,7–9,15,21,25,29,30]), while the uniqueness of positive radial solutions was proved for a large positive constant β [34].

Remark 1.5. Our methods are inspired by the work of [31,33] for scalar nonlinear elliptic equations. For the scalar case, the potential has to be decreasing at infinity since the scalar equations may not have non-radial positive solutions for increasing potential functions. However, we want to point out that for the systems of equations we just need the combined effect from the two potential functions P and Q in the sense that if one dominates the other, the faster decaying function can be increasing or decreasing at infinity and, if the two have the same decay rates at infinity, we just need the combined effect $aB + bC > 0$.

Remark 1.6. There is a gray region of $|\beta| \neq 0$ small in which, for β, synchronized and segregated type solutions exist simultaneously. This, in some cases, may be seen as a continuation from solutions for $\beta = 0$. On the other hand, we want to point out that for some parameters these solutions only exist due to the coupling and do not exist for $\beta = 0$. For example, as we discussed above, a or b can be negative. When $\beta = 0$, an equation with increasing potential cannot have non-radial positive solutions.

Next, we introduce some notations to be used in the proofs of the main theorems and formulate a version of the main results which gives more precise descriptions about the segregated and synchronized character of the solutions. In doing so, we also outline the main idea and approach in the proof of Theorems 1.1 and 1.2.

Hereafter, for any function $K(x) > 0$, the Sobolev space $H^1_K(\mathbb{R}^3)$ is endowed with the standard norm

$$\|u\|_K = \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + K(x)u^2)\right)^{\frac{1}{2}},$$

which is induced by the inner product

$$\langle u,\, v\rangle_K = \int_{\mathbb{R}^3} (\nabla u \nabla v + K(x)uv).$$

Define H to be the product space $H^1_P(\mathbb{R}^3) \times H^1_Q(\mathbb{R}^3)$ with the norm

$$\|(u, v)\| = \|u\|_P + \|v\|_Q.$$

Denote the unique solution of the following problem by W

$$\begin{cases} -\Delta w + w = w^3, \quad w > 0, \quad \text{in } \mathbb{R}^3, \\ w(0) = \max\limits_{x \in \mathbb{R}^3} w(x), \quad w(x) \in H^1(\mathbb{R}^3). \end{cases} \tag{1.5}$$

Note that the limit system for (1.1) is

$$\begin{cases} -\Delta u + u = \mu u^3 + \beta v^2 u, \quad x \in \mathbb{R}^3, \\ -\Delta v + v = \nu v^3 + \beta u^2 v, \quad x \in \mathbb{R}^3, \end{cases} \tag{1.6}$$

and that

$$(U, V) = (\alpha W, \gamma W)$$

solves (1.6), provided $-\sqrt{\mu\nu} < \beta < \min\{\mu, \nu\}$ or $\beta > \max\{\mu, \nu, \}$, where

$$\alpha = \sqrt{\frac{\nu - \beta}{\mu\nu - \beta^2}}, \quad \gamma = \sqrt{\frac{\mu - \beta}{\mu\nu - \beta^2}}.$$

We will use (U, V) to build up the solutions for (1.1).

Let

$$x^j = \left(r \cos \frac{2(j-1)\pi}{\ell}, r \sin \frac{2(j-1)\pi}{\ell}, 0 \right) := \left(x'^j, 0 \right), \quad j = 1, \ldots, \ell, \tag{1.7}$$

where $r \in [r_0 \ell \ln \ell, r_1 \ell \ln \ell]$ for some $r_1 > r_0 > 0$.

Define

$$H_{V,s} = \Big\{ u : u \in H^1_V(\mathbb{R}^3), u \text{ is even in } x_h, h = 2, 3,$$
$$u(r\cos\theta, r\sin\theta, x_3) = u\left(r\cos\left(\theta + \frac{2\pi j}{\ell}\right), r\sin\left(\theta + \frac{2\pi j}{\ell}\right), x_3 \right) \Big\}.$$

We define $H_{Q,s}$ similarly.

Let

$$U_r(x) = \sum_{j=1}^{\ell} U_{x^j}(x), \quad V_r(x) = \sum_{j=1}^{\ell} V_{x^j}(x), \tag{1.8}$$

where $U_\xi(x) = U(x - \xi)$ for $\xi \in \mathbb{R}^3$, $V_\xi(x) = V(x - \xi)$ for $\xi \in \mathbb{R}^3$. It is easy to check that (U_r, V_r) is in $H_{P,s} \times H_{Q,s}$.

We will verify Theorem 1.1 by proving the following result:

Theorem 1.7. *Under the assumptions of Theorem 1.1, there is an integer* $\ell_0 > 0$*, such that for any integer* $\ell \geqq \ell_0$*, (1.1) has a solution* u_ℓ *of the form*

$$(u_\ell, v_\ell) = (U_{r_\ell}(x) + \varphi_\ell, V_{r_\ell}(x) + \psi_\ell),$$

where $(\varphi_\ell, \psi_\ell) \in H_{P,s} \times H_{Q,s}$, $r_\ell \in [r_0 \ell \ln \ell, r_1 \ell \ln \ell]$ *and as* $\ell \to +\infty$,

$$\|(\varphi_\ell, \psi_\ell)\| \to 0.$$

In Theorem 1.7, we construct infinitely many non-radial positive solutions (u_ℓ, v_ℓ) for system (1.1). These are synchronized type solutions as evidenced by the constructions, since the essential supports of the two components u_ℓ and v_ℓ are both placed in the same locations. One can easily see that the larger the value of ℓ, the more synchronized these components are. The next result implies Theorem 1.2 and gives segregated solutions for system (1.1) with the essential support of the two components being separated.

Let U_μ be the unique solution of the following problem

$$\begin{cases} -\Delta u + u = \mu u^3, \quad u > 0, \quad \text{in } \mathbb{R}^3, \\ u(0) = \max_{x \in \mathbb{R}^3} u(x), \quad u(x) \in H^1(\mathbb{R}^3). \end{cases} \tag{1.9}$$

It is well-known that U_μ is non-degenerate and $U_\mu(x) = U_\mu(|x|)$, $U'_\mu < 0$.

We will use (U_μ, U_ν) to build up the approximate solutions for (1.1).

Let x^j be defined in (1.7) and denote

$$y^j = \left(\rho \cos \frac{(2j-1)\pi}{\ell}, \rho \sin \frac{(2j-1)\pi}{\ell}, 0\right) := \left(y'^j,\ 0\right), \quad j = 1, \ldots, \ell, \tag{1.10}$$

where $\rho \in [\bar{r}_0 \ell \ln \ell, \bar{r}_1 \ell \ln \ell]$ for some $\bar{r}_1 > \bar{r}_0 > 0$.

Let

$$\bar{U}_r(x) = \sum_{j=1}^{\ell} U_{\mu, x^j}(x), \quad \bar{V}_\rho(x) = \sum_{j=1}^{\ell} U_{\nu, y^j}(x), \tag{1.11}$$

where $U_{\gamma,\xi}(x) = U_\gamma(x - \xi)$ for $\gamma > 0$ and $\xi \in \mathbb{R}^3$. It is easy to check that $(\bar{U}_r, \bar{V}_\rho)$ is in $H_{P,s} \times H_{Q,s}$.

To prove Theorem 1.2, we need to prove the following result

Theorem 1.8. *Under the assumptions of Theorem 1.2, there exists an integer* $\bar{\ell}_0 > 0$*, such that for any integer* $\ell \geqq \bar{\ell}_0$*, (1.1) has a solution* $(u_\ell,\ v_\ell)$ *of the form*

$$(u_\ell, v_\ell) = (\bar{U}_{r_\ell}(x) + \bar{\varphi}_\ell, \bar{V}_{\rho_\ell}(x) + \bar{\psi}_\ell),$$

where $(\bar{\varphi}_\ell, \bar{\psi}_\ell) \in H_{P,s} \times H_{Q,s}$, $r_\ell \in [\bar{r}_0 \ell \ln \ell, \bar{r}_1 \ell \ln \ell]$ *and as* $\ell \to +\infty$,

$$\|(\bar{\varphi}_\ell, \bar{\psi}_\ell)\| \to 0.$$

Remark 1.9. By Theorem 1.7 and Theorem 1.8 , (1.1) has solutions with a large number of bumps near infinity. Thus, the energy of these solutions can be very large.

We apply the techniques in the singularly perturbed elliptic problems to prove our main results. In particular, we adopt the idea introduced by Wei and Yan [33] by using ℓ, the number of the bumps of the solutions, as a parameter in the construction of spike solutions for (1.1). We encounter some new difficulties in estimates due to the nonlinear coupling.

This paper is organized as follows. In Section 2, we will carry out the reduction to a finite dimensional setting and prove Theorem 1.7. The study of the existence of segregated solutions for system (1.1) and the proof of Theorem 1.8 will appear in Section 3. In Section 4 we discuss some further extensions of our main results by using our framework of methods. We conclude with the energy expansion in the appendix.

2. Synchronized Vector Solutions and the proof of Theorem 1.1

In this section we consider synchronized vector solutions and prove Theorem 1.1 by proving Theorem 1.7. Let

$$Y_j = \frac{\partial U_{x^j}}{\partial r}, \quad Z_j = \frac{\partial V_{x^j}}{\partial r}, \quad j = 1, \ldots, \ell,$$

where x^j is defined in (1.7).

In this section, we always assume

$$r \in \mathcal{D}_\ell =: \left[\left(\frac{\min\{m, n\}}{2\pi} - \delta \right) \ell \ln \ell, \quad \left(\frac{\min\{m, n\}}{2\pi} + \delta \right) \ell \ln \ell \right], \tag{2.1}$$

where $\delta > 0$ is a small constant.

Let

$$\begin{aligned} I(u, v) = &\frac{1}{2} \int_{\mathbb{R}^3} \left(|\nabla u|^2 + P(|x|)u^2 + |\nabla v|^2 + Q(|x|)v^2 \right) \\ &- \frac{1}{4} \int_{\mathbb{R}^3} \left(\mu |u|^4 + \nu |v|^4 \right) - \frac{\beta}{2} \int_{\mathbb{R}^3} u^2 v^2, \quad (u, v) \in H. \end{aligned}$$

Then $I \in C^2$ and its critical points are solutions of (1.1).

Define

$$E = \left\{ (u, v) \in H_{P,s} \times H_{Q,s}, \ \sum_{j=1}^{\ell} \int_{\mathbb{R}^3} W_{x^j}^2 (Y_j u + Z_j v) = 0 \right\}.$$

Let

$$J(\varphi, \psi) = I(U_r + \varphi, V_r + \psi), \quad (\varphi, \psi) \in E.$$

Expand $J(\varphi,\psi)$ as follows:

$$J(\varphi,\psi)=J(0,0)+l(\varphi,\psi)+\frac{1}{2}L(\varphi,\psi)+R(\varphi,\psi),\quad (\varphi,\psi)\in E,\quad (2.2)$$

where

$$\begin{aligned}l(\varphi,\psi)=&\sum_{j=1}^{\ell}\int_{\mathbb{R}^3}(P(|x|)-1)\,U_{x^j}\varphi+\mu\int_{\mathbb{R}^3}\left(U_r^3-\sum_{j=1}^{\ell}U_{x^j}^3\right)\varphi\\&+\sum_{j=1}^{\ell}\int_{\mathbb{R}^3}(Q(|x|)-1)\,V_{x^j}\psi+\nu\int_{\mathbb{R}^3}\left(V_r^3-\sum_{j=1}^{\ell}V_{x^j}^3\right)\psi\\&-\beta\int_{\mathbb{R}^3}\left(V_r^2U_r-\sum_{j=1}^{\ell}V_{x^j}^2U_{x^j}\right)\varphi-\beta\int_{\mathbb{R}^3}\left(V_rU_r^2-\sum_{j=1}^{\ell}V_{x^j}U_{x^j}^2\right)\psi,\end{aligned}$$

$$\begin{aligned}L(\varphi,\psi)=&\int_{\mathbb{R}^3}\left(|\nabla\varphi|^2+P(|x|)\varphi^2-3\mu U_r^2\varphi^2\right)\\&+\int_{\mathbb{R}^3}\left(|\nabla\psi|^2+Q(|x|)\psi^2-3\nu V_r^2\psi^2\right)\\&-\beta\int_{\mathbb{R}^3}\left(U_r^2\psi^2+4U_rV_r\varphi\psi+V_r^2\varphi^2\right),\end{aligned}$$

and

$$\begin{aligned}R(\varphi,\psi)=&\int_{\mathbb{R}^3}\left(\mu U_r\varphi^3+\nu V_r\psi^3+\frac{\mu}{4}\varphi^4+\frac{\nu}{4}\psi^4\right)\\&-\frac{\beta}{2}\int_{\mathbb{R}^3}\Big((U_r+\varphi)^2(V_r+\psi)^2-U_r^2V_r^2-2(U_rV_r^2\varphi+U_r^2V_r\psi)\\&-2(U_r^2\psi^2+V_r^2\psi^2+4U_rV_r\varphi\psi)\Big).\end{aligned}$$

In order to find a critical point $(\varphi,\psi)\in E$ for $J(\varphi,\psi)$, we need to discuss each term in the expansion (2.2).

It is easy to check that

$$\begin{aligned}&\int_{\mathbb{R}^3}\left(\nabla u\nabla\varphi+P(|x|)u\varphi-3\mu U_r^2u\varphi\right)+\int_{\mathbb{R}^3}\left(\nabla v\nabla\psi+Q(|x|)v\psi-3\nu V_r^2v\psi\right)\\&-\beta\int_{\mathbb{R}^3}\left(U_r^2v\psi+V_r^2u\varphi+2U_rV_ru\psi+2U_rV_rv\varphi\right)\end{aligned}$$

is a bounded bi-linear functional in E. Thus, there is a bounded linear operator L from E to E, such that

$$\begin{aligned}\langle L(u,v),(\varphi,\psi)\rangle=&\int_{\mathbb{R}^3}\left(\nabla u\nabla\varphi+P(|x|)u\varphi-3\mu U_r^2u\varphi\right)\\&+\int_{\mathbb{R}^3}\left(\nabla v\nabla\psi+Q(|x|)v\psi-3\nu V_r^2v\psi\right)\\&-\beta\int_{\mathbb{R}^3}\left(U_r^2v\psi+V_r^2u\varphi+2U_rV_ru\psi+2U_rV_rv\varphi\right),\quad (u,v),(\varphi,\psi)\in E.\end{aligned}$$

From the above analysis, we have the following two lemmas.

Lemma 2.1. *There is a constant $C > 0$, independent of ℓ, such that for any $r \in \mathcal{D}_\ell$,*

$$\|L(u, v)\| \leqq C\|(u, v)\|, \quad (u, v) \in E.$$

Next, we discuss the invertibility of L.

Lemma 2.2. *There is a constant $\varrho > 0$, independent of ℓ, such that for any $r \in \mathcal{D}_\ell$,*

$$\|L(u, v)\| \geqq \varrho\|(u, v)\|, \quad (u, v) \in E.$$

Before we prove Lemma 2.2, we need a non-degeneracy result.

