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Ground States and Bound States of a Nonlinear Schrödinger System

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Abstract

This paper concerns existence and multiplicity of ground states and bound states of the time-independent Schrödinger system

$$\begin{cases} -\Delta u_j + \lambda_j u_j = \displaystyle\sum_{i=1}^{N} \beta_{ij} u_i^2 u_j \quad \text{in } \mathbb{R}^n, \\ u_j(x) \to 0 \quad \text{as } |x| \to \infty, \quad j = 1, \ldots, N, \end{cases}$$

where $n = 2, 3$, $N \geq 2$, $\lambda_j > 0$ for $j = 1, \cdots, N$, $\beta_{jj} > 0$ for $j = 1, \cdots, N$, and $\beta_{ij} = \beta_{ji}$. In the attractive case we give sufficient condition for existence of co-existing ground states with large couplings, and in the repulsive case we prove existence of infinitely many co-existing bound state solutions with arbitrary couplings.

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1 Introduction

Consider the time-independent Schrödinger system

$$\begin{cases} -\Delta u_j + \lambda_j u_j = \sum_{i=1}^{N} \beta_{ij} u_i^2 u_j \quad \text{in } \mathbb{R}^n, \\ u_j(x) \to 0 \quad \text{as } |x| \to \infty, \quad j = 1, \ldots, N, \end{cases} \tag{1.1}$$

where $n = 2, 3,\ N \geq 2,\ \lambda_j > 0$ for $j = 1, \cdots, N$, β_{ij} are constants satisfying $\beta_{ij} = \beta_{ji}$ and $\beta_{jj} > 0$ for $j = 1, \cdots, N$. If $\vec{u} = (u_1, \cdots, u_N)$ is a solution of (1.1), then the function $(\Phi_1, \ldots, \Phi_N) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{C}^N$, defined by $\Phi_j(x,t) = e^{i\lambda_j t} u_j(x)$, $j = 1, \ldots, N$, is a standing wave solution of the time-dependent system of N coupled nonlinear Schrödinger equations

$$\begin{cases} -i\frac{\partial}{\partial t}\Phi_j = \Delta\Phi_j + \sum_{i=1}^{N} \beta_{ij} |\Phi_i|^2 \Phi_j \quad \text{for } \ x \in \mathbb{R}^n,\ t > 0, \\ \Phi_j(x,t) \to 0 \quad \text{as } \ |x| \to +\infty,\ t > 0, \quad j = 1, \ldots, N. \end{cases} \tag{1.2}$$

The system (1.2) models naturally many physical problems, especially in nonlinear optics. Physically, the solution Φ_j denotes the j-th component of the beam in Kerr-like photorefractive media ([1]). The positive constant β_{jj} is for self-focusing in the j-th component of the beam. The coupling constant β_{ij} $(i \neq j)$ is the interaction between the i-th and the j-th components of the beam. Problem (1.2) also arises in the Hartree-Fock theory for Bose-Einstein condensates ([11]). Physically, Φ_j are the corresponding condensate amplitudes, β_{jj} and β_{ij} are the intraspecies and interspecies scattering lengths. The sign of the scattering length β_{ij} determines whether the interactions of states $|i\rangle$ and $|j\rangle$ are repulsive or attractive. For more references we refer the reader to [1, 8, 11, 12, 13, 14, 15, 22, 28].

In the last several years there has been intensive work on the existence, multiplicity and qualitative property of ground and bound state nontrivial solutions for systems like (1.1). Here and below by a *nontrivial solution of* (1.1), we mean a solution $\vec{u} = (u_1, \cdots, u_N)$ with each component u_j being nonzero. In the literature these solutions are also referred to as *co-existing solutions*. It is an important feature of the study for these type of systems that one needs to distinguish nontrivial solutions from *semitrivial solutions* (solutions with one or more components being zero). We call a solution a *ground state solution* if it corresponds to the least nonzero critical value of the associated energy functional. Note that this definition is different from the one given in [16], where a solution is called a *ground state solution* if it has the least energy among all the energies of nontrivial positive solutions of (1.1). We will distinguish two cases, the attractive case: $\beta_{ij} > 0$ for $i \neq j$, and the repulsive (or competition) case: $\beta_{ij} < 0$ for $i \neq j$. It turns out the systems have quite different behaviors for the two cases. The existing work mainly has been on systems with two equations (i.e., $N = 2$). For examples, following the work [16] by Lin and Wei about the existence of ground state solutions with small couplings for a general N-system, a number of papers have been devoted to the existence theory of solutions for the 2-system in various different parameter regimes of nonlinear couplings; see [2, 3, 4, 5, 6, 9, 19, 20, 23, 26, 29, 30] for the existence of ground state or bound state solutions and their limiting property with large couplings both for repulsive and attractive cases, [17, 18, 21, 24] for semiclassical states or singularly perturbed settings.

For a general N-system, except the early work in [16], for small couplings (i.e., β_{ij} with $i \neq j$ small), not much has been studied so far. Some partial work was given in [3, 26] (see Remark 2.4 c) below in details). In particular, for a general N-system with large couplings in the attractive case (i.e., β_{ij} with $i \neq j$ tend to plus infinity) the question whether there exists a nontrivial solution and whether the ground state is nontrivial still seem open.

In this paper, we establish a framework to study the general N-system. Our goal is two fold. One is to provide a sufficient condition for the existence of a nontrivial ground state solution for the attractive case with large nonlinear couplings. This would provide an answer to the open question above. The method used to establish this result is by estimates of energies. Another goal of the paper is to establish a multiplicity result of bound state solutions for the repulsive (competition) case. This result generalizes an earlier work of ours in [19] which requires small couplings. The result here applies to arbitrary couplings. We provide two proofs of the result, one based on the method of critical point theory in the setting of invariant sets of the gradient flows and the other based on the minimax method on a Nehari manifold.

The paper is organized as follows. In section 2 we state and prove the result on the existence of a nontrivial ground state solution in the attractive case with large couplings. In section 3 we state and prove the result on the multiplicity of nontrivial bound state solutions in the repulsive case with arbitrary couplings. We finish the paper in section 4 with some further remarks.

2 A nontrivial ground state in the attractive case

Consider the Schrödinger system

$$\begin{cases} -\Delta u_j + \lambda_j u_j = \displaystyle\sum_{i=1}^{N} \beta_{ij} u_i^2 u_j \quad \text{in } \mathbb{R}^n, \\ u_j(x) > 0, \quad u_j(x) \to 0, \quad \text{as } |x| \to \infty, \ j = 1, 2, \cdots, N, \end{cases} \tag{2.1}$$

where $n = 2, 3,$ $N \geq 2$, $\lambda_j > 0$ for $j = 1, \cdots, N$, β_{ij} are constants satisfying $\beta_{ij} = \beta_{ji}$ and $\beta_{ij} > 0$.