Proposition 2.3. *There exists a decreasing sequence $\{\beta_k\} \subset (-\sqrt{\mu\nu}, 0)$ with $\beta_k \to -\sqrt{\mu\nu}$ as $k \to \infty$ such that for $\beta \in (-\sqrt{\mu\nu}, 0) \cup (0, \min\{\mu, \nu\}) \cup (\max\{\mu, \nu\}, \infty)$ and $\beta \neq \beta_k$ for any k, (U, V) is non-degenerate for the system (1.6) in $H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ in the sense that the kernel is given by span$\{(\theta(\beta)\frac{\partial W}{\partial x_j}, \frac{\partial W}{\partial x_j}) \mid j = 1, 2, 3\}$, where $\theta(\beta) \neq 0$.*

Proof of Proposition 2.3. We follow the arguments in [4]. Consider the weighted eigenvalue problem in λ: $-\Delta\Phi + \Phi = \lambda W^2\Phi$ which has a sequence of eigenvalues $1 = \lambda_1 < \lambda_2 = \lambda_3 = \lambda_4 < \lambda_5 \leqq \ldots$ with associated eigenfunctions Φ_k satisfying $\int_{\mathbb{R}^3} W^2\Phi_k\Phi_m dx = 0$ for $k \neq m$. For Φ_k with $k = 2, 3, 4$ we may take them as $\frac{\partial W}{\partial x_1}, \frac{\partial W}{\partial x_2}, \frac{\partial W}{\partial x_3}$. Now for $-\sqrt{\mu\nu} < \beta < 0$ or $0 < \beta < \min\{\mu, \nu\}$, linearization of equations (1.6) at (U, V) gives us

$$\begin{cases} -\Delta\phi + \phi = W^2(a\phi + b\psi), & x \in \mathbb{R}^3, \\ -\Delta\psi + \psi = W^2(b\phi + c\psi), & x \in \mathbb{R}^3, \end{cases} \tag{2.3}$$

where

$$a(\beta) = \frac{3\mu\nu - 2\mu\beta - \beta^2}{\mu\nu - \beta^2}$$

and

$$b(\beta) = 2\beta\frac{\sqrt{(\mu - \beta)(\nu - \beta)}}{\mu\nu - \beta^2}$$

and

$$c(\beta) = \frac{3\mu\nu - 2\nu\beta - \beta^2}{\mu\nu - \beta^2}.$$

Set $\gamma_\pm = \frac{a-c}{2b} \pm \frac{1}{2b}\sqrt{(a-c)^2 + 4b^2}$. For $\beta < 0$ a direct computation shows that

$$-\Delta(\phi - \gamma_+\psi) + (\phi - \gamma_+\psi) = 3W^2(\phi - \gamma_+\psi).$$

Thus $\phi - \gamma_+\psi = \sum_{j=2}^{4}\alpha_j\Phi_j$. Returning to the equation for ψ we get

$$-\Delta\psi + \psi = (b\gamma_+ + c)W^2\psi + bW^2\sum_{j=2}^{4}\alpha_j\Phi_j.$$

Set $\psi = \sum_{j=1}^{\infty}\gamma_j\Phi_j$ and $f(\beta) = b\gamma_+ + c$. Assume $f(\beta) \neq \lambda_k$ for any k. Using orthogonality we see easily that $\gamma_j = 0$ for $j \neq 2, 3, 4$, and $\gamma_j = \frac{b\alpha_j}{3-f(\beta)}$ for $j =, 2, 3, 4$. Thus, the kernel at (U, V) is given by $\operatorname{span}\{((\gamma_+\frac{b}{3-f(\beta)} + 1)\Phi_k, \frac{b}{3-f(\beta)}\Phi_k) \mid k = 2, 3, 4\}$, a three dimensional space. We may take $\theta(\beta) = \gamma_+ + \frac{3-f(\beta)}{b(\beta)}$. Since $b \neq 0$ and $3 - c \neq 0$, we have $\theta(\beta) \neq 0$. If $f(\beta) = \lambda_j$ for some j, then similarly $\gamma_k = 0$ for $k \neq j, k \neq 2, 3, 4$. It is easy to check that Φ_k is in the kernel when $\lambda_k = \lambda_j$. Thus the kernel is generated by $\operatorname{span}\{((\gamma_+\frac{b}{3-f(\beta)} + 1)\Phi_k, \frac{b}{3-f(\beta)}\Phi_k) \mid k = 2, 3, 4\}$ and $\operatorname{span}\{\Phi_k \mid \lambda_k = \lambda_j\}$. For $\min\{\mu, \nu\} > \beta > 0$ we use γ_- instead of γ_+ to get the same result. From [4], $f(\beta) = 3$ if and only if $\beta = 0$ and $f(\beta)$ is monotone decreasing. Also, from [4] there exists a decreasing sequence β_k in $(-\sqrt{\mu\nu}, 0)$ such that $f(\beta) = \lambda_k$ if and only if $\beta = \beta_k$. Thus, for $\beta \notin \{\beta_k\}$, $f(\beta) \neq \lambda_j$ for any j. The same arguments can be used to treat the case $\beta > \max\{\mu, \nu\}$, so we omit it. □

Proof of Lemma 2.2. Suppose to the contrary of Lemma 2.2 assertion, that there are $\ell \to +\infty$, $r_\ell \in \mathcal{D}_\ell$, and $(u_\ell, v_\ell) \in E$, with

$$\big\langle L(u_\ell, v_\ell), (\varphi, \psi)\big\rangle = o(1)\|(u_\ell, v_\ell)\|\|(\varphi, \psi)\|, \quad \forall\, (\varphi, \psi) \in E. \tag{2.4}$$

We may assume that $\|(u_\ell, v_\ell)\|^2 = \ell$. For convenience, we use r to denote r_ℓ.

For $j = 1, \ldots, \ell$, let

$$\Omega_j = \left\{ z = (z', z_3) \in \mathbb{R}^2 \times \mathbb{R} : \left\langle \frac{z'}{|z'|}, \frac{x'^j}{x'^j|} \right\rangle \geqq \cos\frac{\pi}{\ell} \right\}.$$

By symmetry, we see from (2.4),

$$\begin{aligned}\int_{\Omega_1}&\big(\nabla u_\ell\nabla\varphi + P(|x|)u_\ell\varphi - 3\mu U_r^2 u_\ell\varphi\big) + \int_{\Omega_1}\big(\nabla v_\ell\nabla\psi + Q(|x|)v_\ell\psi - 3\nu V_r^2 v_\ell\psi\big)\\ &-\beta\int_{\Omega_1}\big(U_r^2 v_\ell\psi + V_r^2 u_\ell\varphi + 2U_rV_r u_\ell\psi + 2U_rV_r v_\ell\varphi\big)\\ &= \frac{1}{\ell}\langle L(u_\ell, v_\ell), (\varphi, \psi)\rangle = o(1)\Big(\frac{1}{\sqrt{\ell}}\Big)\|(\varphi, \psi)\|, \quad \forall(\varphi, \psi) \in E.\end{aligned} \tag{2.5}$$

In particular,

$$\begin{aligned}\int_{\Omega_1}&\big(|\nabla u_\ell|^2 + P(|x|)u_\ell^2 - 3\mu U_r^2 u_\ell^2\big) + \int_{\Omega_1}\big(|\nabla v_\ell|^2 + Q(|x|)v_\ell^2 - 3\nu V_r^2 v_\ell^2\big)\\ &-\beta\int_{\Omega_1}\big(V_r^2 u_\ell^2 + 4U_rV_r u_\ell v_\ell + U_r^2 v_\ell^2\big) = o(1),\end{aligned} \tag{2.6}$$

and

$$\int_{\Omega_1} \left(|\nabla u_\ell|^2 + P(|x|)u_\ell^2 + |\nabla v_\ell|^2 + Q(|x|)v_\ell^2\right) = 1. \tag{2.7}$$

Let

$$\bar{u}_\ell(x) = u_\ell(x - x^1), \quad \bar{v}_\ell(x) = v_\ell(x - x^1).$$

For any $R > 0$, $B_R(x_1) \subset \Omega_1$ since $|x^j - x^1| \geqq r \sin \frac{\pi}{\ell} \geqq c \ln \ell$ for $j = 2, \ldots, \ell$. Thus, (2.7) implies

$$\int_{B_R(0)} \left(|\nabla \bar{u}_\ell|^2 + \bar{u}_\ell^2 + |\nabla \bar{v}_\ell|^2 + \bar{v}_\ell^2\right) \leqq c.$$

So, we may assume the existence of $u,\ v \in H^1(\mathbb{R}^3)$, such that as $\ell \to +\infty$,

$$\begin{aligned} \bar{u}_\ell &\to u, \quad \text{weakly in } H^1_{\text{loc}}(\mathbb{R}^3), \quad \bar{u}_\ell \to u, \quad \text{strongly in } L^2_{\text{loc}}(\mathbb{R}^3).\\ \bar{v}_\ell &\to v, \quad \text{weakly in } H^1_{\text{loc}}(\mathbb{R}^3), \quad \bar{v}_\ell \to v, \quad \text{strongly in } L^2_{\text{loc}}(\mathbb{R}^3).\end{aligned}$$

Moreover, u and v are even in x_h $(h = 2, 3)$ and satisfy

$$\int_{\mathbb{R}^3} W^2 \left(\frac{\partial U}{\partial x_1} u + \frac{\partial V}{\partial x_1} v\right) = 0. \tag{2.8}$$

Now, we claim that (u, v) satisfies

$$\begin{cases} -\Delta u + u - 3\mu U^2 u - \beta V^2 u - 2\beta U V v = 0, & x \in \mathbb{R}^3,\\ -\Delta v + v - 3\nu V^2 v - \beta U^2 v - 2\beta U V u = 0, & x \in \mathbb{R}^3. \end{cases} \tag{2.9}$$

Define

$$\tilde{E} = \left\{(\varphi, \psi) \in H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) : \quad \int_{\mathbb{R}^3} W^2 \left(\frac{\partial U}{\partial x_1}\varphi + \frac{\partial V}{\partial x_1}\psi\right) = 0\right\}.$$

For any $R > 0$, let $(\varphi, \psi) \in C_0^\infty(B_R(0)) \times C_0^\infty(B_R(0)) \cap \tilde{E}$ and be even in $x_h, h = 2, 3$. Then $(\varphi_\ell(x), \psi_\ell(x)) =: (\varphi(x - x^1), \psi(x - x^1)) \in C_0^\infty(B_R(x^1)) \times C_0^\infty(B_R(x^1)) \subset \Omega_1$ for ℓ, if large enough. We may identify $(\varphi_\ell(x), \psi_\ell(x))$ as elements in E by redefining the values outside Ω_1 with the symmetry. With the argument in [33], we find

$$\int_{\Omega_1} \left(\nabla u_\ell \nabla \varphi_\ell + P(|x|)u_\ell \varphi_\ell - 3\mu U_r^2 u_\ell \varphi_\ell\right) \to \int_{\mathbb{R}^3} \left(\nabla u \nabla \varphi + u\varphi - 3\mu U^2 u \varphi\right), \tag{2.10}$$

$$\int_{\Omega_1} \left(\nabla v_\ell \nabla \psi_\ell + Q(|x|)v_\ell \psi_\ell - 3V_r^2 v_\ell \psi_\ell\right) \to \int_{\mathbb{R}^3} \left(\nabla v \nabla \psi + v\psi - 3\mu U^2 v \psi\right), \tag{2.11}$$

and

$$\begin{aligned}&\int_{\Omega_1}\left(U_r^2 v_\ell \psi_\ell + V_r^2 u_\ell \varphi_\ell + 2U_r V_r u_\ell \psi_\ell + 2U_r V_r v_\ell \varphi_\ell\right)\\ &\to \int_{\mathbb{R}^3}\left(U^2 v\psi + V^2 u\varphi + 2UVu\psi + 2UVv\varphi\right).\end{aligned} \tag{2.12}$$

Inserting (2.10), (2.11) and (2.12) into (2.5), we see

$$\begin{aligned}&\int_{\mathbb{R}^3}\left(\nabla u\nabla\varphi + u\varphi - 3\mu U^2 u\varphi\right) + \int_{\mathbb{R}^3}\left(\nabla v\nabla\psi + v\psi - 3\mu U^2 v\psi\right)\\ &-\beta\int_{\mathbb{R}^3}\left(U^2 v\psi + V^2 u\varphi + 2UVu\psi + 2UVv\varphi\right) = 0.\end{aligned} \tag{2.13}$$

However, since u and v are even in x_h, $h = 2, 3$, (2.13) holds for any function $(\varphi, \psi) \in C_0^\infty(B_R(0)) \times C_0^\infty(B_R(0))$, which is odd in x_h, $h = 2, 3$. Therefore, (2.13) holds for any $(\varphi, \psi) \in C_0^\infty(B_R(0)) \times C_0^\infty(B_R(0)) \cap \tilde{E}$. By the density of $C_0^\infty(B_R(0)) \times C_0^\infty(B_R(0))$ in $H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$, we see

$$\begin{aligned}&\int_{\mathbb{R}^3}\left(\nabla u\nabla\varphi + u\varphi - 3\mu U^2 u\varphi\right) + \int_{\mathbb{R}^3}\left(\nabla v\nabla\psi + v\psi - 3\mu U^2 v\psi\right)\\ &-\beta\int_{\mathbb{R}^3}\left(U^2 v\psi + V^2 u\varphi + 2UVu\psi + 2UVv\varphi\right) = 0, \quad \forall(\varphi, \psi) \in \tilde{E}.\end{aligned} \tag{2.14}$$

Noting that $(U, V) = (\alpha W, \gamma W)$ and W solves (1.9), we can verify that (2.14) holds for $(\varphi, \psi) = (\frac{\partial U}{\partial x_1}, \frac{\partial V}{\partial x_1})$. Thus, (2.14) is true for any $(\varphi, \psi) \in H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$. So, we have proved (2.9).

From Proposition 2.3, (U, V) is nondegenerate. Since we work in the space of functions which are even in x_2 and x_3, the kernel of (U, V) is given by the one dimensional $(\theta(\beta)\frac{\partial W}{\partial x_1}, \frac{\partial W}{\partial x_1})$. Thus, we see $(u, v) = c(\frac{\partial U}{\partial x_1}, \frac{\partial V}{\partial x_1})$ for some c, which implies that $(u, v) = (0, 0)$ since (u, v) satisfies (2.8).