Let $E = H_r^1(\mathbb{R}^n)$ be the space consisting of spherically symmetric functions in

$H^1(\mathbb{R}^n)$ in which we shall use the equivalent inner products

$$(u,v)_j = \int_{\mathbb{R}^n} \nabla u \cdot \nabla v + \lambda_j uv, \quad j = 1, 2, \cdots, N$$

and the induced norms $\|\cdot\|_j$. Here the fact that $\lambda_j > 0$ has been used. The product space $E^N = \overbrace{E \times E \times \cdots \times E}^{N}$ is a subspace of $(H^1(\mathbb{R}^n))^N$ endowed with the inner product

$$(\vec{u},\ \vec{v}) = \sum_{j=1}^{N} (u_j, v_j)_j, \quad \vec{u} = (u_1, \cdots, u_N),\ \vec{v} = (v_1, \cdots, v_N).$$

Solutions of (1.1) correspond to critical points of the functional

$$\Phi(\vec{u}) = \frac{1}{2}\|\vec{u}\|^2 - \frac{1}{4}\sum_{i,j=1}^{N} \beta_{ij} \int_{\mathbb{R}^n} u_i^2 u_j^2, \quad \vec{u} = (u_1, u_2, \cdots, u_N) \in (H^1(\mathbb{R}^n))^N,$$

and spherically symmetric solutions of (1.1) correspond to critical points of the functional

$$J(\vec{u}) = \Phi|_{E^N}(\vec{u}), \quad \vec{u} = (u_1, u_2, \cdots, u_N) \in E^N.$$

Note that $J \in C^2(E^N)$ and J satisfies the (PS) condition. It is easy to check that J has a mountain pass geometry and has a mountain pass critical point. Clearly, any critical value of J is a critical value of Φ according to the principle of symmetric criticality. The functional Φ is also in the class of C^2 and has a mountain pass geometry, but it does not satisfy the (PS) condition. We show that the mountain pass value of Φ is equal to the mountain pass critical value of J, and hence is a critical value. To this end, for any $\vec{u} = (u_1, \cdots, u_n) \in (H^1(\mathbb{R}^n))^N$ with each component being nonnegative, denote $\vec{u}^* = (u_1^*, \cdots, u_n^*)$ with u_j^* being the Schwarz symmetrization of u_j. Then

$$J(\vec{u}^*) \leq \Phi(\vec{u}),$$

since, for all i, j,

$$\int_{\mathbb{R}^n} |\nabla u_j^*|^2 \leq \int_{\mathbb{R}^n} |\nabla u_j|^2,$$

$$\int_{\mathbb{R}^n} (u_j^*)^2 = \int_{\mathbb{R}^n} (u_j)^2,$$

and

$$\int_{\mathbb{R}^n} (u_i^*)^2 (u_j^*)^2 \geq \int_{\mathbb{R}^n} (u_i)^2 (u_j)^2.$$

This implies the mountain pass value of Φ is equal to the mountain pass critical value of J, and hence it is a critical value. It is easy to see the mountain pass critical value of Φ is the least positive critical value of Φ. Due to this reason, we call a solution ground state if it corresponds to the mountain pass critical value of J.

However in general this energy level may not correspond to nontrivial critical points. The case $N = 2$ has been studied extensively in the last few years. Let us call $\beta := \beta_{12}$ in this case which is the lone coupling constant for $N = 2$. In [5] (see also [2, 3, 6, 20, 26] with similar or different arguments) by using mountain pass theorem and Morse theory, it is proved that there is a $\beta_0 > 0$ such that for $\beta > \beta_0$ the ground state solution (i.e., the mountain pass solution) is nontrivial and for $\beta < \beta_0$ the ground state solution is semitrivial in the sense that the solution is of the form $(w, 0)$ or $(0, w)$ with one null component. It seems still an open question under what conditions the ground state solution for a general N-system is nontrivial. We provide a sufficient condition here which guarantees the ground state is nontrivial. To state our result we need some notations.

First we note that the least positive critical value of Φ can also be reformulated as the infimum of the following functional I

$$I(\vec{u}) = \frac{\sum_{j=1}^{N} \int_{\mathbb{R}^n} |\nabla u_j|^2 + \lambda_j u_j^2}{(\sum_{i,j=1}^{N} \beta_{ij} \int_{\mathbb{R}^n} u_i^2 u_j^2)^{1/2}}, \quad \vec{u} \in (H^1(\mathbb{R}^n))^N,\ \vec{u} \neq 0.$$

Define

$$c = \inf_{u \in H^1(\mathbb{R}^n),\ u \neq 0} \frac{\int_{\mathbb{R}^n} |\nabla u|^2 + u^2}{(\int_{\mathbb{R}^n} u^4)^{1/2}} = \inf_{u \in H_r^1(\mathbb{R}^n),\ u \neq 0} \frac{\int_{\mathbb{R}^n} |\nabla u|^2 + u^2}{(\int_{\mathbb{R}^n} u^4)^{1/2}},$$

and let U be the unique positive and spherically symmetric minimizer for c. Then, for $\lambda > 0$,

$$c\lambda^{1-\frac{n}{4}} = \inf_{u \in H^1(\mathbb{R}^n),\ u \neq 0} \frac{\int_{\mathbb{R}^n} |\nabla u|^2 + \lambda u^2}{(\int_{\mathbb{R}^n} u^4)^{1/2}} = \inf_{u \in H_r^1(\mathbb{R}^n),\ u \neq 0} \frac{\int_{\mathbb{R}^n} |\nabla u|^2 + \lambda u^2}{(\int_{\mathbb{R}^n} u^4)^{1/2}},$$

for which $U_\lambda(x) = U(\sqrt{\lambda} x)$ is the unique positive and spherically symmetric minimizer. Denote, for $\lambda > 0$,

$$\alpha(\lambda) = \frac{\int_{\mathbb{R}^n} |\nabla U|^2 + U^2}{\int_{\mathbb{R}^n} |\nabla U|^2 + \lambda U^2}.$$

We need the following assumption (A): For some $\lambda > 0$,

$$\sum_{i,j=1}^{N} \beta_{ij} \alpha(\frac{\lambda_i}{\lambda}) \alpha(\frac{\lambda_j}{\lambda}) > N^2 \Bigg[\max_{1 \leq j \leq N} \beta_{jj} (\frac{\lambda}{\lambda_j})^{2-\frac{n}{2}} + \frac{N-2}{N-1} \max_{1 \leq i,j \leq N,\ i \neq j} \beta_{ij} (\frac{\lambda}{\lambda_i})^{1-\frac{n}{4}} (\frac{\lambda}{\lambda_j})^{1-\frac{n}{4}} \Bigg].$$

Theorem 2.1. *If (A) holds, then* (2.1) *has a nontrivial ground state solution which is spherically symmetric and is given by, up to a Lagrange multiplier, a minimizer of the minimization problem*

$$\inf_{\vec{u} \in (H^1(\mathbb{R}^n))^N,\ \vec{u} \neq 0} I(\vec{u}).$$

Proof. Using Schwarz symmetrization, we see that

$$m(N) := \inf_{\vec{u}\in(H^1(\mathbb{R}^n))^N,\ \vec{u}\neq 0} I(\vec{u}) = \inf_{\vec{u}\in E^N,\ \vec{u}\neq 0} I(\vec{u}).$$

Therefore, the infimum $m(N)$ is achieved by a spherically symmetric element $\vec{u}^0 = (u_1^0, \cdots, u_n^0)$. Replacing u_j^0 with $|u_j^0|$ if necessary, we can assume that $u_j^0 \geq 0$. It suffices to prove that $u_j^0 > 0$ for each j. Choosing $u_j = \sqrt{\alpha(\lambda_j/\lambda)}U_\lambda$, we have, for any j,

$$\begin{aligned}
\int_{\mathbb{R}^n} |\nabla u_j|^2 + \lambda_j u_j^2 &= \alpha(\lambda_j/\lambda) \int_{\mathbb{R}^n} |\nabla U_\lambda|^2 + \lambda_j U_\lambda^2 \\
&= \lambda^{1-\frac{n}{2}} \alpha(\lambda_j/\lambda) \int_{\mathbb{R}^n} |\nabla U|^2 + (\lambda_j/\lambda) U^2 \\
&= \lambda^{1-\frac{n}{2}} \int_{\mathbb{R}^n} |\nabla U|^2 + U^2 \\
&= \int_{\mathbb{R}^n} |\nabla U_\lambda|^2 + \lambda U_\lambda^2,
\end{aligned}$$

which implies

$$\begin{aligned}
&\sum_{i,j=1}^N \beta_{ij} \int_{\mathbb{R}^n} u_i^2 u_j^2 \\
&\quad = \sum_{i,j=1}^N \beta_{ij} \alpha(\lambda_i/\lambda)\alpha(\lambda_j/\lambda) \int_{\mathbb{R}^n} U_\lambda^4 \\
&\quad = \frac{1}{\lambda^{2-\frac{n}{2}} c^2} \sum_{i,j=1}^N \beta_{ij} \alpha(\lambda_i/\lambda)\alpha(\lambda_j/\lambda) \left(\int_{\mathbb{R}^n} |\nabla U_\lambda|^2 + \lambda U_\lambda^2 \right)^2 \\
&\quad = \frac{1}{\lambda^{2-\frac{n}{2}} c^2 N^2} \sum_{i,j=1}^N \beta_{ij} \alpha(\lambda_i/\lambda)\alpha(\lambda_j/\lambda) \left(\sum_{k=1}^N \int_{\mathbb{R}^n} |\nabla u_k|^2 + \lambda_k u_k^2 \right)^2.
\end{aligned}$$

Therefore,

$$m(N) \leq \frac{cN\lambda^{1-\frac{n}{4}}}{\sqrt{\sum_{i,j=1}^N \beta_{ij}\alpha(\lambda_i/\lambda)\alpha(\lambda_j/\lambda)}}.$$

Using Schwarz symmetrization again, we infer that

$$m(N,s) := \inf_{\vec{u}\in(H^1(\mathbb{R}^n))^N,\ \vec{u}\neq 0,\ u_s=0} I(\vec{u}) = \inf_{\vec{u}\in E^N,\ \vec{u}\neq 0,\ u_s=0} I(\vec{u}), \quad s = 1, 2, \cdots, N.$$

Note that if $u_N = 0$ then

$$\begin{aligned}
&\sum_{i,j=1}^{N} \beta_{ij} \int_{\mathbb{R}^n} u_i^2 u_j^2 \le \sum_{i,j=1}^{N-1} \beta_{ij} \left(\int_{\mathbb{R}^n} u_i^4 \right)^{1/2} \left(\int_{\mathbb{R}^n} u_j^4 \right)^{1/2} \\
\le & \frac{1}{c^2} \sum_{i,j=1}^{N-1} \frac{\beta_{ij}}{\lambda_i^{1-\frac{n}{4}} \lambda_j^{1-\frac{n}{4}}} \int_{\mathbb{R}^n} (|\nabla u_i|^2 + \lambda_i u_i^2) \int_{\mathbb{R}^n} (|\nabla u_j|^2 + \lambda_j u_j^2) \\
\le & \frac{1}{c^2} \Bigg\{ \max_{1 \le j \le N-1} \frac{\beta_{jj}}{\lambda_j^{2-\frac{n}{2}}} \sum_{k=1}^{N-1} \left(\int_{\mathbb{R}^n} |\nabla u_k|^2 + \lambda_k u_k^2 \right)^2 \\
& + \max_{i \ne j,\ 1 \le i,j \le N-1} \frac{\beta_{ij}}{\lambda_i^{1-\frac{n}{4}} \lambda_j^{1-\frac{n}{4}}} \sum_{i \ne j}^{N-1} \int_{\mathbb{R}^n} (|\nabla u_i|^2 + \lambda_i u_i^2) \int_{\mathbb{R}^n} (|\nabla u_j|^2 + \lambda_j u_j^2) \Bigg\} \\
\le & \frac{1}{c^2} \left\{ \max_{1 \le j \le N-1} \frac{\beta_{jj}}{\lambda_j^{2-\frac{n}{2}}} + \frac{N-2}{N-1} \max_{i \ne j,\ 1 \le i,j \le N-1} \frac{\beta_{ij}}{\lambda_i^{1-\frac{n}{4}} \lambda_j^{1-\frac{n}{4}}} \right\} \\
& \times \left(\sum_{k=1}^{N} \int_{\mathbb{R}^n} |\nabla u_k|^2 + \lambda_k u_k^2 \right)^2 .
\end{aligned}$$

Therefore,

$$m(N,N) \ge \frac{c}{\sqrt{\max_{1 \le j \le N-1} \frac{\beta_{jj}}{\lambda_j^{2-\frac{n}{2}}} + \frac{N-2}{N-1} \max_{i \ne j,\ 1 \le i,j \le N-1} \frac{\beta_{ij}}{\lambda_i^{1-\frac{n}{4}} \lambda_j^{1-\frac{n}{4}}}}}.$$

The same estimate applies also to $m(N,s)$, $s = 1, 2, \cdots, N-1$. Now, if (A) is satisfied then

$$m(N) < \min\{m(N,1),\ m(N,2),\ \cdots,\ m(N,N)\},$$

which implies $u_j^0 \ne 0$ and therefore $u_j^0 > 0$ for each j. The maximum principle implies that $u_j^0(x) > 0$ for all j and $x \in \mathbb{R}^n$. Now $(m(N)/\|\vec{u}^0\|)\vec{u}^0$ is a nontrivial ground solution of (2.1) which is spherically symmetric. The proof is complete.

Remark 2.2. If λ and β_{jj} $(j = 1, 2, \cdots, N)$ are fixed, then there exist $\delta > 0$ and $\beta^* > 0$ such that (A) holds provided that $|\lambda_j - \lambda| < \delta$ for all j and $|\beta_{ij} - \beta| < \delta$ for all $i \ne j$ and for some $\beta > \beta^*$. In particular, if $\lambda_1 = \cdots = \lambda_N = \lambda$ and $\beta_{ij} = \beta$ for $i \ne j$, then (A) reduces to $\beta > N(N-1)\max_{1 \le j \le N} \beta_{jj} - N^{-1}(N-1)\sum_{j=1}^{N} \beta_{jj}$. Therefore, we have the following corollary.

Corollary 2.3. *Let $\lambda_1 = \cdots = \lambda_N = \lambda$ and β_{jj} $(j = 1, 2, ..., N)$ be fixed. Assume $\beta_{ij} = \beta$ for $i \ne j$. If $\beta > N(N-1)\max_{1 \le j \le N} \beta_{jj} - N^{-1}(N-1)\sum_{j=1}^{N} \beta_{jj}$, then* (2.1) *has a nontrivial ground state solution which is spherically symmetric and is given by, up to a Lagrange multiplier, a minimizer of the minimization problem*

$$\inf_{\vec{u} \in (H^1(\mathbb{R}^n))^N,\ \vec{u} \ne 0} I(\vec{u}).$$

Remark 2.4. a). It seems quite natural and necessary in some sense to require the couplings $\beta_{ij}, i \neq j$, large relative to $\beta_{jj}, j = 1, ..., N$, as the non-existence result on positive solutions in [5] asserts that in the attractive case if the λ_j's are non-increasing and the β_{ij}'s are non-decreasing in i and j, then (2.1) admits no positive solutions unless $\lambda_j = \lambda$ and $\beta_{ij} = \beta$ for all $i, j = 1, ..., N$, some positive constants λ, β.

b). We believe the condition (A) is not sharp for the existence of a nontrivial ground state. In the case of the 2-system (there is only one coupling constant $\beta := \beta_{12} = \beta_{21}$), in [2, 3, 5, 6, 20, 26] more precise estimates on the size of a β_0 have been given to assure for $\beta > \beta_0$ the existence of a nontrivial ground state solution.

c). Partial results on the existence of nontrivial ground states of higher dimensional systems were also given in [3, 26]. In [3] a 3-system was discussed under conditions that assure all three semi-trivial solutions with single nonzero component are saddle points on the Nehari manifold so the minimizer on the Nehari manifold has at least two nonzero components. In [26] a set of structure conditions involving matrix (β_{ij}) and vector (λ_j) (Hypotheses 1-6 there) are assumed to assert the existence of a nontrivial ground state. It does not seem easy to compare our condition (A) with theirs.