As a result,

$$\int_{B_R(x^1)} u_\ell^2 + v_\ell^2 = o(1), \quad \forall\, R > 0.$$

On the other hand, using Lemma A.1, we obtain

$$U_{r_\ell}(x) \leqq C\mathrm{e}^{-\frac{|x-x_1|}{2}}, \quad V_{r_\ell}(x) \leqq C\mathrm{e}^{-\frac{|x-x_1|}{2}} \quad x \in \Omega_1.$$

Thus,

$$\begin{aligned}&\int_{\Omega_1}\left(|\nabla u_\ell|^2 + P(|x|)u_\ell^2 - 3\mu U_r^2 u_\ell^2\right) + \int_{\Omega_1}\left(|\nabla v_\ell|^2 + Q(|x|)v_\ell^2 - 3\nu V_r^2 v_\ell^2\right)\\ &= \int_{\Omega_1}\left(|\nabla u_\ell|^2 + P(|x|)u_\ell^2\right) + o(1) + O(\mathrm{e}^{-\frac{R}{2}})\int_{\Omega_1} u_\ell^2\\ &\quad + \int_{\Omega_1}\left(|\nabla v_\ell|^2 + Q(|x|)v_\ell^2\right) + o(1) + O(\mathrm{e}^{-\frac{R}{2}})\int_{\Omega_1} v_\ell^2\end{aligned} \tag{2.15}$$

$$\int_{\Omega_1}\left(V_r^2 u_\ell^2 + 4U_r V_r u_\ell v_\ell + U_r^2 v_\ell^2\right) = o(1) + O(\mathrm{e}^{-\frac{R}{2}})\int_{\Omega_1}\left(u_\ell^2 + v_\ell^2\right). \tag{2.16}$$

Inserting (2.15), (2.16) and (2.7) into (2.6), we find

$$\begin{aligned} o(1) &= \int_{\Omega_1} \left(|\nabla u_\ell|^2 + P(|x|)u_\ell^2 - 3\mu U_r^2 u_\ell^2\right) + \int_{\Omega_1} \left(|\nabla v_\ell|^2 + Q(|x|)v_\ell^2 - 3\nu V_r^2 v_\ell^2\right) \\ &\quad -\beta \int_{\Omega_1} \left(V_r^2 u_\ell^2 + 4U_r V_r u_\ell v_\ell + U_r^2 v_\ell^2\right) \\ &= 1 + O(\mathrm{e}^{-\frac{R}{2}}), \end{aligned}$$

which is impossible for large ℓ and large R.

As a result, we complete the proof. □

Lemma 2.4. *There exist constants $C > 0$, independent of ℓ, such that*

$$\|R^{(i)}(\varphi, \psi)\| \leqq C\|(\varphi, \psi)\|^{3-i}, \quad i = 0, 1, 2.$$

Proof. Calculating directly, we have that for any $(u, v), (w, \omega) \in E$,

$$\begin{aligned} |R(\varphi, \psi)| &\leqq \left| \int_{\mathbb{R}^3} \left(\mu U_r \varphi^3 + \nu V_r \psi^3 + \frac{\mu}{4}\varphi^4 + \frac{\nu}{4}\psi^4\right) \right| \\ &\quad + \frac{|\beta|}{2} \left| \int_{\mathbb{R}^3} \left((U_r+\varphi)^2(V_r+\psi)^2 - U_r^2 V_r^2 - 2(U_r V_r^2 \varphi + U_r^2 V_r \psi)\right.\right. \\ &\quad \left.\left. -2(U_r^2\psi^2 + V_r^2\psi^2 + 4U_r V_r \varphi\psi)\right) \right| \\ &\leqq C \int_{\mathbb{R}^3} \left(|\varphi|^3 + |\psi|^3 + |\varphi|^4 + |\psi|^4 + \varphi^2|\psi| + \psi^2|\varphi|\right) \\ &\leqq C\left(\|(\varphi, \psi)\|^3 + \|(\varphi, \psi)\|^4\right), \\ |\langle R'(\varphi, \psi), (u, v)\rangle| &= \left| \int_{\mathbb{R}^3} \left(3\mu U_r \varphi^2 u + \mu\varphi^3 u - \beta U_r \psi^3 u - 2\beta V_r \varphi^2 \psi u - \beta\varphi\psi^2 u \right.\right. \\ &\quad \left.\left. + 3\nu V_r \psi^2 v + \nu\psi^3 v - 2\beta U_r \varphi\psi v - \beta V_r \varphi^2 v - \beta\varphi^2\psi v\right) \right| \\ &\leqq C\left(\|\varphi\|_P^2 + \|\varphi\|_P^3 + \|\psi\|_P^2 + \|\psi\|_P^3 + \|\varphi\|_P^2\|\psi\|_Q + \|\varphi\|_P\|\psi\|_Q^2\right) \\ &\quad \times (\|u\|_P + \|v\|_Q) \\ &\leqq C\left(\|(\varphi, \psi)\|^2 + \|(\varphi, \psi)\|^3\right)\|(u, v)\|, \end{aligned}$$

and, similarly,

$$|\langle R''(\varphi, \psi)(u, v), (w, \omega)\rangle| \leqq C\left(\|(\varphi, \psi)\| + \|(\varphi, \psi)\|^2\right)\|(u, v)\|\|(w, \omega)\|.$$

Hence, the result follows. □

Proposition 2.5. *There is an integer $\ell_0 > 0$ such that, for each $\ell \geqq \ell_0$, there is a C^1 map from $\mathcal{D}_\ell$ to $H_{P,s} \times H_{Q,s}$: $(\varphi, \psi) = (\varphi(r), \psi(r))$, $r = |x^1|$, satisfying $(\varphi, \psi) \in E$, and*

$$\left\langle \frac{\partial J(\varphi, \psi)}{\partial(\varphi, \psi)}, (g, h) \right\rangle = 0, \quad \forall\ (g, h) \in E.$$

Moreover, there is a constant C, such that

$$\|(\varphi, \psi)\| \leqq C\left(\frac{\ell}{r^m} + \frac{\ell}{r^n} + \ell^{\frac{1}{2}}\mathrm{e}^{-\frac{2\pi}{\ell}r}\frac{\ell}{r}\right). \tag{2.17}$$

Proof. It follows from Lemma 2.6 below, that $l(\varphi, \psi)$ is a bounded linear functional in E. Thus, there is an $l_\ell \in E$, such that

$$l(\varphi, \psi) = \langle l_\ell, (\varphi, \psi) \rangle.$$

Thus, finding a critical point for $J(\varphi, \psi)$ is equivalent to solving

$$l_\ell + L(\varphi, \psi) + R'(\varphi, \psi) = 0. \tag{2.18}$$

By Lemma 2.2, L is invertible. Thus, (2.18) can be rewritten as

$$(\varphi, \psi) = A(\varphi, \psi) =: -L^{-1} l_\ell - L^{-1} R'(\varphi, \psi). \tag{2.19}$$

Set

$$D = \left\{ (\varphi, \psi) : (\varphi, \psi) \in E, \|(\varphi, \psi)\| \leqq \frac{\ell^{1+\sigma}}{r^m} + \frac{\ell^{1+\sigma}}{r^n} + \ell^{\frac{1+\sigma}{2}} e^{-\frac{2\pi}{\ell} r} \frac{\ell}{r} \right\},$$

where $\sigma > 0$ is small.

From Lemma 2.4 and Lemma 2.6, below, for ℓ large,

$$\begin{aligned} \|A(\varphi, \psi)\| &\leqq C\|l_\ell\| + C\|\varphi, \psi\|^2 \\ &\leqq C\left(\frac{\ell}{r^m} + \frac{\ell}{r^n} + \ell^{\frac{1}{2}} e^{-\frac{2\pi}{\ell} r} \frac{\ell}{r} \right) + C\left(\frac{\ell^{1+\sigma}}{r^m} + \frac{\ell^{1+\sigma}}{r^n} + \ell^{\frac{1+\sigma}{2}} e^{-\frac{2\pi}{\ell} r} \frac{\ell}{r} \right)^2 \\ &\leqq \frac{\ell^{1+\sigma}}{r^m} + \frac{\ell^{1+\sigma}}{r^n} + \ell^{\frac{1+\sigma}{2}} e^{-\frac{2\pi}{\ell} r} \frac{\ell}{r}, \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} \|A(\varphi_1, \psi_1) - A(\varphi_2, \psi_2)\| &= \|L^{-1} R'(\varphi_1, \psi_1) - L^{-1} R'(\varphi_2, \psi_2)\| \\ &\leqq C\big(\|(\varphi_1, \psi_1)\| + \|(\varphi_1, \psi_1)\|^2\big) \|(\varphi_1, \psi_1) - (\varphi_2, \psi_2)\| \\ &\leqq \frac{1}{2} \|(\varphi_1, \psi_1) - (\varphi_2, \psi_2)\|. \end{aligned}$$

Therefore, A maps D into D and is a contraction map. So, by the contraction mapping theorem, there exists $(\varphi, \psi) \in E$, such that $(\varphi, \psi) = A(\varphi, \psi)$. Finally, by (2.19), we have

$$\|(\varphi, \psi)\| \leqq C\left(\frac{\ell}{r^m} + \frac{\ell}{r^n} + \ell^{\frac{1}{2}} e^{-\frac{2\pi}{\ell} r} \frac{\ell}{r} \right).$$

□

Lemma 2.6. *There is a constant $C > 0$ independent of ℓ, such that*

$$\|l_\ell\| \leqq C\left(\frac{\ell}{r^m} + \frac{\ell}{r^n} + \ell^{\frac{1}{2}} e^{-\frac{2\pi}{\ell} r} \frac{\ell}{r} \right).$$

Proof. First, we see

$$\begin{aligned}
&\left|\sum_{j=1}^{\ell}\int_{\mathbb{R}^3}\big(P(|x|)-1\big)U_{x^j}\varphi+\sum_{j=1}^{\ell}\int_{\mathbb{R}^3}\big(Q(|x|)-1\big)V_{x^j}\psi\right|\\
&\quad=\ell\left|\int_{\mathbb{R}^3}\big(P(|x|)-1\big)U_{x^1}\varphi+\int_{\mathbb{R}^3}\big(Q(|x|)-1\big)V_{x^1}\psi\right|\\
&\quad=\ell\left|\int_{\mathbb{R}^3}\big(P(x-x^1)-1\big)U_{\mu}\varphi(x-x^1)+\int_{\mathbb{R}^3}\big(Q(x-x^1)-1\big)U_{\nu}\psi(x-x^1)\right|\\
&\quad\leqq C\ell\left(\frac{1}{r^m}+\frac{1}{r^n}\right)\|(\varphi,\psi)\|.
\end{aligned}\tag{2.21}$$

Next, we estimate

$$\mu\int_{\mathbb{R}^3}\left(U_r^3-\sum_{j=1}^{\ell}U_{x^j}^3\right)\varphi+\nu\int_{\mathbb{R}^3}\left(V_r^3-\sum_{j=1}^{\ell}V_{x^j}^3\right)\psi.$$

By symmetry, we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3}\big(U_r^3-\sum_{j=1}^{\ell}U_{x^j}^3\big)\varphi\right|=\ell\left|\int_{\Omega_1}\big(U_r^3-\sum_{j=1}^{\ell}U_{x^j}^3\big)\varphi\right|\\
&\quad=C\ell\int_{\Omega_1}\left(U_{x_1}^2\sum_{j=2}^{\ell}U_{x^j}+O\left(U_{x_1}\sum_{j=2}^{\ell}U_{x^j}^2\right)\right)|\varphi|\\
&\quad\leqq C\ell\sum_{j=1}^{\ell}\frac{\mathrm{e}^{-|x^1-x^j|}}{|x^1-x^j|}\left(\int_{\Omega_1}|\varphi|^2\right)^{\frac12}\leqq C\ell^{\frac12}\sum_{j=1}^{\ell}\frac{\mathrm{e}^{-|x^1-x^j|}}{|x^1-x^j|}\|\varphi\|_P\\
&\quad\leqq C\ell^{\frac12}\mathrm{e}^{-\frac{2\pi}{\ell}r}\frac{\ell}{r}\|(\varphi,\psi)\|.
\end{aligned}\tag{2.22}$$

Similarly,

$$\left|\int_{\mathbb{R}^3}\left(V_r^3-\sum_{j=1}^{\ell}V_{x^j}^3\right)\psi\right|\leqq C\ell^{\frac12}\mathrm{e}^{-\frac{2\pi}{\ell}r}\frac{\ell}{r}\|(\varphi,\psi)\|.\tag{2.23}$$

Finally, since $U=\frac{\alpha}{\gamma}V$, we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3}\left(V_r^2U_r-\sum_{j=1}^{\ell}V_{x^j}^2U_{x^j}\right)\varphi+\int_{\mathbb{R}^3}\left(V_rU_r^2-\sum_{j=1}^{\ell}V_{x^j}U_{x^j}^2\right)\psi\right|\\
&=\left|\left(\frac{\gamma}{\alpha}\right)^2\int_{\mathbb{R}^3}\left(U_r^3-\sum_{j=1}^{\ell}U_{x^j}^3\right)\varphi+\left(\frac{\alpha}{\gamma}\right)^2\int_{\mathbb{R}^3}\left(V_r^3-\sum_{j=1}^{\ell}V_{x^j}^3\right)\psi\right|\\
&\leqq C\ell^{\frac12}\mathrm{e}^{-\frac{2\pi}{\ell}r}\frac{\ell}{r}\|(\varphi,\psi)\|.
\end{aligned}\tag{2.24}$$

The result follows from (2.21), (2.22), (2.23) and (2.24). □

Now we are ready to prove Theorem 1.1. Let $(\varphi_r, \psi_r) = (\varphi(r), \psi(r))$ be the map obtained in Proposition 2.5. Define

$$F(r) = I(U_r + \varphi_r, V_r + \psi_r), \quad \forall\, r \in \mathcal{D}_\ell.$$

With the same argument used in [13,22], we can easily check that for ℓ sufficiently large, if r is a critical point of $F(r)$, then $(U_r + \varphi_r, V_r + \psi_r)$ is a critical point of I.

Proof of Theorem 1.7. It follows from Lemmas 2.1 and 2.4 that

$$\|L(\varphi_r, \psi_r)\| \leqq C\|(\varphi_r, \psi_r)\|, \quad |R(\varphi_r, \psi_r)| \leqq C\|(\varphi_r, \psi_r)\|^3.$$

So, Proposition 2.5 and A.2 give

$$\begin{aligned}
F(r) =& I(U_r, V_r) + l(\varphi_r, \psi_r) + \frac{1}{2}\big\langle L(\varphi_r, \psi_r), (\varphi_r, \psi_r)\big\rangle + R(\varphi_r, \psi_r)\\
=& I(U_r, V_r) + O\big(\|l_\ell\|\|(\varphi_r, \psi_r)\| + \|(\varphi_r, \psi_r)\|^2\big)\\
=& I(U_r, V_r) + O\big(\frac{1}{\ell^{m-1+\sigma}} + \frac{1}{\ell^{n-1+\sigma}}\big)\\
=& \ell\left(A + \left(\frac{aB}{r^m} + \frac{bC}{r^n} - (D + \beta H)\mathrm{e}^{-\frac{2\pi r}{\ell}}\frac{\ell}{r}\right) + O\left(\frac{1}{\ell^{m+\sigma}} + \frac{1}{\ell^{n+\sigma}}\right)\right).
\end{aligned}$$

We prove the theorem only for the case $m = n$, since the other case is similar. If $m = n$, then

$$F(r) = \ell\left(A + \left(\frac{aB + bC}{r^m} - (D + \beta H)\mathrm{e}^{-\frac{2\pi r}{\ell}}\frac{\ell}{r}\right) + O\left(\frac{1}{\ell^{m+\sigma}}\right)\right).$$

Note that $\beta \neq \beta_k$ with $\{\beta_k\}$ is given in Proposition 2.3 and $(D + \beta H) > 0$. Let $\mathcal{D}_\ell$ be defined in (2.1). Consider the following maximization problem

$$\max_{r\in\mathcal{D}_\ell} F(r). \tag{2.25}$$

Assume that (2.25) is achieved by some r_ℓ in $\mathcal{D}_\ell$. We will prove that r_ℓ is an interior point of $\mathcal{D}_\ell$.