3 Multiple bound states in the repulsive case

Again consider the Schrödinger system

$$\begin{cases} -\Delta u_j + \lambda_j u_j = \displaystyle\sum_{i=1}^{N} \beta_{ij} u_i^2 u_j \quad \text{in } \mathbb{R}^n, \\ u_j(x) \to 0 \quad \text{as } |x| \to \infty, \quad j = 1, \ldots, N, \end{cases} \tag{3.1}$$

where $n = 2, 3$, $N \geq 2$, $\lambda_j > 0$ for $j = 1, \cdots, N$, β_{ij} are constants satisfying $\beta_{ij} = \beta_{ji}$, $\beta_{jj} > 0$ for $j = 1, \cdots, N$ and $\beta_{ij} \leq 0$ for $i \neq j$, $i, j = 1, ..., N$. We again look for nontrivial solutions with each component nonzero.

We shall prove the following theorem.

Theorem 3.1. *Assume $N \geq 2$, $n = 2, 3$, $\lambda_j > 0$, $\beta_{jj} > 0$ for $j = 1, \cdots, N$, and $\beta_{ij} \leq 0$ for $i \neq j$, $i, j = 1, ..., N$. Then* (3.1) *has infinitely many nontrivial spherically symmetric solutions.*

Remark 3.2. This result extends an earlier one in [19] where the nonlinear couplings β_{ij} for $i \neq j$ are assumed to be small. Some interesting multiplicity results on nontrivial positive solutions have been given recently in [4, 9, 27, 30]. We remark that in [9, 30] multiplicity results on positive solutions were proved for a 2-system with a symmetric structure: $\lambda_1 = \lambda_2$, $\mu_1 = \mu_2$. In this case there is symmetry in the system in that if (u_1, u_2) is a solution so is (u_2, u_1). A result of a similar nature but without requiring the symmetric condition $\mu_1 = \mu_2$ was given in [4] by using a global bifurcation approach. In [27] a multiplicity result on positive solutions is given for a symmetric case of the general N-system where it is assumed that

$\lambda_j = \mu_j = 1$ for all j and $\beta_{ij} = \beta$ for $i \neq j$, and solutions are constructed for $-\beta$ sufficiently large, with each component separating in many pulses from the others. The solutions constructed in Theorem 3.1 of our paper may be of different types from that in the above mentioned papers and are potentially nodal type solutions.

3.1 A proof based on invariant sets of the gradient flow

As in the last section, we work in $E = H_r^1(\mathbb{R}^n)$. Spherically symmetric solutions of (3.1) correspond to critical points of the functional

$$J(\vec{u}) = \frac{1}{2}\|\vec{u}\|^2 - \frac{1}{4}\sum_{i,j=1}^{N} \beta_{ij}\int_{\mathbb{R}^n} u_i^2 u_j^2, \quad \vec{u} = (u_1, u_2, \cdots, u_N) \in E^N.$$

Note that $J \in C^2(E^N)$, J satisfies the (PS) condition, and

$$\nabla J(\vec{u}) = \vec{u} - A(\vec{u}), \quad \vec{u} = (u_1, u_2, \cdots, u_N) \in E^N,$$

where $A(\vec{u}) = ((A(\vec{u}))_1, (A(\vec{u}))_2, \cdots, (A(\vec{u}))_N)$ and

$$(A(\vec{u}))_j = (-\Delta + \lambda_j I)^{-1}(\sum_{i=1}^{N} \beta_{ij} u_i^2 u_j).$$

Proof of Theorem 3.1. Let $\varphi^t(\vec{u})$ with the maximal interval of existence $[0, \eta(\vec{u}))$ be the solution of the initial value problem

$$\begin{cases} \frac{d}{dt}\varphi^t = -\nabla J(\varphi^t), & \text{for } t \geq 0, \\ \varphi^0 = \vec{u}. \end{cases}$$

We say the map $\varphi : \{(t, \vec{u}) |\ \vec{u} \in E^N, t \in [0, \eta(\vec{u}))\} \to E^N$ is the *gradient flow of* J. A subset $\mathcal{F}$ of E^N is said to be an *invariant set* for the flow if $\varphi(t, \vec{u}) \in \mathcal{F}$ for all $\vec{u} \in \mathcal{F}$ and $t \in [0, \eta(\vec{u}))$. For two invariant sets $\mathcal{F} \subset \mathcal{G}$, we say $\mathcal{F}$ is *strictly invariant with respect to* $\mathcal{G}$ if $\varphi(t, \vec{u}) \in \mathrm{int}_{\mathcal{G}}\mathcal{F}$ for all $\vec{u} \in \mathcal{F}$ and $t \in (0, \eta(\vec{u}))$ where $\mathrm{int}_{\mathcal{G}}\mathcal{F}$ is the interior of $\mathcal{F}$ in $\mathcal{G}$. Define

$$\mathcal{A}_0 = \{\vec{u} \in E^N \mid \lim_{t \to \eta(\vec{u})-0} \varphi^t(\vec{u}) = 0\}.$$

It is clear that 0 is a strict local minimizer of J, $\mathcal{A}_0$ is an open neighborhood of 0 in E^N, $\partial\mathcal{A}_0$ is an invariant set, and $\inf_{\partial\mathcal{A}_0} J > 0$; see [19].

For any $\vec{u} \in E^N$, since $\beta_{ij} \leq 0$ for $i \neq j$, we have

$$(u_j, (A(\vec{u}))_j)_j = \sum_{i=1}^{N} \beta_{ij}\int_{\mathbb{R}^n} u_i^2 u_j^2 \leq \beta_{jj}\int_{\mathbb{R}^n} u_j^4 \leq C_j\|u_j\|_j^4,$$

where C_j is a positive constant. Therefore, if $0 < C_j\|u_j\|_j^2 < 1$ then for $h > 0$ sufficiently small

$$\begin{aligned}
&\|u_j + h(-u_j + (A(\vec{u}))_j)\|_j^2 \\
&\qquad =\|u_j\|_j^2 - 2h(\|u_j\|_j^2 - (u_j, (A(\vec{u}))_j)_j) + h^2\|u_j - (A(\vec{u}))_j\|_j^2 \\
&\qquad \le\|u_j\|_j^2 - 2h\|u_j\|_j^2(1 - C_j\|u_j\|_j^2) + h^2\|u_j - (A(\vec{u}))_j\|_j^2 \\
&\qquad <\|u_j\|_j^2.
\end{aligned}$$

Choose $r > 0$ such that $B_r(0) \subset \mathcal{A}_0$. Set

$$\varepsilon_0 = \frac{1}{2}\min\left\{C_1^{-\frac{1}{2}},\ \cdots,\ C_N^{-\frac{1}{2}},\ rN^{-\frac{1}{2}}\right\}.$$

Define, for $\varepsilon \in (0, \varepsilon_0]$ and $j = 1, 2, \cdots, N$,

$$\mathcal{D}_j^\varepsilon = \{\vec{u} \mid \vec{u} = (u_1, u_2, \cdots, u_N) \in E^N,\ \|u_j\|_j \le \varepsilon\}.$$

For any $\vec{u} \in \mathcal{D}_j^\varepsilon$, $j = 1, 2, \cdots, N$, if $h > 0$ is sufficiently small then the discussion above shows that

$$\vec{u} + h(-\nabla J(\vec{u})) \in \operatorname{int}(\mathcal{D}_j^\varepsilon).$$

According to [10, Section 4], for any $\vec{u} \in \mathcal{D}_j^\varepsilon$ there exists $t_0 = t_0(\vec{u}, \varepsilon, j) > 0$ such that $\varphi^t(\vec{u}) \in \operatorname{int}(\mathcal{D}_j^\varepsilon)$ for $t \in (0, t_0)$. This implies that $\varphi^t(\vec{u}) \in \partial\mathcal{A}_0 \cap \operatorname{int}(\mathcal{D}_j^\varepsilon)$ for $\vec{u} \in \partial\mathcal{A}_0 \cap \mathcal{D}_j^\varepsilon$, $t \in (0, \eta(\vec{u}))$, $j = 1, 2, \cdots, N$, and $0 < \varepsilon \le \varepsilon_0$. Therefore, $\partial\mathcal{A}_0 \cap \mathcal{D}_j^\varepsilon$, $j = 1, 2, \cdots, N$, $0 < \varepsilon \le \varepsilon_0$, are strictly invariant sets with respect to $\partial\mathcal{A}_0$ and $\eta(\vec{u}) = +\infty$ for $\vec{u} \in \partial\mathcal{A}_0 \cap \mathcal{D}_j^\varepsilon$. Define

$$\mathcal{A}_1 = \{\vec{u} \in \partial\mathcal{A}_0 \mid \exists\ t > 0 \text{ such that } \varphi^t(\vec{u}) \in \cup_{j=1}^N \operatorname{int}(\mathcal{D}_j^{\varepsilon_0})\}.$$

Then $\mathcal{A}_1$ is an open subset of $\partial\mathcal{A}_0$, and $\partial\mathcal{A}_0 \setminus \mathcal{A}_1$ is closed and invariant for the flow. We want to prove

$$\operatorname{gen}(\partial\mathcal{A}_0 \setminus \mathcal{A}_1) = +\infty,$$

where $\operatorname{gen}(\cdot)$ is the genus of a closed symmetric subset of E^N.

For any $k \in \mathbb{N}$, since $\beta_{jj} > 0$, there exist k-dimensional subspaces $F_1, \cdots, F_N$ of $E = H_r^1(\mathbb{R}^n)$ such that $\mathcal{A}_0 \cap (F_1 \times \cdots \times F_N)$ is bounded. Indeed, we may, for example, choose a k-dimensional subspace F_j from $H_{0,r}^1(\Omega_j)$, with $\Omega_1, \cdots, \Omega_N$ being mutually disjoint spherically symmetric domains. Then

$$J(\vec{u}) = \frac{1}{2}\|\vec{u}\|^2 - \frac{1}{4}\sum_{j=1}^N \beta_{jj}\int_{\mathbb{R}^n} u_j^4, \quad \vec{u} = (u_1, u_2, \cdots, u_N) \in F_1 \times \cdots \times F_N,$$

which implies $J(\vec{u}) \to -\infty$ as $\vec{u} \in F_1 \times \cdots \times F_N$ and $\|\vec{u}\| \to \infty$, and therefore $\mathcal{A}_0 \cap (F_1 \times \cdots \times F_N)$ is bounded. Denote

$$F = F_1 \times \cdots \times F_N.$$

Then

$$\mathrm{gen}(\partial\mathcal{A}_0 \cap F) = kN.$$

For $\vec{u} \in \mathcal{A}_1$, since $\partial\mathcal{A}_0 \cap \mathcal{D}_j^{\varepsilon}$, $j = 1, 2, \cdots, N$, $0 < \varepsilon \le \varepsilon_0$, are strictly invariant sets with respect to $\partial\mathcal{A}_0$, there exists $t > 0$ such that $\varphi^t(\vec{u}) \in \cup_{j=1}^N \mathcal{D}_j^{\varepsilon_0/2}$ and the function $\tau : \mathcal{A}_1 \to \mathbb{R}^+$ defined by

$$\tau(\vec{u}) = \inf\{t \ge 0 : \ \varphi^t(\vec{u}) \in \cup_{j=1}^N \mathcal{D}_j^{\varepsilon_0/2}\}$$

is even and continuous. Since $\varepsilon_0 \le \frac{1}{2} r N^{-\frac{1}{2}}$,

$$\cap_{j=1}^N \mathcal{D}_j^{\varepsilon_0/2} \subset B_{\sqrt{N}\varepsilon_0/2}(0) \subset B_r(0) \subset \mathcal{A}_0.$$

Therefore, since $\mathcal{A}_1 \subset \partial\mathcal{A}_0$ and since $\mathcal{A}_1$ is invariant, $\varphi^t(\vec{u}) \notin \cap_{j=1}^N \mathcal{D}_j^{\varepsilon_0/2}$ for $\vec{u} \in \mathcal{A}_1$ and $t > 0$.

Define a map $h : \mathcal{A}_1 \cap F \to F$ as

$$h(\vec{u}) = (\gamma_1(\vec{u})u_1, \gamma_2(\vec{u})u_2, \cdots, \gamma_N(\vec{u})u_N),$$

where

$$\gamma_i(\vec{u}) = \begin{cases} 1, & \text{if } \|\varphi_i^{\tau(\vec{u})}(\vec{u})\|_i \ge \varepsilon_0, \\ \frac{2}{\varepsilon_0}\|\varphi_i^{\tau(\vec{u})}(\vec{u})\|_i - 1, & \text{if } \varepsilon_0/2 < \|\varphi_i^{\tau(\vec{u})}(\vec{u})\|_i < \varepsilon_0, \\ 0, & \text{if } \|\varphi_i^{\tau(\vec{u})}(\vec{u})\|_i \le \varepsilon_0/2, \end{cases}$$

and φ_i^t is the i^{th} component of φ^t. Then $h : F \cap \mathcal{A}_1 \to F$ is odd and continuous. For any $\vec{u} \in \mathcal{A}_1 \cap F$, the definition of $\tau(\vec{u})$ implies that

$$\|\varphi_{i_1}^{\tau(\vec{u})}(\vec{u})\|_{i_1} \le \varepsilon_0/2, \quad \text{for at least one } i_1 \in \{1, \cdots, N\},$$

while the fact that $\varphi^t(\vec{u}) \notin \cap_{j=1}^N \mathcal{D}_j^{\varepsilon_0/2}$ for any $t > 0$ implies that

$$\|\varphi_{i_2}^{\tau(\vec{u})}(\vec{u})\|_{i_2} > \varepsilon_0/2, \quad \text{for at least one } i_2 \in \{1, \cdots, N\}.$$

Note that $\|\varphi_{i_2}^{\tau(\vec{u})}(\vec{u})\|_{i_2} > \varepsilon_0/2$ implies $\|u_{i_2}\|_{i_2} > \varepsilon_0/2$. Therefore

$$\gamma_{i_1}(\vec{u})u_{i_1} = 0, \quad \gamma_{i_2}(\vec{u})u_{i_2} \ne 0, \qquad \text{for any } \vec{u} \in \mathcal{A}_1 \cap F. \tag{3.2}$$

Let $\{e_{j1}, \cdots, e_{jk}\}$ be a base of F_j for $j = 1, \cdots, N$. Using this base we can define an isomorphism $T_j : F_j \to \mathbb{R}^k$ as

$$T_j u = (\alpha_1, \cdots, \alpha_k) \quad \text{if } u = \sum_{i=1}^k \alpha_i e_{ji}.$$

Define

$$W = \{\vec{u} = (u_1, \cdots, u_N) \in F : T_1 u_1 = \cdots = T_N u_N\}$$

and let V be the orthogonal complement of W in F. Then $\dim W = k$ and $\dim V = k(N-1)$. Denote by $g : F \to V$ the orthogonal projection from F to V. From (3.2) we see that

$$W \cap h(\mathcal{A}_1 \cap F) = \emptyset,$$

and $g \circ h : \mathcal{A}_1 \cap F \to V \setminus \{0\}$ is odd and continuous. Therefore

$$\operatorname{gen}(\mathcal{A}_1 \cap F) \le \dim V = (N-1)k,$$

which implies

$$\begin{aligned}\operatorname{gen}(\partial\mathcal{A}_0 \setminus \mathcal{A}_1) \ge \operatorname{gen}((\partial\mathcal{A}_0 \setminus \mathcal{A}_1) \cap F) &\ge \operatorname{gen}(\partial\mathcal{A}_0 \cap F) - \operatorname{gen}(\mathcal{A}_1 \cap F)\\ &\ge Nk - (N-1)k = k.\end{aligned}$$

Since k is arbitrary, we have

$$\operatorname{gen}(\partial\mathcal{A}_0 \setminus \mathcal{A}_1) = \infty.$$

Define

$$d_i = \inf_{A \in \Sigma_i} \sup_{\vec{u} \in A} J(\vec{u}),$$

where

$$\Sigma_i = \{A \mid A \subset \partial\mathcal{A}_0 \setminus \mathcal{A}_1,\ \operatorname{gen}(A) \ge i\}, \quad i = 1, 2, \cdots .$$

Now standard arguments (see, for example, [25]) can be used to obtain the conclusion. The proof is complete.