Define

$$g(t) = \frac{aB + bC}{t^m \ell^m} - \frac{(D + \beta H)\mathrm{e}^{-2\pi t}}{t}.$$

Then,

$$g'(t) = -\frac{m(aB + bC)}{t^{m+1}\ell^m} + \frac{2\pi(D + \beta H)\mathrm{e}^{-2\pi t}}{t} + \frac{(D + \beta H)\mathrm{e}^{-2\pi t}}{t^2}.$$

It is easy to check that $g(t)$ has a maximum point t_ℓ, satisfying

$$\frac{m(aB + bC)}{t^{m+1}\ell^m} = \frac{2\pi(D + \beta H)\mathrm{e}^{-2\pi t}}{t} + \frac{(D + \beta H)\mathrm{e}^{-2\pi t}}{t^2}. \tag{2.26}$$

Thus,

$$t_\ell = \left(\frac{m}{2\pi} + o(1)\right) \ln \ell.$$

So, the function

$$\bar{g}(r) := \frac{aB + bC}{r^m} - \frac{(D + \beta H)\ell e^{-\frac{2\pi r}{\ell}}}{r}$$

has a maximum point

$$\bar{r}_\ell = \ell t_\ell = \left(\frac{m}{2\pi} + o(1)\right) \ell \ln \ell.$$

Hence, it follows from the expression of $F(r)$ that the maximizer r_ℓ is an interior point of $\mathcal{D}_\ell$, if we choose $\delta > 0$ small.

Now we prove that $u_{r_\ell} = U_{r_\ell} + \varphi_{r_\ell}$ and $v_{r_\ell} = V_{r_\ell} + \psi_{r_\ell}$ are positive.

First, by regularity theory, we have that $(\varphi_{r_\ell}, \psi_{r_\ell})$ tends to zero in L^∞-norm as $\ell \to \infty$. Set $(u_{r_\ell})_- = \max\{-u_{r_\ell}, 0\}$, $(v_{r_\ell})_- = \max\{-v_{r_\ell}, 0\}$. Then we have that $(u_{r_\ell})_-$ and $(v_{r_\ell})_-$ tends to zero as $\ell \to \infty$. Due to the form of the solution, we see that when $u_{r_\ell}(x) < 0$ we have $V_{r_\ell}(x) \leqq -\psi_{r_\ell}(x) = o(1)$ as $\ell \to \infty$. We see from $\langle I'(u_{r_\ell}, v_{r_\ell}), ((u_{r_\ell})_-, 0)\rangle = 0$ that

$$\|(u_{r_\ell})_-\|_P^2 = \mu \int_{\mathbb{R}^3} |(u_{r_\ell})_-|^4 + \beta \int_{\mathbb{R}^3} v_{r_\ell}^2 |(u_{r_\ell})_-|^2 \leqq o(1) \|(u_{r_\ell})_-\|_P^2.$$

Hence, $u_{r_\ell} > 0$.

Similarly, we can prove $v_{r_\ell} > 0$. As a result, $(U_{r_\ell} + \varphi_{r_\ell}, V_{r_\ell} + \psi_{r_\ell})$ is a positive solution of (1.1). □

3. Segregated Vector Solutions and The proof of Theorem 1.2

In this section we consider synchronized vector solutions and prove Theorem 1.2 by proving Theorem 1.8. Let

$$\bar{Y}_j = \frac{\partial U_{\mu, x^j}}{\partial r}, \quad \bar{Z}_j = \frac{\partial U_{\nu, y^j}}{\partial \rho}, \quad j = 1, \ldots, \ell,$$

where x^j is defined in (1.7) and y^j is defined in (1.10).

For simplicity of notation, in the sequel we use $\bar{U}_{x^j}$ and $\bar{V}_{y^j}$ to replace U_{μ, x^j} and U_{ν, y^j}, respectively. In this section, we assume

$$(r, \rho) \in \mathbb{D}_\ell \times \mathbb{D}_\ell =: \left[\left(\frac{m}{2\pi} - \bar{\delta}\right) \ell \ln \ell, \quad M\ell \ln \ell\right] \times \left[\left(\frac{m}{2\pi} - \bar{\delta}\right) \ell \ln \ell, \quad M\ell \ln \ell\right], \tag{3.1}$$

where $\bar{\delta} > 0$ is a small but $M > 0$ is a large constant depending only on $m, a, b, \bar{B} and \bar{G}$ (from the Appendix).

Define

$$\mathbb{E} = \left\{ (u, v) \in H_{P,s} \times H_{Q,s},\ \sum_{j=1}^{\ell} \int_{\mathbb{R}^3} \bar{U}_{x^j}^2 \bar{Y}_j u = 0,\quad \sum_{j=1}^{\ell} \int_{\mathbb{R}^3} \bar{V}_{y^j}^2 \bar{Z}_j v = 0 \right\}.$$

Let

$$\bar{J}(\bar{\varphi}, \bar{\psi}) = I(\bar{U}_r + \bar{\varphi}, \bar{V}_\rho + \bar{\psi}), \quad (\bar{\varphi}, \bar{\psi}) \in \mathbb{E}.$$

Then, similar to (2.2), $\bar{J}(\bar{\varphi}, \bar{\psi})$ has the following expansion

$$\bar{J}(\bar{\varphi}, \bar{\psi}) = \bar{J}(0, 0) + \bar{l}(\bar{\varphi}, \bar{\psi}) + \frac{1}{2}\bar{L}(\bar{\varphi}, \bar{\psi}) + \bar{R}(\bar{\varphi}, \bar{\psi}), \quad (\bar{\varphi}, \bar{\psi}) \in \mathbb{E}, \tag{3.2}$$

where $\bar{L}(\bar{\varphi}, \bar{\psi})$ and $\bar{R}(\bar{\varphi}, \bar{\psi})$ are exactly defined exactly as $L(\varphi, \psi)$ and $R(\varphi, \psi)$ in Section 2, but with U_r, V_r, φ and ψ being replaced by $\bar{U}_r$, $\bar{V}_\rho$, $\bar{\varphi}$ and $\bar{\psi}$, respectively. Moreover, we can find a bounded linear operator $\bar{L} : \mathbb{E} \to \mathbb{E}$ corresponding to the quadratic part $\bar{L}(\bar{\varphi}, \bar{\psi})$. However, $\bar{l}(\bar{\varphi}, \bar{\psi})$ has the following form

$$\begin{aligned}\bar{l}(\bar{\varphi}, \bar{\psi}) = &\sum_{j=1}^{\ell} \int_{\mathbb{R}^3} \big(P(|x|) - 1\big)\bar{U}_{x^j}\bar{\varphi} + \mu \int_{\mathbb{R}^3} \Big(\bar{U}_r^3 - \sum_{j=1}^{\ell} \bar{U}_{x^j}^3\Big)\bar{\varphi} \\ &+ \sum_{j=1}^{\ell} \int_{\mathbb{R}^3} \big(Q(|x|) - 1\big)\bar{V}_{y^j}\bar{\psi} + \nu \int_{\mathbb{R}^3} \Big(\bar{V}_\rho^3 - \sum_{j=1}^{\ell} \bar{V}_{y^j}^3\Big)\bar{\psi} \\ &- \beta \int_{\mathbb{R}^3} \big(\bar{U}_r \bar{V}_\rho^2 \bar{\varphi} + \bar{U}_r^2 \bar{V}_\rho \bar{\psi}\big).\end{aligned}$$

The above analysis gives the following lemma.

Lemma 3.1. *There is a constant $C > 0$, independent of ℓ, such that for any $(r, \rho) \in \mathbb{D}_\ell \times \mathbb{D}_\ell$,*

$$\|\bar{L}(u, v)\| \leqq C\|(u, v)\|, \quad (u, v) \in \mathbb{E}.$$

Lemma 3.2. *There exist $\bar{\beta}^* > 0$, $\bar{\varrho} > 0$ independent of ℓ, such that for any $(r, \rho) \in \mathbb{D}_\ell \times \mathbb{D}_\ell$, if $\beta < \bar{\beta}^*$, then*

$$\|\bar{L}(u, v)\| \geqq \bar{\varrho}\|(u, v)\|, \quad (u, v) \in \mathbb{E}.$$

Proof. The argument is similar to the proof of Lemma 2.2. Arguing by contradiction, we suppose that there are $\ell \to +\infty$, $r_\ell, \rho_\ell \in \mathbb{D}_\ell$, and $(u_\ell, v_\ell) \in \mathbb{E}$, with $\|(u_\ell, v_\ell)\|^2 = \ell$, and

$$\big\langle L(u_\ell, v_\ell), (\bar{\varphi}, \bar{\psi})\big\rangle = o(1)\|(u_\ell, v_\ell)\|\|(\bar{\varphi}, \bar{\psi})\|, \quad \forall\, (\bar{\varphi}, \bar{\psi}) \in \mathbb{E}. \tag{3.3}$$

For $j = 1, \dots, \ell$, let

$$\Omega_j = \left\{ z = (z', z_3) \in \mathbb{R}^2 \times \mathbb{R} : \left\langle \frac{z'}{|z'|}, \frac{x'^j}{|x'^j|} \right\rangle \geqq \cos\frac{\pi}{\ell} \right\},$$

$$\tilde{\Omega}_j = \left\{ z = (z', z_3) \in \mathbb{R}^2 \times \mathbb{R} : \left\langle \frac{z'}{|z'|}, \frac{y'^j}{|y'^j|} \right\rangle \geqq \cos\frac{\pi}{\ell} \right\}.$$

We will use r, ρ to replace r_ℓ, ρ_ℓ, respectively, in the sequel. By symmetry, we see from (3.3),

$$\begin{aligned}&\int_{\Omega_1}\left(\nabla u_\ell\nabla\bar\varphi+P(|x|)u_\ell\bar\varphi-3\mu\bar U_r^2u_\ell\bar\varphi\right)+\int_{\Omega_1}\left(\nabla v_\ell\nabla\bar\psi+Q(|x|)v_\ell\bar\psi-3\nu\bar V_\rho^2v_\ell\bar\psi\right)\\&\quad-\beta\int_{\Omega_1}\left(\bar U_r^2v_\ell\bar\psi+\bar V_\rho^2u_\ell\bar\varphi+2\bar U_r\bar V_\rho u_\ell\bar\psi+2\bar U_r\bar V_\rho v_\ell\bar\varphi\right)\\&\quad=\frac{1}{\ell}\langle L(u_\ell,v_\ell),(\bar\varphi,\bar\psi)\rangle=o(1)\left(\frac{1}{\sqrt{\ell}}\right)\|(\bar\varphi,\bar\psi)\|,\quad\forall(\bar\varphi,\bar\psi)\in\mathbb{E}.\end{aligned}\tag{3.4}$$

In particular,

$$\begin{aligned}&\int_{\Omega_1}\left(|\nabla u_\ell|^2+P(|x|)u_\ell^2-3\mu\bar U_r^2u_\ell^2\right)+\int_{\Omega_1}\left(|\nabla v_\ell|^2+Q(|x|)v_\ell^2-3\nu\bar V_\rho^2v_\ell^2\right)\\&\quad-\beta\int_{\Omega_1}\left(\bar V_\rho^2u_\ell^2+4\bar U_r\bar V_\rho u_\ell v_\ell+\bar U_r^2v_\ell^2\right)=o(1),\end{aligned}\tag{3.5}$$

and

$$\int_{\Omega_1}\left(|\nabla u_\ell|^2+P(|x|)u_\ell^2+|\nabla v_\ell|^2+Q(|x|)v_\ell^2\right)=1.\tag{3.6}$$

Obviously, estimates (3.4), (3.5) and (3.6) are also true on $\tilde\Omega_1$.

Let

$$\bar u_\ell(x)=u_\ell(x-x^1),\quad \bar v_\ell(x)=v_\ell(x-y^1).$$

Now we consider $\bar u_\ell(x)$ in detail. The analysis on $\bar v_\ell(x)$ is similar for v_ℓ and u_ℓ are also even with respect to the axis y^1.

We may assume the existence of $\bar u\in H^1(\mathbb{R}^3)$, such that as $\ell\to+\infty$,

$$\bar u_\ell\to\bar u,\quad\text{weakly in } H^1_{\text{loc}}(\mathbb{R}^3),\quad \bar u_\ell\to\bar u,\quad\text{strongly in } L^2_{\text{loc}}(\mathbb{R}^3).$$

Let $\bar\varphi\in C_0^\infty(B_R(0))$ and be even in x_h, $h=2,3$. Define $\bar\varphi_\ell(x)=:\bar\varphi(x-x^1)\in C_0^\infty(B_R(x^1))$. Then choosing $(\bar\varphi,\bar\psi)=(\bar\varphi_\ell,0)$ in (3.4) and proceeding as we did in Lemma 2.2, we can see that $\bar u$ satisfies

$$-\Delta\bar u+\bar u-3\mu U_{1,\mu}^2\bar u=0,\quad\text{in }\mathbb{R}^3.\tag{3.7}$$

Also, by the nondegeneracy of $U_{1,\mu}$, we find $\bar u=0$.

Using the same argument on $\tilde\Omega_1$, we can prove that as $\ell\to+\infty$,

$$\bar v_\ell\to 0,\quad\text{weakly in } H^1_{\text{loc}}(\mathbb{R}^3),\quad \bar v_\ell\to 0,\quad\text{strongly in } L^2_{\text{loc}}(\mathbb{R}^3).$$

As a result,

$$\int_{B_R(x^1)}u_\ell^2=o(1),\quad\int_{B_R(y^1)}v_\ell^2=o(1),\quad\forall\ R>0.$$

On the other hand, using Lemma A.1, we obtain

$$U_{r_\ell}(x)\leqq C\mathrm{e}^{-\frac{|x-x_1|}{2}},\quad x\in\Omega_1;\quad V_{\rho_\ell}(x)\leqq C\mathrm{e}^{-\frac{|x-y_1|}{2}},\quad x\in\tilde\Omega_1.$$

Thus, from (3.3), we see

$$\begin{aligned} o(1)\ell &= \int_{\mathbb{R}^3}\big(|\nabla u_\ell|^2+P(|x|)u_\ell^2-3\mu\bar{U}_r^2u_\ell^2\big)+\int_{\mathbb{R}^3}\big(|\nabla v_\ell|^2+Q(|x|)v_\ell^2-3\nu\bar{V}_\rho^2v_\ell^2\big)\\ &\quad -\beta\int_{\mathbb{R}^3}\big(\bar{V}_\rho^2u_\ell^2+4\bar{U}_r\bar{V}_\rho u_\ell v_\ell+\bar{U}_r^2v_\ell^2\big)\\ &= \int_{\mathbb{R}^3}\big(|\nabla u_\ell|^2+P(|x|)u_\ell^2+|\nabla v_\ell|^2+Q(|x|)v_\ell^2\big)-\beta\int_{\mathbb{R}^3}\big(\bar{V}_\rho^2u_\ell^2+\bar{U}_r^2v_\ell^2\big)\\ &\quad -4\beta\ell\int_{\Omega_1}\bar{U}_r\bar{V}_\rho u_\ell v_\ell-\ell\int_{\Omega_1}3\mu\bar{U}_r^2u_\ell^2-\ell\int_{\tilde{\Omega}_1}3\nu\bar{V}_\rho^2v_\ell^2\\ &= \int_{\mathbb{R}^3}\big(|\nabla u_\ell|^2+P(|x|)u_\ell^2+|\nabla v_\ell|^2+Q(|x|)v_\ell^2\big)-\beta\int_{\mathbb{R}^3}\big(\bar{V}_\rho^2u_\ell^2+\bar{U}_r^2v_\ell^2\big)\\ &\quad +O(\mathrm{e}^{-|x^1-y^1|})\|u_\ell\|_P\|v_\ell\|_Q+\Big(o(1)+O(\mathrm{e}^{-R})\Big)\int_{\mathbb{R}^3}\Big(u^2+v_\ell^2\Big)\\ &= \ell-\beta\int_{\mathbb{R}^3}\big(\bar{V}_\rho^2u_\ell^2+\bar{U}_r^2v_\ell^2\big)+O(\mathrm{e}^{-\frac{\pi}{\ell}r_\ell})\ell+\Big(o(1)+O(\mathrm{e}^{-R})\Big)\ell\\ &\geqq \ell-C\beta\ell+O(\mathrm{e}^{-\frac{\pi}{\ell}r_\ell})\ell+\Big(o(1)+O(\mathrm{e}^{-R})\Big)\ell, \end{aligned} \tag{3.8}$$

since

$$0\leqq\int_{\mathbb{R}^3}\big(\bar{V}_\rho^2u_\ell^2+\bar{U}_r^2v_\ell^2\big)\leqq C\int_{\mathbb{R}^3}\big(u_\ell^2+v_\ell^2\big)\leqq C\ell,$$

where C is independent of ℓ.

If we choose $\beta<\bar{\beta}^*=:\frac{1}{C}$, then (3.8) is impossible for large R and ℓ. Consequently, we complete the proof. □

Now we apply the above reduction process to the functional $\bar{J}(\bar{\varphi},\bar{\psi})$.

Proposition 3.3. *There is an integer $\bar{\ell}_0>0$, such that for each $\ell\geqq\bar{\ell}_0$, there is a C^1 map from $\mathbb{D}_\ell\times\mathbb{D}_\ell$ to $H_{P,s}\times H_{Q,s}$: $(\bar{\varphi},\bar{\psi})=(\bar{\varphi}(r,\rho),\bar{\psi}(r,\rho))$, $r=|x^1|$, $\rho=|y^1|$, satisfying $(\bar{\varphi},\bar{\psi})\in\mathbb{E}$, and*

$$\left\langle\frac{\partial\bar{J}(\bar{\varphi},\bar{\psi})}{\partial(\bar{\varphi},\bar{\psi})},(h,g)\right\rangle=0,\quad\forall\,(h,g)\in\mathbb{E}.$$

Moreover,

$$\|(\bar{\varphi},\bar{\psi})\|\leqq C\ell\left(\frac{1}{r^m}+\frac{1}{r^n}\right)+C|\beta|\ell^{\frac{1}{2}}\frac{\ell}{r}\mathrm{e}^{-\sqrt{(\rho-r\cos\frac{\pi}{\ell})^2+r^2(\frac{\pi}{\ell})^2}}. \tag{3.9}$$

Proof. We see that $\bar{l}(\bar{\varphi},\bar{\psi})$ is a bounded linear functional in $\mathbb{E}$. Thus, there is $\bar{l}_\ell\in\mathbb{E}$, such that

$$\bar{l}(\bar{\varphi},\bar{\psi})=\langle\bar{l}_\ell,(\bar{\varphi},\bar{\psi})\rangle.$$

Hence, to verify the proposition, we only need to use the argument in the proof of Proposition 2.5 and the following estimate on $\|\bar{l}_\ell\|$:

$$\|\bar{l}_\ell\| \leqq C\ell\left(\frac{1}{r^m}+\frac{1}{r^n}\right)+C|\beta|\ell^{\frac{1}{2}}\frac{\ell}{r}\mathrm{e}^{-\sqrt{(\rho-r\cos\frac{\pi}{\ell})^2+r^2(\frac{\pi}{\ell})^2}}, \tag{3.10}$$

where C is independent of ℓ and β.

Now we prove (3.10). Indeed, since in Lemma 2.6, we have a similar estimate on the first four terms of $\bar{l}(\bar{\varphi},\bar{\psi})$, we need to estimate only

$$\int_{\mathbb{R}^3}\left(\bar{U}_r\bar{V}_\rho^2\bar{\varphi}+\bar{U}_r^2\bar{V}_\rho\bar{\psi}\right).$$

By symmetry, we see

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3}\bar{U}_r\bar{V}_\rho^2\bar{\varphi}\right| = \ell\left|\int_{\Omega_1}\bar{U}_r\bar{V}_\rho^2\bar{\varphi}\right|\\
&= C\ell\int_{\Omega_1}\left(\bar{U}_{x^1}\bar{V}_{y^1}^2+\bar{U}_{x^1}\bar{V}_{y^\ell}^2+\bar{U}_{x^1}\sum_{j=2}^{\ell-1}\bar{V}_{y^j}^2+\bar{V}_{y_1}^2\sum_{j=2}^{\ell}\bar{U}_{x^j}+\sum_{j=2}^{\ell}\bar{U}_{x^j}\sum_{j=2}^{\ell}\bar{V}_{y^j}^2\right)|\bar{\varphi}|\\
&\leqq C\ell\int_{\Omega_1}\left(\bar{V}_{y^1}\frac{\mathrm{e}^{-|y^1-x^1|}}{|y^1-x^1|}+\bar{V}_{y^\ell}\frac{\mathrm{e}^{-|y^\ell-x^1|}}{|y^\ell-x^1|}+\bar{U}_{x^1}\sum_{j=2}^{\ell}\mathrm{e}^{-\frac{1}{2}|x^j-y^1|}+\bar{V}_{y^1}^2\sum_{j=2}^{\ell}\mathrm{e}^{-\frac{1}{2}|x^j-x^1|}\right)|\bar{\varphi}|\\
&\leqq C\ell\frac{\mathrm{e}^{-|y^1-x^1|}}{|y^1-x^1|}\left(\int_{\Omega_1}|\bar{\varphi}|^2\right)^{\frac{1}{2}} \leqq C\ell^{\frac{1}{2}}\frac{\ell}{r}\mathrm{e}^{-\sqrt{(\rho-r\cos\frac{\pi}{\ell})^2+r^2(\frac{\pi}{\ell})^2}}\|(\bar{\varphi},\bar{\psi})\|,
\end{aligned}\tag{3.11}$$

since $|x-y^j|\geqq\frac{1}{2}|y^j-y^1|$, $|x-x^j|\geqq\frac{1}{2}|x^j-x^1|$ if $x\in\Omega_1$ and $j=2,\ldots,\ell$, and for ℓ large $\mathrm{e}^{-|y^1-x^1|}\leqq 2\mathrm{e}^{-\sqrt{(\rho-r\cos\frac{\pi}{\ell})^2+r^2(\frac{\pi}{\ell})^2}}$.

Similarly,

$$\left|\int_{\mathbb{R}^3}\bar{U}_r^2\bar{V}_\rho\bar{\varphi}\right|\leqq C\ell^{\frac{1}{2}}\frac{\ell}{r}\mathrm{e}^{-\sqrt{(\rho-r\cos\frac{\pi}{\ell})^2+r^2(\frac{\pi}{\ell})^2}}\|(\bar{\varphi},\bar{\psi})\|. \tag{3.12}$$

Hence, we complete the proof. □

Proof of Theorem 1.8. Let $(\bar{\varphi}_{r,\rho},\bar{\psi}_{r,\rho})=(\bar{\varphi}(r,\rho),\bar{\psi}(r,\rho))$ be the map obtained in Proposition 3.3. Define

$$\bar{F}(r,\rho)=I(\bar{U}_r+\bar{\varphi}_{r,\rho},\bar{V}_\rho+\bar{\psi}_{r,\rho}),\quad\forall\, r,\rho\in\mathbb{D}_\ell.$$

We can easily check that for ℓ sufficiently large, if (r,ρ) is a critical point of $\bar{F}(r,\rho)$, then $(\bar{U}_r+\bar{\varphi}_{r,\rho},\bar{V}_\rho+\bar{\psi}_{r,\rho})$ is a critical point of I.

It follows from Propositions 3.3 and Proposition A.3 that

$$\begin{aligned}\bar{F}(r,\rho) &= I(\bar{U}_r,\bar{V}_\rho)+\bar{l}(\bar{\varphi}_{r,\rho},\bar{\psi}_{r,\rho})+\frac{1}{2}\langle \bar{L}(\bar{\varphi}_{r,\rho},\bar{\psi}_{r,\rho}),(\bar{\varphi}_{r,\rho},\bar{\psi}_{r,\rho})\rangle+\bar{R}(\bar{\varphi}_{r,\rho},\bar{\psi}_{r,\rho})\\
&= I(\bar{U}_r,\bar{V}_\rho)+O\left(\|\bar{l}_\ell\|\|(\bar{\varphi}_{r,\rho},\bar{\psi}_{r,\rho})\|+\|(\bar{\varphi}_{r,\rho},\bar{\psi}_{r,\rho})\|^2\right)\\
&= I(\bar{U}_r,\bar{V}_\rho)+O\left(\beta^2\frac{\ell^3}{r^2}\mathrm{e}^{-2\sqrt{(\rho-r\cos\frac{\pi}{\ell})^2+r^2(\frac{\pi}{\ell})^2}}\right)+O\left(\frac{1}{r^{m-1+\sigma}}+\frac{1}{\rho^{m-1+\sigma}}\right)\\
&= \ell\Bigg(\bar{A}+\frac{a\bar{B}}{r^m}-\bar{D}\frac{\ell}{r}\mathrm{e}^{-\frac{2\pi r}{\ell}}-\left(o(1)\beta+O\left(\frac{\beta^2}{\ln\ell}\right)\right)\frac{\ell}{r}\mathrm{e}^{-2\sqrt{(\rho-r\cos\frac{\pi}{\ell})^2+r^2(\frac{\pi}{\ell})^2}}\\
&\quad+\frac{b\bar{C}}{\rho^m}-\bar{G}\frac{\ell}{\rho}\mathrm{e}^{-\frac{2\pi\rho}{\ell}}+O\left(\frac{1}{r^{m+\sigma}}+\frac{1}{\rho^{m+\sigma}}\right)\Bigg).\end{aligned}$$

For any $\beta<\bar{\beta}^*$, where $\bar{\beta}^*$ is defined in Lemma 3.2, we can choose $\ell^*>0$ such that $\bar{D}+o(1)\beta+O(\frac{\beta^2}{\ln\ell})>0$ for $\ell\geqq\ell^*$. Consider the maximization problem

$$\max_{(r,\rho)\in\mathbb{D}_\ell\times\mathbb{D}_\ell}\bar{F}(r,\rho). \tag{3.13}$$

Assume that (3.13) is achieved by some (r_ℓ,ρ_ℓ) in $\mathbb{D}_\ell\times\mathbb{D}_\ell$. We will prove that (r_ℓ,ρ_ℓ) is an interior point of $\mathbb{D}_\ell\times\mathbb{D}_\ell$.

When $o(1)\beta+O(\frac{\beta^2}{\ln\ell})>0$, we define

$$\bar{g}(t)=\frac{a\bar{B}}{t^m\ell^m}-\left(\bar{D}+o(1)\beta+O\left(\frac{\beta^2}{\ln\ell}\right)\right)\frac{\mathrm{e}^{-2\pi t}}{t},\quad \bar{h}(s)=\frac{b\bar{C}}{s^m\ell^m}-\frac{\bar{G}\mathrm{e}^{-2\pi s}}{s}.$$

Then,

$$\begin{aligned}\bar{g}'(t)=&-\frac{ma\bar{B}}{t^{m+1}\ell^m}+\frac{2\pi\left(\bar{D}+o(1)\beta+O\left(\frac{\beta^2}{\ln\ell}\right)\right)\mathrm{e}^{-2\pi t}}{t}\\
&+\left(\bar{D}+o(1)\beta+O\left(\frac{\beta^2}{\ln\ell}\right)\right)\frac{\mathrm{e}^{-2\pi t}}{t^2},\end{aligned}$$

and

$$\bar{h}'(s)=-\frac{mb\bar{C}}{s^{m+1}\ell^m}+\frac{2\pi\bar{G}\mathrm{e}^{-2\pi s}}{s}+\frac{\bar{G}\mathrm{e}^{-2\pi s}}{s^2}.$$

It is easy to check that $\bar{g}(t)$ has a maximum point, t_ℓ, satisfying

$$\frac{ma\bar{B}}{t^{m+1}\ell^m}=\frac{\left(\bar{D}+o(1)\beta+O\left(\frac{\beta^2}{\ln\ell}\right)\right)\mathrm{e}^{-2\pi t}}{t}\left(2\pi+\frac{1}{t}\right). \tag{3.14}$$

Thus,

$$t_\ell=\left(\frac{m}{2\pi}+o(1)\right)\ln\ell,$$

and by (3.14),

$$\bar{g}(t_\ell) = \frac{a\bar{B}}{t_\ell^m \ell^m}\left(1 + O\left(\frac{1}{t_\ell}\right)\right).$$

So, the function

$$\tilde{g}(r) := \frac{a\bar{B}}{r^m} - \left(\bar{D} + o(1)\beta + O\left(\frac{\beta^2}{\ln \ell}\right)\right)\frac{\ell e^{-\frac{2\pi r}{\ell}}}{r}$$

has a maximum point

$$\bar{r}_\ell = \ell t_\ell = \left(\frac{m}{2\pi} + o(1)\right)\ell \ln \ell,$$

with

$$\tilde{g}(\bar{r}_\ell) = \bar{g}(t_\ell) = \frac{c_1 + o(1)}{|\ln \ell|^m \ell^m}$$

for some constant $c_1 > 0$ depending only on $a, \bar{B}, m$.

If $o(1)\beta + O(\frac{\beta^2}{\ln \ell}) \leqq 0$, we define $\tilde{g}_1(t) = \frac{a\bar{B}}{r^m} - \frac{\bar{D}e^{\frac{-2\pi r}{\ell}}}{r}$. Then we still get that the maximum of $\tilde{g}_1$ is $\frac{c_1 + o(1)}{|\ln \ell|^m \ell^m}$.

Similarly, the function

$$\tilde{h}(\rho) := \frac{b\bar{C}}{\rho^m} - \frac{\bar{G}\ell e^{-\frac{2\pi\rho}{\ell}}}{\rho}$$

has a maximum point

$$\bar{\rho}_\ell = \left(\frac{m}{2\pi} + o(1)\right)\ell \ln \ell,$$

with

$$\tilde{h}(\bar{\rho}_\ell) = \frac{c_2 + o(1)}{|\ln \ell|^m \ell^m}$$

for some constant $c_2 > 0$ depending on $b, \bar{C}, m$.