3.2 A proof based on Nehari manifold

Define

$$\mathcal{N} = \{\vec{u} = (u_1, \cdots, u_N) \in E^N \mid u_j \ne 0,\ \|u_j\|_j^2 = \sum_{i=1}^N \beta_{ij} \int_{\mathbb{R}^n} u_i^2 u_j^2,\ j = 1, \cdots, N\}.$$

We shall use $\mathcal{N}$ to find nontrivial spherically symmetric solutions of (1.1).

Note that $\mathcal{N}$ is not the classical Nehari manifold and, generally, a critical point of $J|_{\mathcal{N}}$ need not be a critical point of J. However, we have the following lemma stating that $\mathcal{N}$ is a natural constraint of J.

Lemma 3.3. *Critical points of $J|_{\mathcal{N}}$ are critical points of J under the assumptions of Theorem* 3.1.

Proof. Let $\vec{u} \in \mathcal{N}$ be a critical point of $J|_{\mathcal{N}}$. Denote

$$\Phi_j(\vec{u}) = \|u_j\|_j^2 - \sum_{i=1}^N \beta_{ij} \int_{\mathbb{R}^n} u_i^2 u_j^2, \quad j = 1, \cdots, N.$$

Then there exist real numbers $\alpha_1, \cdots, \alpha_N$ such that

$$\nabla J(\vec{u}) - \sum_{j=1}^{N} \alpha_j \nabla \Phi_j(\vec{u}) = 0.$$

Multiplying the jth component of the last equation with u_j and taking integral for $j = 1, \cdots, N$ we have

$$\begin{cases} \hat{a}_{11}\alpha_1 + \cdots + \hat{a}_{1N}\alpha_N & = & \Phi_1(\vec{u}) = 0 \\ \vdots & \vdots & \\ \hat{a}_{N1}\alpha_1 + \cdots + \hat{a}_{NN}\alpha_N & = & \Phi_N(\vec{u}) = 0, \end{cases} \tag{3.3}$$

where

$$\hat{a}_{ij} = \left(\frac{\partial \Phi_i(\vec{u})}{\partial u_j},\ u_j \right)_j = -2\beta_{ij} \int_{\mathbb{R}^n} u_i^2 u_j^2.$$

Denote

$$a_{ij} = -\frac{\hat{a}_{ij}}{2} = \beta_{ij} \int_{\mathbb{R}^n} u_i^2 u_j^2, \quad i, j = 1, \cdots, N.$$

Then a_{ij} satisfy

$$\begin{cases} a_{jj} > 0 & \text{for } j = 1, \cdots, N, \\ a_{ij} \le 0 & \text{for } i, j = 1, \cdots, N \text{ with } i \ne j, \\ a_{ij} = a_{ji} & \text{for } i, j = 1, \cdots, N, \\ \sum_{i=1}^{N} a_{ij} > 0 & \text{for } j = 1, \cdots, N. \end{cases} \tag{3.4}$$

Denote by Δ the determinant of the matrix (a_{ij}). Then, clearly, $\Delta > 0$ for $N = 1, 2$. We shall use an induction argument to show that $\Delta > 0$ for $N \ge 3$ and thus assume it is true for matrices of order $N-1$ and consider a matrix of order N. For $i = 2, \cdots, N$, subtracting the first row multiplied with a_{i1}/a_{11} from the ith row and then subtracting the first column multiplied with a_{i1}/a_{11} from the ith column, we see that

$$\Delta = \begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & \tilde{a}_{22} & \cdots & \tilde{a}_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \tilde{a}_{N2} & \cdots & \tilde{a}_{NN} \end{vmatrix},$$

where

$$\tilde{a}_{ij} = a_{ij} - \frac{a_{1i}a_{1j}}{a_{11}}, \quad \text{for } i, j = 2, \cdots, N.$$

Since a_{ij} satisfy (3.4), $\tilde{a}_{ij}$ satisfy

$$\tilde{a}_{jj} = a_{jj} - \frac{a_{1j}a_{1j}}{a_{11}} = \frac{1}{a_{11}}[a_{11}(a_{1j} + a_{jj}) - a_{1j}(a_{11} + a_{1j})] > 0, \quad \text{for } j = 2, \cdots, N,$$

$$\tilde{a}_{ij} = a_{ij} - \frac{a_{1i}a_{1j}}{a_{11}} \le 0, \quad \text{for } i, j = 2, \cdots, N \text{ with } i \ne j,$$

$$\tilde{a}_{ij} = \tilde{a}_{ji}, \quad \text{for } i, j = 2, \cdots, N,$$

and for $j = 2, \cdots, N$,

$$\sum_{i=2}^{N} \tilde{a}_{ij} = \sum_{i=2}^{N} a_{ij} - \frac{a_{1j}}{a_{11}} \sum_{i=2}^{N} a_{1i} = \frac{1}{a_{11}} \left(a_{11} \sum_{i=1}^{N} a_{ij} - a_{1j} \sum_{i=1}^{N} a_{1i} \right) > 0.$$

Therefore, $\tilde{a}_{ij}$, $i, j = 2, \cdots, N$, satisfy (3.4), and by the induction assumption

$$\begin{vmatrix} \tilde{a}_{22} & \cdots & \tilde{a}_{2N} \\ \vdots & \ddots & \vdots \\ \tilde{a}_{N2} & \cdots & \tilde{a}_{NN} \end{vmatrix} > 0,$$

which implies $\Delta > 0$.

In view of (3.3), we then see that $\alpha_1 = \cdots = \alpha_N = 0$. Therefore, $\nabla J(\vec{u}) = 0$ and $\vec{u}$ is a critical point of J. The proof is complete.

Remark 3.4. It is still possible to show that $\mathcal{N}$ is a natural constraint of J when β_{ij} $(i \neq j)$ are positive and small. For example, if for some constant $\nu > N - 1$,

$$|\beta_{ij}| \leq \frac{1}{\nu} \sqrt{\beta_{ii}\beta_{jj}} \quad \text{for } i \neq j, \tag{3.5}$$

then for any $(\alpha_1, \cdots, \alpha_N) \in \mathbb{R}^n$ with $(\alpha_1, \cdots, \alpha_N) \neq 0$ and $\vec{u} \in \mathcal{N}$

$$\begin{aligned} &\sum_{i,j=1}^{N} \alpha_i \alpha_j \beta_{ij} \int_{\mathbb{R}^n} u_i^2 u_j^2 \\ &\geq \sum_{j=1}^{N} \alpha_j^2 \beta_{jj} \int_{\mathbb{R}^n} u_j^4 - \frac{1}{\nu} \sum_{i \neq j} |\alpha_i \alpha_j| \sqrt{\beta_{ii}\beta_{jj}} \int_{\mathbb{R}^n} u_i^2 u_j^2 \\ &\geq \left(1 - \frac{N-1}{\nu}\right) \sum_{j=1}^{N} \alpha_j^2 \beta_{jj} \int_{\mathbb{R}^n} u_j^4 > 0. \end{aligned}$$

Therefore, critical points of $J|_{\mathcal{N}}$ are critical points of J if (3.5) is satisfied. However, the constrained functional $J|_{\mathcal{N}}$ may not satisfy the (PS) condition even if J always satisfies the (PS) condition. For example, it was pointed out in [26, Theorem 2(iii)] that when $N = 2$ and $\beta_{11} \leq \beta_{12} < \sqrt{\beta_{11}\beta_{22}}$, $J|_{\mathcal{N}}$ does not satisfy the (PS) condition though critical points of $J|_{\mathcal{N}}$ are critical points of J. Nevertheless, we have the following lemma in the repulsive case.