Hence, if $o(1)\beta + O(\frac{\beta^2}{\ln \ell}) > 0$ we see

$$\begin{aligned}\bar{F}(r_\ell, \rho_\ell) &\geqq \ell\left(\bar{A} + \tilde{g}(\bar{r}_\ell) + \tilde{h}(\bar{\rho}_\ell) + \frac{o(1)}{|\ln \ell|^m \ell^m}\right)\\ &= \ell\left(\bar{A} + \frac{c_1 + c_2 + o(1)}{|\ln \ell|^m \ell^m}\right).\end{aligned} \tag{3.15}$$

If $o(1)\beta + O(\frac{\beta^2}{\ln \ell}) \leqq 0$, we have

$$\begin{aligned}\bar{F}(r_\ell, \rho_\ell) &\geqq \ell\left(\bar{A} + \tilde{g}_1(\bar{r}_\ell) + \tilde{h}(\bar{\rho}_\ell) + \frac{o(1)}{|\ln \ell|^m \ell^m}\right)\\ &= \ell\left(\bar{A} + \frac{c_1 + c_2 + o(1)}{|\ln \ell|^m \ell^m}\right).\end{aligned} \tag{3.16}$$

Now we show that the maximum cannot be on the boundary of $\mathbb{D}_\ell \times \mathbb{D}_\ell$. Suppose that $r_\ell = \left(\frac{m}{2\pi} - \bar{\delta}\right)\ell \ln \ell$. Then if $o(1)\beta + O(\frac{\beta^2}{\ln \ell}) > 0$, for any $\rho \in \mathbb{D}_\ell$,

$$\begin{aligned}
\bar{F}(r_\ell, \rho) &\leqq \ell\left(\bar{A} + \frac{C}{|\ln \ell|^m \ell^m} - \bar{D}\frac{e^{-2\pi(\frac{m}{2\pi}-\bar{\delta})\ln \ell}}{(\frac{m}{2\pi} - \bar{\delta})\ln \ell} + \tilde{h}(\bar{\rho}_\ell)\right) \\
&\leqq \ell\left(\bar{A} + \frac{C}{|\ln \ell|^m \ell^m} - \bar{D}\frac{1}{\ell^{2\pi(\frac{m}{2\pi}-\bar{\delta})}(\frac{m}{2\pi} - \bar{\delta})\ln \ell}, + \frac{c_2 + o(1)}{|\ln \ell|^m \ell^m}\right) \\
&< \ell\left(\bar{A} + \frac{c_2 + o(1)}{|\ln \ell|^m \ell^m}\right).
\end{aligned}$$

This is a contradiction to (3.15). If $o(1)\beta + O(\frac{\beta^2}{\ln \ell}) \leqq 0$ for any $\rho \in \mathbb{D}_\ell$

$$\begin{aligned}
\bar{F}(r_\ell, \rho) &\leqq \ell\left(\bar{A} + \frac{C}{|\ln \ell|^m \ell^m} - \left(\bar{D} + o(1)\beta + O\left(\frac{\beta^2}{\ln \ell}\right)\right)\right. \\
&\qquad \left. \times \frac{e^{-2\pi(\frac{m}{2\pi}-\bar{\delta})\ln \ell}}{(\frac{m}{2\pi} - \bar{\delta})\ln \ell} + \tilde{h}(\bar{\rho}_\ell)\right) \\
&\leqq \ell\left(\bar{A} + \frac{C}{|\ln \ell|^m \ell^m} - \left(\bar{D} + o(1)\beta + O\left(\frac{\beta^2}{\ln \ell}\right)\right)\right. \\
&\qquad \left. \times \frac{1}{\ell^{2\pi(\frac{m}{2\pi}-\bar{\delta})}(\frac{m}{2\pi} - \bar{\delta})\ln \ell} + \frac{c_2 + o(1)}{|\ln \ell|^m \ell^m}\right) \\
&< \ell\left(\bar{A} + \frac{c_2 + o(1)}{|\ln \ell|^m \ell^m}\right).
\end{aligned}$$

This is a contradiction to (3.16). Suppose that $r_\ell = M\ell \ln \ell$. Then for any $\rho \in \mathbb{D}_\ell$, we see

$$\begin{aligned}
\bar{F}(r_\ell, \rho) &\leqq \ell\left(\bar{A} + \frac{a\bar{B}}{(M\ell \ln \ell)^m} + \tilde{h}(\bar{\rho}_\ell)\right) + \frac{o(1)\ell}{|\ln \ell|^m \ell^m} \\
&\leqq \ell\left(\bar{A} + \frac{a\bar{B}}{(M\ell \ln \ell)^m} + \frac{c_2 + o(1)}{|\ln \ell|^m \ell^m}\right) < \ell\left(\bar{A} + \frac{c_1 + c_2 + o(1)}{|\ln \ell|^m \ell^m}\right),
\end{aligned}$$

which is also a contradiction to (3.15), if $M > 0$ is large.

Similarly, we can also verify that $\rho_\ell \neq \left(\frac{m}{2\pi} - \gamma\right)\ell \ln \ell$ and $\rho_\ell \neq M\ell \ln \ell$ for suitable $\gamma > 0$ and $M > 0$.

So we have proved that (r_ℓ, ρ_ℓ) is an interior point of $\mathbb{D}_\ell \times \mathbb{D}_\ell$ for large ℓ, thus (r_ℓ, ρ_ℓ) is a critical point of $\bar{F}(r, \rho)$.

Proceeding as in the proof of Theorem 1.7, for $\beta < 0$ we can check that $u_{r_\ell,\rho_\ell} = \bar{U}_{r_\ell} + \bar{\varphi}_{r_\ell,\rho_\ell}$ and $v_{r_\ell,\rho_\ell} = \bar{V}_{\rho_\ell} + \bar{\psi}_{r_\ell,\rho_\ell}$ are positive, and hence a solution of (1.1). When $\beta > 0$, we note that the estimates on $\bar{\varphi}_{r_\ell,\rho_\ell}$ and $\bar{\psi}_{r_\ell,\rho_\ell}$ can be done independently of $\beta \leqq \beta^*$ so that u_{r_ℓ,ρ_ℓ} and v_{r_ℓ,ρ_ℓ} are uniformly bounded independently of β. Again we can then use the same argument as before to show the solutions are positive when $\beta > 0$ is small enough.

As a result, we complete the proof. □

4. Extensions

Remark 4.1. First of all, we remark that our main result on synchronized solutions can be stated and proved for the case of $\mathbb{R}^2$ with little change to the proof. However, we do not know whether our result on segregated solutions is valid for $\mathbb{R}^2$. It would be to examine this case.

Remark 4.2. Radial symmetry can be replaced by the following weaker symmetry assumption: after suitably rotating the coordinate system,

(P1) $P(x) = P(x', x_3) = P(|x'|, |x_3|)$, where $x = (x', x_3) \in \mathbb{R}^2 \times \mathbb{R}$,
(P2) $P(x) = p^2 + \frac{a}{|x|^m} + O\big(\frac{1}{|x|^{m+\theta}}\big)$ as $|x| \to +\infty$, where $p > 0$, $a \in \mathbb{R}$, $m > 1$ and $\theta > 0$ are some constants.
(Q1) $Q(x) = Q(x', x_3) = Q(|x'|, |x_3|)$, where $x = (x', x_3) \in \mathbb{R}^2 \times \mathbb{R}$,
(Q2) $Q(x) = q^2 + \frac{b}{|x|^n} + O\big(\frac{1}{|x|^{n+\varepsilon}}\big)$ as $|x| \to +\infty$, where $b \in \mathbb{R}$, $n > 1$, $\varepsilon > 0$, and $q > 0$ are some constants.

Remark 4.3. Our methods allow us to treat sign-changing solutions, also. The solutions (u, v) are constructed in the form with u and v components both having alternating sign-changing bumps at infinity.

For sign-changing solutions to problem 1.1, we have the following result.

Theorem 4.4. *Suppose that $P(r)$ satisfies (P) and $Q(r)$ satisfies (Q), then there exists a decreasing sequence $\{\beta_k\} \subset (-\sqrt{\mu\nu}, 0)$ with $\beta_k \to -\sqrt{\mu\nu}$ as $k \to \infty$ such that for $\beta \in (-\sqrt{\mu\nu}, 0) \cup (0, \min\{\mu, \nu\}) \cup (\max\{\mu, \nu\}, \infty)$ and $\beta \neq \beta_k$ for any k, problem (1.1) has infinitely many non-radial sign-changing synchronized solutions (u_ℓ, v_ℓ), whose energy can be arbitrarily large, provided one of the following two conditions holds:*

(i) *$m < n$, $a < 0$ and $b \in \mathbb{R}$; or $m > n$, $a \in \mathbb{R}$ and $b < 0$;*
(ii) *$m = n$, $aB + bC < 0$, where B and C are defined in Proposition A.2.*

Furthermore, as $\ell \to \infty$

$$||\sqrt{|\mu - \beta|}u_\ell - \sqrt{|\nu - \beta|}v_\ell||_{H^1} + ||\sqrt{|\mu - \beta|}u_\ell - \sqrt{|\nu - \beta|}v_\ell||_{L^\infty} \to 0.$$

The sketch of proof for Theorem 4.4. For any positive even number ℓ, set

$$U_r(x) = \sum_{j=1}^{\ell} (-1)^j U_{x^j}, \quad V_r(x) = \sum_{j=1}^{\ell} (-1)^j V_{x^j}.$$

We will find a solution for system (1.1) of the form $(U_r + \varphi, V_r + \psi)$ with $(\varphi, \psi) \in E$. To this end, we should also perform the same procedure as the proof of Theorem 1.7.

Expanding $J(U_r + \varphi, V_r + \psi)$, analyzing each term of the expansion and performing the reduction process, we conclude that finding a critical point with the form $(U_r+\varphi, V_r+\psi)$ can be reduced to finding a minimum point of the following function in the interior of $\mathcal{D}_\ell$:

$$F^*(r)=\ell\left(A+\left(\frac{aB}{r^m}+\frac{bC}{r^n}+(\tilde{D}+\beta\tilde{E})e^{-\frac{2\pi r}{\ell}}\frac{\ell}{r}\right)+O\left(\frac{1}{\ell^{m+\sigma}}+\frac{1}{\ell^{n+\sigma}}\right)\right),$$

where A, B and C are defined in Proposition A.2 and $\tilde{D}$ and $\tilde{E}$ are two positive constants.

The rest of the proof can be finished as in the proof of Theorem 1.7. □

Theorem 1.8 can be generalized to the following general system

$$\begin{cases}-\Delta u+P(|x|)u=\mu u^3+\dfrac{\beta p}{p+q}|v|^q|u|^{p-2}u, & x\in\mathbb{R}^3,\\ -\Delta v+Q(|x|)v=\nu v^3+\dfrac{\beta q}{p+q}|u|^p|v|^{q-2}v, & x\in\mathbb{R}^3,\end{cases}\tag{4.1}$$

where $2\leqq p\leqq 5$, $2\leqq q\leqq 5$ and $p+q\leqq 6$.

Theorem 4.5. *Suppose that $P(r)$ satisfies (P), $Q(r)$ satisfies (Q) and $m=n$, $a>0$, $b>0$. Then there exists $\tilde{\beta}>0$ such that, for $\beta<\tilde{\beta}$, problem (4.1) has infinitely many nonradial positive segregated solutions, whose energy can be arbitrarily large.*

The proof for the case $p=2$ or $q=2$ is the same as that of Theorem 1.8. To prove the case $p>2$ and $q>2$, we notice that for any $\tau>0$ small,

$$\int_{\mathbb{R}^3}U_r^pV_\rho^q=\tilde{C}\ell\left(\frac{\ell}{r}\right)^{\min(p,q)}e^{-\min(p-\tau,q-\tau)\sqrt{(\rho-r\cos\frac{\pi}{\ell})^2+r^2(\frac{\pi}{\ell})^2}}.$$

Hence, the energy expansion has the form

$$\begin{aligned}\tilde{F}(r,\rho)=\ell\Bigg(&\bar{A}+\frac{a\bar{B}}{r^m}-\bar{D}\frac{\ell}{r}e^{-\frac{2\pi r}{\ell}}-\tilde{C}\beta\left(\frac{\ell}{r}\right)^{\min(p,q)}\\ &\times e^{-\min(p-\tau,q-\tau)\sqrt{(\rho-r\cos\frac{\pi}{\ell})^2+r^2(\frac{\pi}{\ell})^2}}\\ &+\frac{b\bar{C}}{\rho^m}-\bar{G}\frac{\ell}{\rho}e^{-\frac{2\pi\rho}{\ell}}+O\left(\frac{1}{r^{m+\sigma}}+\frac{1}{\rho^{m+\sigma}}\right)\Bigg).\end{aligned}$$

The rest part of the proof is similar to that of Theorem 1.8. We point out here that the condition $\beta<\tilde{\beta}$ can guarantee that the reduction programm works (see Lemma 3.2).

Finally, arguing as we prove Theorem 4.4, we can obtain infinitely many sign-changing solutions to problem (4.1).

Theorem 4.6. *Suppose that $P(r)$ satisfies (P), $Q(r)$ satisfies (Q) and $m=n$, $a<0$, $b<0$. Then there exists $\tilde{\beta}^*>0$ such that, for $\beta<\tilde{\beta}^*$, problem (4.1) has infinitely many nonradial sign-changing segregated solutions, whose energy can be arbitrarily large.*

Acknowledgements. The authors are grateful to the referee's thoughtful reading of details of the paper and nice suggestions to improve the results. S. Peng was supported by the fund from NSFC (11071092; 11125101).

Appendix A. Energy Expansions

In this section, we will expand the energies $I(U_r, V_r)$ and $I(\bar{U}_r, \bar{V}_\rho)$, where

$$I(u,v) = \frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2 + P(|x|)u^2 + |\nabla v|^2 + Q(|x|)v^2\right) \\ -\frac{1}{4}\int_{\mathbb{R}^3}(\mu|u|^4 + \nu|v|^4) - \frac{\beta}{2}\int_{\mathbb{R}^3} u^2v^2, \quad (u,v) \in H.$$

First, we estimate U_r, V_r, $\bar{U}_r$ and $\bar{V}_\rho$.