Lemma 3.5. *The constrained functional $J|_{\mathcal{N}}$ satisfies the (PS) condition under the assumptions of Theorem* 3.1.

Proof. Let $(\vec{u}^m) \subset \mathcal{N}$ be a (PS) sequence for $J|_{\mathcal{N}}$. Then $(\vec{u}^m)$ is bounded in E^N and there exist real numbers $\alpha_1^m, \cdots, \alpha_N^m$ such that

$$\nabla J(\vec{u}^m) - \sum_{j=1}^{N} \alpha_j^m \nabla \Phi_j(\vec{u}^m) = o(1).$$

Therefore, we have

$$\begin{cases} a_{11}^m\alpha_1^m+\cdots+a_{1N}^m\alpha_N^m & =o(1)\\ \qquad\vdots & \quad\vdots\\ a_{N1}^m\alpha_1^m+\cdots+a_{NN}^m\alpha_N^m & =o(1), \end{cases} \tag{3.6}$$

where

$$a_{ij}^m=\beta_{ij}\int_{\mathbb{R}^n}(u_i^m)^2(u_j^m)^2.$$

For any $\vec{u}\in\mathcal{N}$, since $\beta_{ij}\le 0$ $(i\neq j)$,

$$\|u_j\|_j^2=\sum_{i=1}^N\beta_{ij}\int_{\mathbb{R}^n}u_i^2u_j^2\le\beta_{jj}\int_{\mathbb{R}^n}u_j^4,$$

and therefore there exists $\delta>0$ such that for any $\vec{u}\in\mathcal{N}$

$$\|u_j\|_j^2>\delta,\qquad j=1,\cdots,N.$$

Let λ^m be the smallest eigenvalue of the matrix

$$\begin{pmatrix} a_{11}^m & \cdots & a_{1N}^m\\ \vdots & \ddots & \vdots\\ a_{N1}^m & \cdots & a_{NN}^m \end{pmatrix},$$

with corresponding eigenvector $x^m=(x_1^m,\cdots,x_N^m)^T$. Suppose

$$|x_j^m|=\max\{|x_1^m|,\cdots,|x_N^m|\}.$$

Since $a_{ij}^m\le 0$ $(i\neq j)$ and

$$a_{j1}^mx_1^m+\cdots+a_{jN}^mx_N^m=\lambda^mx_j^m,$$

we see that

$$\lambda^m=a_{jj}^m+\sum_{i,\ i\neq j}a_{ji}^m\frac{x_i^m}{x_j^m}\ge a_{jj}^m-\sum_{i,\ i\neq j}|a_{ji}^m|=\sum_{i=1}^N a_{ji}^m=\|u_j^m\|_j^2>\delta.$$

From this observation and (3.6) we deduce that

$$\alpha_j^m=o(1),\quad j=1,\cdots,N.$$

Then $(\vec{u}^m)$ is a (PS) sequence for J and since J satisfies the (PS) condition the result follows. The proof is complete.

Proof of Theorem 3.1. Define

$$d_i'=\inf_{A\in\Sigma_i'}\sup_{\vec{u}\in A}J(\vec{u}),$$

where

$$\Sigma_i'=\{A\mid A\subset\mathcal{N},\ \mathrm{gen}(A)\ge i\},\quad i=1,2,\cdots.$$

Now standard arguments (see, for example, [25]) can be used to obtain the conclusion. The proof is complete.

4 Further remarks

a). In [19] for a 2-system a multiplicity result was given for the attractive case. More precisely, it was shown that for any k integer there is $\beta^k > 0$ such that for $\beta := \beta_{12} > \beta^k$ the 2-system has at least k nontrivial bound state solutions. It is not clear how to extend that result to a general N-system. Our Theorem 2.1 asserts the existence of one such solution with large coupling constants. The idea described in section 2 provides also an alternative proof for the result on multiple bound states for the 2-system in [19]. As in [19], rewriting $\mu_i = \beta_{ii}$ and $\beta = \beta_{12}$, we reformulate the 2-system as

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 u^3 + \beta v^2 u, & \text{in } \mathbb{R}^n, \\ -\Delta v + \lambda_2 v = \mu_2 v^3 + \beta u^2 v, & \text{in } \mathbb{R}^n, \\ \quad u(x) \to 0, \quad v(x) \to 0, & \text{as } |x| \to \infty. \end{cases} \tag{4.1}$$

The following theorem was first proved in [19] by comparing Morse indices of solutions obtained via a Ljusternik-Schnirelman minimax procedure with Morse indices of semitrivial solutions. We now give a different proof.

Theorem 4.1 ([19])**.** *Let $n = 2, 3$ and for $i = 1, 2$ let λ_i and μ_i be fixed positive constants. Then for any $k \in \mathbb{N}$, there exists $\beta^k > 0$ such that for $\beta > \beta^k$ the system* (4.1) *has at least k pairs of nontrivial spherically symmetric solutions.*

Proof. Using the symbols and estimates from the proof of Theorem 2.1, we have

$$m(2,1) \geq \frac{c\lambda_2^{1-\frac{n}{4}}}{\sqrt{\mu_2}}, \qquad m(2,2) \geq \frac{c\lambda_1^{1-\frac{n}{4}}}{\sqrt{\mu_1}}.$$

For any $k \in \mathbb{N}$, define

$$c_k := \inf_{H_k \subset E,\ \dim H_k = k} \ \max_{u \in H_k,\ u \neq 0} \frac{\int_{\mathbb{R}^n} |\nabla u|^2 + u^2}{(\int_{\mathbb{R}^n} u^4)^{1/2}}.$$

For any $\varepsilon > 0$, there exists a k-dimensional subspace $H_k \subset E$ such that

$$\max_{u \in H_k,\ u \neq 0} \frac{\int_{\mathbb{R}^n} |\nabla u|^2 + u^2}{(\int_{\mathbb{R}^n} u^4)^{1/2}} < c_k + \varepsilon.$$

Denote

$$\bar{\lambda} = \frac{\lambda_1 + \lambda_2}{2}, \qquad \hat{H}_k = \{u(\sqrt{\bar{\lambda}}\,\cdot) \mid u \in H_k\}.$$

It is then easy to see that

$$\max_{u \in \hat{H}_k,\ u \neq 0} \frac{\int_{\mathbb{R}^n} |\nabla u|^2 + \bar{\lambda} u^2}{(\int_{\mathbb{R}^n} u^4)^{1/2}} < \bar{\lambda}^{1-\frac{n}{4}}(c_k + \varepsilon).$$

Set

$$G_k = \{(u, u) \mid u \in \hat{H}_k\}.$$

Then G_k is a k-dimensional subspace of E^2 and

$$\max_{\vec{u}\in G_k,\ \vec{u}\neq 0} I(\vec{u}) = \frac{2}{\sqrt{\mu_1+\mu_2+2\beta}} \max_{u\in \hat{H}_k,\ u\neq 0} \frac{\int_{\mathbb{R}^n} |\nabla u|^2 + \bar{\lambda}u^2}{(\int_{\mathbb{R}^n} u^4)^{1/2}} < \frac{2\bar{\lambda}^{1-\frac{n}{4}}(c_k+\varepsilon)}{\sqrt{\mu_1+\mu_2+2\beta}}.$$