Lemma A.1. *For any $\theta \geqq 1$, there is a $\sigma > 0$, such that*

$$U_r^\theta(x) = U_{x^j}^\theta(x) + O\left(\frac{1}{r^\sigma}\mathrm{e}^{-\frac{1}{2}|x-x^j|}\right), \quad \forall x \in \Omega_j,$$

$$V_r^\theta(x) = V_{x^j}^\theta(x) + O\left(\frac{1}{r^\sigma}\mathrm{e}^{-\frac{1}{2}|x-x^j|}\right), \quad \forall\, x \in \Omega_j, \tag{A.1}$$

$$\bar{U}_r^\theta(x) = \bar{U}_{x^j}^\theta(x) + O\left(\frac{1}{r^\sigma}\mathrm{e}^{-\frac{1}{2}|x-x^j|}\right), \quad \forall x \in \Omega_j,$$

$$\bar{V}_\rho^\theta(x) = \bar{V}_{y^j}^\theta(x) + O\left(\frac{1}{\rho^\sigma}\mathrm{e}^{-\frac{1}{2}|x-y^j|}\right), \quad \forall \quad x \in \tilde{\Omega}_j, \tag{A.2}$$

where

$$\Omega_j = \left\{z = (z', z_3) \in \mathbb{R}^3 : \left\langle \frac{z'}{|z'|}, \frac{x'^j}{|x'^j|}\right\rangle \geqq \cos\frac{\pi}{\ell}\right\}, \quad j = 1, \ldots, \ell,$$

$$\tilde{\Omega}_j = \left\{z = (z', z_3) \in \mathbb{R}^3 : \left\langle \frac{z'}{|z'|}, \frac{y'^j}{|y'^j|}\right\rangle \geqq \cos\frac{\pi}{\ell}\right\}, \quad j = 1, \ldots, \ell.$$

Proof. Without loss of generality, and in view of the symmetry, we need to estimate only U_r^θ in Ω_1.
For any $x \in \Omega_1$, we have

$$|x - x^j| \geqq |x - x^1|, \quad \forall\, x \in \Omega_1,$$

which gives $|x - x^j| \geqq \frac{1}{2}|x^j - x^1|$ if $|x - x^1| \geqq \frac{1}{2}|x^j - x^1|$. On the other hand, if $|x - x^1| \leqq \frac{1}{2}|x^j - x^1|$, then

$$|x - x^j| \geqq |x^j - x^1| - |x - x^1| \geqq \frac{1}{2}|x^j - x^1|.$$

So, we find

$$|x - x^j| \geqq \frac{1}{2}|x^j - x^1|, \quad \forall\, x \in \Omega_1. \tag{A.3}$$

Now,

$$U_r^\theta(x) = U_{x^1}^\theta(x) + O\left(U_{x^1}^{\theta-1}(x)\sum_{j=2}^{\ell} U_{x^j}(x) + \left(\sum_{j=2}^{\ell} U_{x^j}(x)\right)^\theta\right).$$

But for any $\kappa > 0$, using (A.3), we find

$$\begin{aligned}\sum_{j=2}^{\ell} U_{x^j}^\kappa(x) &\leqq C\sum_{j=2}^{\ell} \mathrm{e}^{-\kappa|x-x^j|}\\ &\leqq C\sum_{j=2}^{\ell} \mathrm{e}^{-\frac{1}{2}\kappa|x^1-x^j|} \leqq C\mathrm{e}^{-\frac{\kappa\pi r}{\ell}} \leqq \frac{C}{\ell^\sigma}, \quad \forall\, x \in \Omega_1.\end{aligned}$$

As a result,

$$U_{x^1}^{\theta-1}(x)\sum_{j=1}^{\ell} U_{x^j}(x) \leqq U_{x^1}^{\theta-\frac{1}{2}}(x)\sum_{j=2}^{\ell} U_{x^j}^{\frac{1}{2}}(x) \leqq \frac{C}{\ell^\sigma}\mathrm{e}^{-\frac{1}{2}|x-x^1|}, \quad \forall\, x \in \Omega_1,$$

and

$$\left(\sum_{j=2}^{\ell} U_{x^j}(x)\right)^\theta \leqq U_{x^1}^{\frac{\theta}{2}}(x)\left(\sum_{j=2}^{\ell} U_{x^j}^{\frac{1}{2}}(x)\right)^\theta \leqq \frac{C}{\ell^\sigma}\mathrm{e}^{-\frac{1}{2}|x-x^1|}, \quad \forall\, x \in \Omega_1.$$

So, (A.2) follows. □

Proposition A.2. *There is a small constant $\sigma > 0$, such that*

$$\begin{aligned}I(U_r, V_r) = A + \ell\left(\frac{aB}{r^m} + \frac{bC}{r^n} - (D+\beta H)\mathrm{e}^{-\frac{2\pi r}{\ell}}\frac{\ell}{r}\right)\\ + O\left(\frac{\ell}{r^m\ell^\sigma} + \frac{\ell}{r^n\ell^\sigma} + \ell\mathrm{e}^{-\frac{3\pi r}{\ell}}\right),\end{aligned}$$

where

$$A = \frac{\mu+\nu-2\beta}{4(\mu\nu-\beta^2)}\int_{\mathbb{R}^3} W^4, \quad B = \frac{\alpha^2}{2}\int_{\mathbb{R}^3} W^2, \quad C = \frac{\gamma^2}{2}\int_{\mathbb{R}^3} W^2,$$

D, H are positive constants independent of ℓ and for $\beta > -\sqrt{\mu\nu}$ it holds that $D + \beta H > 0$.

Proof. Write

$$
\begin{aligned}
&I(U_r+V_r)\\
&\quad=\frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla U_r|^2+P(|x|)U_r^2+|\nabla V_r|^2+Q(|x|)V_r^2\right)\\
&\qquad-\frac{1}{4}\int_{\mathbb{R}^3}\left(\mu U_r^4+\nu V_r^4\right)-\frac{\beta}{2}\int_{\mathbb{R}^3}U_r^2V_r^2\\
&\quad=\ell\left\{\frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla U|^2+U^2+|\nabla V|^2+V^2\right)\right.\\
&\qquad\left.-\frac{1}{4}\int_{\mathbb{R}^3}\left(\mu U^4+\nu V^4\right)-\frac{\beta}{2}\int_{\mathbb{R}^3}U^2V^2\right\}\\
&\qquad+\frac{1}{2}\int_{\mathbb{R}^3}\left((P(|x|)-1)\sum_{i,j=1}^{\ell}U_{x^i}U_{x^j}+(Q(|x|)-1)\sum_{i,j=1}^{\ell}V_{x^i}V_{x^j}\right)\\
&\qquad-\frac{\mu}{4}\int_{\mathbb{R}^3}\left(\left(\sum_{i=1}^{\ell}U_{x^i}\right)^4-\sum_{i=1}^{\ell}U_{x^i}^4-2\sum_{i\neq j}^{\ell}U_{x^i}^3U_{x^j}\right)\\
&\qquad-\frac{\nu}{4}\int_{\mathbb{R}^3}\left(\left(\sum_{i=1}^{\ell}V_{x^i}\right)^4-\sum_{i=1}^{\ell}V_{x^i}^4-2\sum_{i\neq j}^{\ell}V_{x^i}^3V_{x^j}\right)\\
&\qquad-\frac{\beta}{2}\int_{\mathbb{R}^3}\left(\left(\sum_{i=1}^{\ell}U_{x^i}\right)^2\left(\sum_{i=1}^{\ell}V_{x^i}\right)^2-\sum_{i=1}^{\ell}U_{x^i}^2V_{x^i}^2-\sum_{i\neq j}^{\ell}V_{x^i}^2U_{x^i}U_{x^j}\right.\\
&\qquad\left.-\sum_{i\neq j}^{\ell}U_{x^i}^2V_{x^i}V_{x^j}\right).
\end{aligned}
\tag{A.4}
$$

Now we estimate each term in (A.4).
By symmetry and Lemma A.1, we see

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\left(P(|x|)-1\right)\sum_{i,j=1}^{\ell}U_{x^i}U_{x^j}\\
&\quad=\ell\int_{\Omega_1}\left(P(|x|)-1\right)\left(U_{x^1}^2+2U_{x^1}\sum_{i=2}^{\ell}U_{x^i}+\left(\sum_{i=2}^{\ell}U_{x^i}\right)^2\right)\\
&\quad=\ell\int_{\Omega_1}\left(P(|x|)-1\right)U_{x^1}^2+\ell O\left(\frac{1}{\ell^\sigma}\right)\int_{\Omega_1}\left|P(|x|)-1\right|U_{x^1}\mathrm{e}^{-|x-x^1|}\\
&\quad=\ell\left(\frac{a\alpha^2}{r^m}\int_{\mathbb{R}^3}W^2+O\left(\frac{1}{r^m\ell^\sigma}\right)\right),
\end{aligned}
\tag{A.5}
$$

where $\sigma>0$ is a small constant.

Similarly,

$$\int_{\mathbb{R}^3} \big(Q(|x|)-1\big) \sum_{i,j=1}^{\ell} V_{x^i} V_{x^j} = \ell \left(\frac{b\gamma^2}{r^n} \int_{\mathbb{R}^3} W^2 + O\left(\frac{1}{r^n \ell^\sigma}\right)\right). \tag{A.6}$$

Employing Lemma A.1 and (A.3), we see

$$\begin{aligned}
&\int_{\mathbb{R}^3} \left(\left(\sum_{i=1}^{\ell} U_{x^i}\right)^4 - \sum_{i=1}^{\ell} U_{x^i}^4 - 2\sum_{i\neq j}^{\ell} U_{x^i}^3 U_{x^j} \right) \\
&= \ell \int_{\Omega_1} \Bigg(\left(U_{x^1} + \sum_{i=2}^{\ell} U_{x^i}\right)^4 - \left(U_{x^1}^4 + \sum_{i=2}^{\ell} U_{x^i}^4 \right) \\
&\qquad -2\left(U_{x^1}^3 \sum_{i=2}^{\ell} U_{x^i} + U_{x^1} \sum_{i=2}^{\ell} U_{x^i}^3 + \sum_{i,j\geqq 2, i\neq j}^{\ell} U_{x^i}^3 U_{x^j} \right)\Bigg) \\
&= 2\ell \int_{\Omega_1} U_{x^1}^3 \sum_{i=2}^{\ell} U_{x^i} + O\big(\ell e^{-\frac{3\pi r}{\ell}}\big),
\end{aligned} \tag{A.7}$$

and similarly

$$\int_{\mathbb{R}^3} \left(\left(\sum_{i=1}^{\ell} V_{x^i}\right)^4 - \sum_{i=1}^{\ell} V_{x^i}^4 - 2\sum_{i\neq j}^{\ell} V_{x^i}^3 V_{x^j} \right) = 2\ell \int_{\Omega_1} V_{x^1}^3 \sum_{i=2}^{\ell} V_{x^i} + O\left(\ell e^{-\frac{3\pi r}{\ell}}\right), \tag{A.8}$$

$$\begin{aligned}
&\int_{\mathbb{R}^3} \left(\left(\sum_{i=1}^{\ell} U_{x^i}\right)^2 \left(\sum_{i=1}^{\ell} V_{x^i}\right)^2 - \sum_{i=1}^{\ell} U_{x^i}^2 V_{x^i}^2 - \sum_{i\neq j}^{\ell} V_{x^i}^2 U_{x^i} U_{x^j} - \sum_{i\neq j}^{\ell} U_{x^i}^2 V_{x^i} V_{x^j} \right) \\
&= \ell \int_{\Omega_1} \left(V_{x^1}^2 U_{x^1} \sum_{i=2}^{\ell} U_{x^j} + U_{x^1}^2 V_{x^1} \sum_{i=2}^{\ell} V_{x^i} \right) + O\left(\ell e^{-\frac{3\pi r}{\ell}}\right).
\end{aligned} \tag{A.9}$$

Noting that

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^3} (|\nabla U|^2 + U^2 + |\nabla V|^2 + V^2) - \frac{1}{4}\int_{\mathbb{R}^3} (\mu U^4 + \nu V^4) - \frac{\beta}{2}\int_{\mathbb{R}^3} U^2 V^2 \\
&= \frac{\mu+\nu-2\beta}{4(\mu\nu-\beta^2)} \int_{\mathbb{R}^3} W^4,
\end{aligned}$$

we insert (A.5)-(A.9) into (A.4) and find

$$\begin{aligned}
&I(U_r, V_r) \\
&= \frac{\mu+\nu-2\beta}{4(\mu\nu-\beta^2)} \int_{\mathbb{R}^3} W^4 + \ell \left(\frac{a\alpha^2}{2r^m} \int_{\mathbb{R}^3} W^2 + \frac{b\gamma^2}{2r^n} \int_{\mathbb{R}^3} W^2 \right)
\end{aligned}$$

$$
\begin{aligned}
&-\ell\left\{\frac{\mu}{2}\int_{\Omega_1} U_{x^1}^3\sum_{i=2}^{\ell} U_{x^i}+\frac{\nu}{2}\int_{\Omega_1} V_{x^1}^3\sum_{i=2}^{\ell} V_{x^i}\right.\\
&\left.+\frac{\beta}{2}\int_{\Omega_1}\left(V_{x^1}^2 U_{x^1}\sum_{i=2}^{\ell} U_{x^i}+U_{x^1}^2 V_{x^1}\sum_{i=2}^{\ell} V_{x^i}\right)\right\}\\
&+O\left(\frac{\ell}{r^m\ell^\sigma}+\frac{\ell}{r^n\ell^\sigma}+\ell \mathrm{e}^{-\frac{3\pi r}{\ell}}\right)\\
=&\frac{\mu+\nu-2\beta}{4(\mu\nu-\beta^2)}\int_{\mathbb{R}^3} W^4+\ell\left(\frac{a\alpha^2}{2r^m}\int_{\mathbb{R}^3} W^2+\frac{b\gamma^2}{2r^n}\int_{\mathbb{R}^3} W^2\right) \qquad \text{(A.10)}\\
&-\left\{\int_{\Omega_1}\left(\frac{\mu\alpha}{2}U_{x^1}^3+\frac{\nu\gamma}{2}V_{x^1}^3\right)\sum_{i=2}^{\ell}\frac{\mathrm{e}^{-|x^1-x^i|}}{|x^1-x^i|}\right\}\\
&-\frac{\beta}{2}\left\{\int_{\Omega_1}\left(\alpha V_{x^1}^2 U_{x^1}+\gamma U_{x^1}^2 V_{x^1}\right)\sum_{i=2}^{\ell}\frac{\mathrm{e}^{-|x^1-x^i|}}{|x^1-x^i|}\right\}\\
&+O\left(\frac{\ell}{r^m\ell^\sigma}+\frac{\ell}{r^n\ell^\sigma}+\ell \mathrm{e}^{-\frac{3\pi r}{\ell}}\right)\\
=&\frac{\mu+\nu-2\beta}{4(\mu\nu-\beta^2)}\int_{\mathbb{R}^3} W^4+\ell\left(\frac{a\alpha^2}{2r^m}\int_{\mathbb{R}^3} W^2+\frac{b\gamma^2}{2r^n}\int_{\mathbb{R}^3} W^2\right)\\
&-\ell(D+\beta H)\mathrm{e}^{-\frac{2\pi r}{\ell}}\frac{\ell}{r}+O\left(\frac{\ell}{r^m\ell^\sigma}+\frac{\ell}{r^n\ell^\sigma}+\ell \mathrm{e}^{-\frac{3\pi r}{\ell}}\right),
\end{aligned}
$$

since $U_{x^i}=\frac{\alpha}{\gamma}V_{x^i}$, where $D, H>0$ are constants independent of ℓ. For $\beta>-\sqrt{\mu\nu}$ using the expression of U_{x^1} and V_{x^1} by the Hölder inequality it is easy to see $D+\beta H>0$. As a result, we can complete the proof. □

Proposition A.3. *There is a small constant $\sigma>0$, such that*

$$
\begin{aligned}
I(\bar U_r,\bar V_\rho)=\ell\Bigg(&\bar A+\frac{a\bar B}{r^m}-\bar D e^{-\frac{2\pi r}{\ell}}\frac{\ell}{r}-o(1)\beta \mathrm{e}^{-2\sqrt{(\rho-r\cos\frac{\pi}{\ell})^2+r^2(\frac{\pi}{\ell})^2}}\frac{\ell}{r}\\
&+\frac{b\bar C}{\rho^n}-\bar G \mathrm{e}^{-\frac{2\pi\rho}{\ell}}\frac{\ell}{\rho}+O\left(\frac{1}{r^{m+\sigma}}+\frac{1}{\rho^{m+\sigma}}\right)\Bigg),
\end{aligned}
$$

where

$$
\bar A=\frac14\int_{\mathbb{R}^3}\left(\mu U_{1,\mu}^4+\nu U_{1,\nu}^4\right),\quad \bar B=\frac12\int_{\mathbb{R}^3} U_{1,\mu}^2,\quad \bar C=\frac12\int_{\mathbb{R}^3} U_{1,\nu}^2,
$$

$\bar D$, $\bar G$ are positive constants independent of ℓ.