Let β^k be the positive number defined by

$$\frac{2c_k\bar{\lambda}^{1-\frac{n}{4}}}{\sqrt{\mu_1+\mu_2+2\beta^k}} = \min\left\{\frac{c\lambda_1^{1-\frac{n}{4}}}{\sqrt{\mu_1}},\quad \frac{c\lambda_2^{1-\frac{n}{4}}}{\sqrt{\mu_2}}\right\}.$$

If $\beta > \beta^k$ then for $\varepsilon > 0$ small enough

$$\frac{2(c_k+\varepsilon)\bar{\lambda}^{1-\frac{n}{4}}}{\sqrt{\mu_1+\mu_2+2\beta}} < \min\left\{\frac{c\lambda_1^{1-\frac{n}{4}}}{\sqrt{\mu_1}},\quad \frac{c\lambda_2^{1-\frac{n}{4}}}{\sqrt{\mu_2}}\right\}.$$

Therefore, according to the discussion above, there is a k-dimensional subspace G_k of E^2 such that

$$\max_{\vec{u}\in G_k,\ \vec{u}\neq 0} I(\vec{u}) < \min\{m(2,1),\ m(2,2)\}.$$

This inequality together with standard arguments yields at least k pairs of nontrivial spherically symmetric solutions of (4.1). The proof is complete.

b). When $\lambda_1 = \cdots = \lambda_N$ and the matrix $B = (\beta_{ij})$ is in the form of the product of a row-stochastic matrix and a diagonal matrix both with positives entries, special type nontrivial solutions of (1.1) for attractive case can be constructed ([5, 26]). We do not know whether these special type solutions are the ground states or not. For 2-systems a uniqueness result on positive solutions in the attractive case was given recently in [31].

c). Our main results in both sections still hold when we replace the entire space by a bounded domain in $\mathbb{R}^n$.

d). We do not know whether the solutions given in Theorem 3.1 are positive ones. We suspect that some of these solutions are nodal solutions.

References

[1] N. Akhmediev and A. Ankiewicz, *Partially coherent solitons on a finite background,* Phys. Rev. Lett. **82** (1999), 2661-2664.

[2] A. Ambrosetti and E. Colorado, *Bound and ground states of coupled nonlinear Schrödinger equations,* C. R. Math. Acad. Sci. Paris **342** (2006), 453–458.

[3] A. Ambrosetti and E. Colorado, *Standing waves of some coupled nonlinear Schrödinger equations,* J. London Math. Soc. **75** (2007), 67-82.

[4] T. Bartsch, E.N. Dancer and Z.-Q. Wang, A Liouville theorem, a-priori bounds, and bifurcating branches of positive solutions for a nonlinear elliptic system, preprint.

[5] T. Bartsch and Z.-Q. Wang,*Note on ground states of nonlinear Schrödinger systems,* J. Partial Differential Equations **19** (2006), 200-207.

[6] T. Bartsch, Z.-Q. Wang, and J.C. Wei, *Bound states for a coupled Schrödinger system,* J. Fixed Point Theory Appl. **2** (2007), 353-367.

[7] K.C. Chang, Infinite Dimensional Morse Theory and Multiple Solution Problems, Progress in Nonlinear Differential Equations and Their Applications **6**, Birkhäuser, Boston, 1993.

[8] D. N. Christodoulides, T. H. Coskun, M. Mitchell, and M. Segev, *Theory of incoherent self-focusing in biased photorefractive media,* Phys. Rev. Lett. **78** (1997), 646-649.

[9] E.N. Dancer, J.C. Wei, and T. Weth, A priori bounds versus multiple existence of positive solutions for a nonlinear Schrödinger system, preprint.

[10] K. Deimling, Ordinary Differential Equations in Banach Spaces, Lecture Notes in Mathematics, Vol. 596. Springer-Verlag, Berlin, 1977.

[11] B. D. Esry, C. H. Greene, J. P. Burke Jr., and J. L. Bohn, *Hartree-Fock theory for double condensates,* Phys. Rev. Lett. **78** (1997), 3594-3597.

[12] G. M. Genkin,*Modification of superfluidity in a resonantly strongly driven Bose-Einstein condensate,* Phys. Rev. A **65** (2002), No.035604.

[13] F. T. Hioe, *Solitary waves for N coupled nonlinear Schrödinger equations, Phys. Rev. Lett.* **82** (1999), 1152-1155.

[14] F. T. Hioe and T. S. Salter, *Special set and solutions of coupled nonlinear Schrödinger equations,* J. Phys. A: Math. Gen. **35** (2002), 8913-8928.

[15] T. Kanna and M. Lakshmanan, *Exact soliton solutions, shape changing collisions, and partially coherent solitons in coupled nonlinear Schrödinger equations,* Phys. Rev. Lett. **86** (2001), 5043-5046.

[16] T.-C. Lin and J.C. Wei, *Ground state of N coupled nonlinear Schrödinger equations in R^n, $n \leq 3$,* Comm. Math. Phys. **255** (2005), 629-653.

[17] T.-C. Lin and J.C. Wei, *Spikes in two coupled nonlinear Schrödinger equations,* Ann. Inst. H. Poincaré Anal. Non Linéaire **22** (2005), 403-439.

[18] T.-C. Lin and J.C. Wei, *Spikes in two-component systems of nonlinear Schrödinger equations with trapping potentials,* J. Differential Equations **229** (2006), 538–569.

[19] Z.L. Liu and Z.-Q. Wang, *Multiple bound states of nonlinear Schrödinger systems,* Comm. Math. Phys. **282** (2008), 721-731.

[20] L.A. Maia, E. Montefusco, and B. Pellacci, *Positive solutions for a weakly coupled nonlinear Schrödinger system,* J. Differential Equations **229** (2006), 743-767.

[21] E. Montefusco, B. Pellacci, and M. Squassina, *Semiclassical states for weakly coupled nonlinear Schrödinger systems, J. Eur. Math. Soc.* **10** (2008), 47-71.

[22] M. Mitchell, Z. Chen, M. Shih, and M. Segev, *Self-Trapping of partially spatially incoherent light,* Phys. Rev. Lett. **77** (1996) 490-493.

[23] B. Noris, H. Tavares, S. Terracini, and G. Verzini, Uniform Hölder bounds for nonlinear Schrödinger systems with strong competition, preprint.

[24] A. Pomponio, *Coupled nonlinear Schrödinger systems with potentials, J. Differential Equations* **227** (2006), 258–281.

[25] P.H. Rabinowitz, Minimax Methods in Critical Point Theory with Applications to Differential Equations, CBMS Reg. Conf. Ser. Math. **65**, AMS, Providence, R.I., 1986.

[26] B. Sirakov, *Least energy solitary waves for a system of nonlinear Schrödinger equations in* $\mathbb{R}^n$*,* Comm. Math. Phys. **271** (2007), 199–221.

[27] S. Terracini and G. Verzini, Multipulses phases in k-mixtures of Bose-Einstein condensates, preprint.

[28] E. Timmermans, *Phase separation of Bose-Einstein condensates,* Phys. Rev. Lett. **81** (1998), 5718-5721.

[29] J.C. Wei and T. Weth, *Nonradial symmetric bound states for a system of two coupled Schrödinger equations,* Rend. Lincei Mat. Appl., to appear.

[30] J.C. Wei and T. Weth, *Radial solutions and phase seperation in a system of two coupled Schrödinger equations,* Arch. Rational Mech. Anal. **190** (2008), 83-106.

[31] J.C. Wei and W. Yao, Note on uniqueness of positive solutions for osme coupled nonlinear Schrödinger equations, preprint.