Proof. Since

$$
\begin{aligned}
&I(\bar U_r+\bar V_\rho)\\
&=\frac12\int_{\mathbb{R}^3}\left(|\nabla\bar U_r|^2+P(|x|)\bar U_r^2+|\nabla\bar V_\rho|^2+Q(|x|)\bar V_\rho^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{4}\int_{\mathbb{R}^3}\left(\mu\bar{U}_r^4+\nu\bar{V}_\rho^4\right)-\frac{\beta}{2}\int_{\mathbb{R}^3}\bar{U}_r^2\bar{V}_\rho^2\\
&=\frac{\ell}{4}\int_{\mathbb{R}^3}\left(\mu\bar{U}_{1,\mu}^4+\nu\bar{U}_{1,\nu}^4\right)-\frac{\beta}{2}\int_{\mathbb{R}^3}\left(\sum_{i=1}^{\ell}\bar{U}_{x^i}\right)^2\left(\sum_{i=1}^{\ell}\bar{V}_{y^i}\right)^2\\
&+\frac{1}{2}\int_{\mathbb{R}^3}\left((P(|x|)-1)\sum_{i,j=1}^{\ell}\bar{U}_{x^i}\bar{U}_{x^j}+\left(Q(|x|)-1\right)\sum_{i,j=1}^{\ell}\bar{V}_{y^i}\bar{V}_{y^j}\right)\\
&-\frac{\mu}{4}\int_{\mathbb{R}^3}\left(\left(\sum_{i=1}^{\ell}\bar{U}_{x^i}\right)^4-\sum_{i=1}^{\ell}\bar{U}_{x^i}^4-2\sum_{i\neq j}^{\ell}\bar{U}_{x^i}^3\bar{U}_{x^j}\right)\\
&-\frac{\nu}{4}\int_{\mathbb{R}^3}\left(\left(\sum_{i=1}^{\ell}\bar{V}_{y^i}\right)^4-\sum_{i=1}^{\ell}\bar{V}_{y^i}^4-2\sum_{i\neq j}^{\ell}\bar{V}_{y^i}^3\bar{V}_{y^j}\right),
\end{aligned}
$$

the last three terms can be estimated exactly as done in the proof of Proposition A.2. We need to estimate the term $\int_{\Omega_1}\left(\sum_{i=1}^{\ell}\bar{U}_{x^i}\right)^2\left(\sum_{i=1}^{\ell}\bar{V}_{y^i}\right)^2$, which is done in the next lemma, claiming

$$
\int_{\Omega_1}U_{x^1}^2V_{y^1}^2=o(1)\mathrm{e}^{-2\sqrt{(\rho-r\cos\frac{\pi}{\ell})^2+r^2(\frac{\pi}{\ell})^2}}\frac{\ell}{r}.
$$

Now, to complete the proof, we have the estimate

$$
\begin{aligned}
&\int_{\Omega_1}\left(\sum_{i=1}^{\ell}\bar{U}_{x^i}\right)^2\left(\sum_{i=1}^{\ell}\bar{V}_{y^i}\right)^2\\
&=\int_{\Omega_1}\left(U_{x^1}+U_{x^2}+U_{x^\ell}+\sum_{i=3}^{\ell-1}\bar{U}_{x^i}\right)^2\left(V_{y^1}+V_{y^\ell}+\sum_{i=2}^{\ell-1}\bar{V}_{y^i}\right)^2\\
&=\int_{\Omega_1}\left(U_{x^1}^2V_{y^1}^2+U_{x^1}^2V_{y^\ell}^2+U_{x^2}^2V_{y^1}+U_{x^\ell}^2V_{y^\ell}^2\right)+O\left(\mathrm{e}^{-\frac{3\pi}{\ell}r}\right)\\
&=o(1)\frac{\ell}{r}\mathrm{e}^{-2\sqrt{(\rho-r\cos\frac{\pi}{\ell})^2+r^2(\frac{\pi}{\ell})^2}}+O\left(\mathrm{e}^{-\frac{3\pi}{\ell}r}\right).
\end{aligned}
$$

□

Lemma A.4. *As $\ell\to\infty$,*

$$
\int_{\Omega_1}U_{x^1}^2V_{y^1}^2=o(1)\mathrm{e}^{-2\sqrt{(\rho-r\cos\frac{\pi}{\ell})^2+r^2(\frac{\pi}{\ell})^2}}\frac{\ell}{r}.
$$

Proof. Let $\Omega_1'=\left\{z=(z',z_3)\in\mathbb{R}^3:\langle\frac{z'}{|z'|},\frac{x'^1}{|x'^1|}\rangle\geqq\cos\frac{\pi}{2\ell}\right\}$. Then $\int_{\Omega_1}U_{x^1}^2V_{y^1}^2=2\int_{\Omega_1'}U_{x^1}^2V_{y^1}^2$. We divide Ω_1' into two parts: $\omega_1=\{x\in\Omega_1'\mid|x-x^1|\geqq|x^1-$

$y^1|\}$, $\omega_2 = \{x \in \Omega_1' \mid |x - x^1| \leqq |x^1 - y^1|\}$. Noticing that $|x - y^1| \geqq \frac{1}{2}|x^1 - y^1|$ for $x \in \Omega_1'$, we find for some $C > 0$

$$\int_{\omega_1} U_{x^1}^2 V_{y^1}^2 \leqq C \frac{\mathrm{e}^{-2|x^1-y^1|}}{|x^1 - y^1|^2} \int_{\omega_1} \mathrm{e}^{-2|x-y^1|} = o(1) \left(\frac{\mathrm{e}^{-2|x^1-y^1|}}{|x^1 - y^1|} \right),$$

and

$$\begin{aligned} \int_{\omega_2} U_{x^1}^2 V_{y^1}^2 &\leqq C \frac{\mathrm{e}^{-2|x^1-y^1|}}{|x^1 - y^1|^2} \int_{\omega_2} \frac{\mathrm{e}^{-2|x-x^1|-2|x-y^1|+2|x^1-y^1|}}{|x - x^1|^2} \\ &\leqq C \frac{\mathrm{e}^{-2|x^1-y^1|}}{|x^1 - y^1|^2} \int_{|x| \leqq |x^1-y^1|} \frac{\mathrm{e}^{-2|x|-2|x-(x^1-y^1)|+2|x^1-y^1|}}{|x|^2} \\ &= C \frac{\mathrm{e}^{-2|x^1-y^1|}}{|x^1 - y^1|} \int_{|x| \leqq 1} \frac{\mathrm{e}^{-2|x^1-y^1|(|x|+|x-\frac{(x^1-y^1)}{|x^1-y^1|}|-1)}}{|x|^2}. \end{aligned}$$

Here we have used the fact that

$$U_{1,\mu} \leqq C|x|^{-1}\mathrm{e}^{-|x|}, \quad U_{1,\nu} \leqq C|x|^{-1}\mathrm{e}^{-|x|}.$$

Without loss of generality we assume $x^1 - y^1 = (x^1 - y^1, 0, 0)$, with $x^1 - y^1 > 0$, and we write $\mathbf{1} = (1, 0, 0)$. Using convexity, we have $c > 0$ such that $|x| + |x - \mathbf{1}| - 1 \geqq c(x_2^2 + x_3^2)$ for $|x| \leqq 1$. Then, using the dominated convergence theorem, we have

$$\int_{|x| \leqq 1} \frac{\mathrm{e}^{-2|x^1-y^1|(|x|+|x-\mathbf{1}|-1)}}{|x|^2} \leqq \int_{|x| \leqq 1} \frac{\mathrm{e}^{-2c|x^1-y^1|(x_2^2+x_3^2)}}{|x|^2} \to 0 \text{ as } \ell \to +\infty.$$

Hence, the lemma is proved. □

References

1. Akhmediev, N., Ankiewicz, A.: Partially coherent solitons on a finite background. *Phys. Rev. Lett.* **82**(1–4), 2661 (1999)
2. Ambrosetti, A., Colorado, E.: Bound and ground states of coupled nonlinear Schrödinger equations. *C. R. Math. Acad. Sci. Paris* **342**, 453–458 (2006)
3. Ambrosetti, A., Colorado, E.: Standing waves of some coupled nonlinear Schrödinger equations. *J. Lond. Math. Soc.* **75**, 67–82 (2007)
4. Bartsch, T., Dancer, N., Wang, Z.-Q.: A Liouville theorem, a-priori bounds, and bifurcating branches of positive solutions for a nonlinear elliptic system. *Calc. Var. Partial Diff. Equ.* **37**, 345–361 (2010)
5. Bartsch, T., Wang, Z.-Q.: Note on ground states of nonlinear Schrödinger systems. *J. Part. Diff. Equ.* **19**, 200–207 (2006)
6. Bartsch, T., Wang, Z.-Q., Wei, J.: Bound states for a coupled Schrödinger system. *J. Fixed Point Theory Appl.*, **2**, 353–367 (2007)
7. Chang, S., Lin, C.S., Lin, T.C., Lin, W.: Segregated nodal domains of two-dimensional multispecies Bose–Einstein condensates. *Phys. D*, **196**(3–4), 341–361 (2004)
8. Conti, M., Terracini, S., Verzini, G.: Nehari's problem and competing species systems. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **19**, 871–888 (2002)

9. DANCER, E.N., WEI, J., WETH, T.: A priori bounds versus multiple existence of positive solutions for a nonlinear Schrödinger system. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **27**, 953–969 (2010)
10. DE FIGUEIREDO, D.G., LOPES, O.: Solitary waves for some nonlinear Schrödinger systems. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **25**, 149–161 (2008)
11. ESRY, B.D., GREENE, C.H., BURKE, J.P. Jr., BOHN, J.L.: Hartree–Fock theory for double condensates. *Phys. Rev. Lett.* **78**, 3594–3597 (1997)
12. BENCI, V., FORTUNATO, D.: Solitary waves of the nonlinear Klein-Gordon equation coupled with the Maxwell equations. *Rev. Math. Phys.*, **14**, 409–420 (2002)
13. CAO, D., NOUSSAIR, E.S., YAN, S.: Solutions with multiple peaks for nonlinear elliptic equations. *Proc. R. Soc. Edinburgh Sect. A*, **129**, 235–264 (1999)
14. LIN, T.C., WEI, J.: Ground state of N coupled nonlinear Schrödinger equations in $\mathbb{R}^n$, $n \leqq 3$. *Commun. Math. Phys.*, **255**, 629–653 (2005)
15. LIN, T.C., WEI, J.: Spikes in two coupled nonlinear Schrödinger equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **22**, 403–439 (2005)
16. LIN, T.C., WEI, J.: Solitary and self-similar solutions of two-component systems of nonlinear Schrödinger equations. *Physica D: Nonlinear Phenomena*, **220**, 99–115 (2006)
17. LIU, Z., WANG, Z.-Q.: Multiple bound states of nonlinear Schrödinger systems. *Comm. Math. Phys.*, **282**, 721–731 (2008)
18. LIU, Z., WANG, Z.-Q.: Ground states and bound states of a nonlinear Schrödinger system. *Adv. Nonlinear Stud.*, **10**, 175–193 (2010)
19. MITCHELL, M., SEGEV, M.: Self-trapping of inconherent white light. *Nature*, **387**, 880–882 (1997)
20. MONTEFUSCO, E., PELLACCI, B., SQUASSINA, M.: Semiclassical states for weakly coupled nonlinear Schrödinger systems. *J. Eur. Math. Soc.*, **10**, 41–71 (2008)
21. NORIS, B., TAVARES, H., TERRACINI, S., VERZINI, G.: Uniform Hölder bounds for nonlinear Schrödinger systems with strong competition. *Comm. Pure Appl. Math.*, **63**, 267–302 (2010)
22. REY, O.: The role of the Green's function in a non-linear elliptic equation involving the critical Sobolev exponent. *J. Funct. Anal.*, **89**, 1–52 (1990)
23. RÜEGG, Ch. et al.: Bose–Einstein condensation of the triple states in the magnetic insulator $TlCuCl_3$. *Nature*, **423**, 62–65 (2003)
24. SIRAKOV, B.: Least energy solitary waves for a system of nonlinear Schrödinger equations in $\mathbb{R}^n$. *Comm. Math. Phys.*, **271**, 199–221 (2007)
25. TERRACINI, S., VERZINI, G.: Multipulse Phase in k-mixtures of Bose–Einstein condenstates. *Arch. Rational Mech. Anal.*, **194**, 717–741 (2009)
26. TIAN, R., WANG, Z.-Q.: Multiple solitary wave solutions of nonlinear Schrödinger systems. *Top Methods Nonlinear Anal.*, **37**, 203–223 (2011)
27. TIMMERMANS, E.: Phase seperation of Bose–Einstein condensates. *Phys. Rev. Lett.*, **81**, 5718–5721 (1998)
28. WANG, L., WEI, J., YAN, S.: A Neumann problem with critical exponent in non-convex domain and Lin–Ni's conjecture. *Trans. Am. Math. Soc.*, **361**, 1189–1208 (2009)
29. WEI, J., WETH, T.: Nonradial symmetric bound states for s system of two coupled Schrödinger equations. *Rend. Lincei Mat. Appl.*, **18**, 279–293 (2007)
30. WEI, J., WETH, T.: Radial solutions and phase separation in a system of two coupled Schrödinger equations. *Arch. Rational Mech. Anal.*, **190**, 83–106 (2008)
31. WEI, J., YAN, S.: Infinitely many solutions for the prescribed scalar curvature problem on S^n. *J. Funct. Anal.*, **258**, 3048–3081 (2010)
32. WEI, J., YAN, S.: An elliptic problem with critical growth and Lazer–Mckenna conjecture. *Ann. Sci. Norm. Super. Pisa Cl. Sci.* **9**(5), 423–457 (2010)
33. WEI, J., YAN, S.: Infinitely many solutions for the nonlinear Schrödinger equations in $\mathbb{R}^N$. *Calc. Var. Partial Diff. Equ.*, **37**, 423–439 (2010)
34. WEI, J., YAO, W.: Uniqueness of positive solutions to some coupled nonlinear Schrödinger equations. *Commun. Pure Appl. Anal.*, **11**, 1003–1011 (2012)

School of Mathematics and Statistics
Central China Normal University
430079 Wuhan,
People's Republic of China.
e-mail: sjpeng@mail.ccnu.edu.cn

and

Chern Institute of Mathematics
Nankai University
300071 Tianjin,
People's Republic of China

and

Department of Mathematics and Statistics
Utah State University
Logan, UT 84322, USA.
e-mail: zhi-qiang.wang@usu.edu

(*Received March 19, 2012 / Accepted October 29, 2012*)
Published online December 11, 2012 – © Springer-Verlag Berlin Heidelberg (*2012*)