Existence and symmetry of multi-bump solutions for nonlinear Schrödinger equations

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Abstract. *We study the existence and symmetry property of multi-bump solutions of*

$$(\mathcal{P}_\lambda) \quad -\Delta v + \lambda V(x) v = v^p, v > 0, \quad in \ \mathbf{R}^N.$$

Especially when the potential V is radially symmetric, multi-bump solutions are constructed with bumps concentrating on a single connected component of the set of global minimum points of V. Furthermore, conditions are given to assure that the multi-bump solutions obtained have prescribed subgroups of $\mathbf{O}(N)$ as their exact symmetry.

1. Introduction and Main Results

This paper is concerned with the existence of standing wave solutions to the following nonlinear Schrödinger equation

$$ih\psi_t = -\frac{h^2}{2m}\Delta\psi + V(x)\psi - |\psi|^{p-1}\psi$$

where $x \in \mathbf{R}^N$, h, m and p are positive constants with $p > 1$. A solution of the following form is called a standing wave solution

$$\psi(x,t) = \exp(-iEt/h)v(x),$$

which leads to a nonlinear elliptic PDE for v on $\mathbf{R}^N$:

$$-\frac{h^2}{2m}\Delta v + (V(x) - E)v = |v|^{p-1}v.$$

We will assume V to be bounded from below so with suitable choice of E we may assume $E = 0$ and V is bounded from below by 1. We shall seek solutions tending to zero as $|x| \to \infty$. With suitable rescalings this equation can be written in different forms. For convenience we consider the following problem in which the equation is equivalent to the above with suitable change of variables

$$(\mathcal{P}_\lambda) \quad \begin{cases} -\Delta v + \lambda V(x) v = v^p, v > 0 \quad \text{in } \mathbf{R}^N \\ u \to 0, \quad |x| \to \infty \end{cases}$$

where $\lambda > 0$ is a parameter, $V(x) \in C(\mathbf{R}^N, \mathbf{R})$ satisfies $V(x) \geq 1$, for all $x \in \mathbf{R}^N$. In this paper, we assume $1 < p < \frac{N+2}{N-2}$ if $N \geq 3$ and $1 < p < \infty$ for $N = 1, 2$.

Following the pioneering work of Strauss ([Sw]), there have been many papers in this area (See [BL] [DN] [Lp1] [R] [AL] [HKS] [J] [BWm] [BWz] [M] [TW] [KS] and references therein), and in recent years more papers have been devoted to concerning the existence of so-called multi-bump solutions, e.g., [ABC] [CZR] [DF1-2] [FW] [G] [Ly] [O1-2] [Wx] and references therein. Several methods, which are quite different in nature, have been used in these papers to handle the problem under different conditions on V. However, in all these papers multi-bump solutions are constructed in such a way that bumps of the solutions as $\lambda \to \infty$ are concentrated more and more near a subset of the nontrivial critical points of V which are *disjoint from each other*. It seems to be crucial in all these papers that the critical point sets of V where multiple bumps are located are disjoint. In this paper we shall consider a situation where the above mentioned results do not seem to apply: multi-bump solutions will be constructed with multiple bumps located near *a single connected component of the set of critical points of* V. For example, we may consider V to be radially symmetric, i.e., $V(x) = V(|x|)$. Assuming that V achieves its minimum at $|x| = 1$. Then for $N \geq 2$ we shall be able to obtain k-bump solutions of $(\mathcal{P}_\lambda)$ for any positive integer k with bumps concentrated more and more near $|x| = 1$ provided λ is large.

Another question we shall address in this paper is the *symmetry property* of solutions of $(\mathcal{P}_\lambda)$ when V is symmetric with respect to a subgroup of $\mathbf{O}(N)$. For example, again as a special case of our general theory we may consider radially symmetric V. Then we can construct symmetric multi-bump solutions of $(\mathcal{P}_\lambda)$ having *prescribed symmetry.* More precisely, in this situation, for subgroups G of $\mathbf{O}(N)$ satisfying certain conditions we shall obtain multi-bump solutions with exact symmetry G. Apparently, the symmetry problem for non-autonomous Schrödinger equations has not been studied very much so far. We shall give conditions on G under which *multi-bump solutions with prescribed symmetry* can be constructed. Consequently we obtain many *non-radial* solutions of the problem with *prescribed exact symmetry.*

To state our results, let us first introduce some notations. Let $1 \leq l \leq N$ be an integer. We write

$$\mathbf{R}^N = \mathbf{R}^l \times \mathbf{R}^{N-l}. \tag{1.1}$$

Let $\mathbf{O}(N)$ be the group of orthogonal transformations in $\mathbf{R}^N$. Let $G_1 \subset \mathbf{O}(l)$ and $G_2 \subset \mathbf{O}(N-l)$ be two closed subgroups, and define $G \subset \mathbf{O}(N)$ by

$$G = G_1 \times G_2, \tag{1.2}$$

which acts canonically on $\mathbf{R}^l \times \mathbf{R}^{N-l} = \mathbf{R}^N$. We denote by $\mathrm{Fix}_G(\mathbf{R}^N)$ the fixed point set of an action $G \subset \mathbf{O}(N)$ in $\mathbf{R}^N$, i.e., $\mathrm{Fix}_G(\mathbf{R}^N) = \{x \in \mathbf{R}^N \mid gx = x \ \ \forall g \in G\}$, and by $\#G(x)$ the cardinal number of $G(x)$, the orbit of x under the action G. We assume

(S1) $\mathrm{Fix}_{G_2}(\mathbf{R}^N) = \mathbf{R}^l \times \{0\}$, and

$$k := \min\{\#G(x) \mid x \in \mathbf{R}^l \times \{0\}, |x| = 1\} < \infty.$$

Remark 1.1. Note that $k = \min\{\#G_1(x) \mid x \in \mathbf{R}^l \times \{0\}, |x| = 1\}$ since for $g \in G_2$, $gx = x$ for any $x \in \mathbf{R}^l \times \{0\}$.

On the potential function V we assume

(V1) $V \in C(\mathbf{R}^N, \mathbf{R})$ is radially symmetric, i.e., $V(x) = V(|x|)$, and $\inf_{\mathbf{R}^N} V(x) = 1$. Furthermore, V achieves its infimum on $\{x \mid |x| = 1\}$, and there exists $\sigma_0 > 0$ such that

$$V(x) > 1, \ \text{ for } \ 0 < ||x| - 1| \leq 2\sigma_0.$$

In order to deal with the issue of exact symmetry, we assume

(S2) For all $x \in \mathbf{R}^l \times \{0\}$ with $|x| = 1$ and $\#G(x) = k$, and all nontrivial supergroups $\tilde{G}$ of G in $\mathbf{O}(N)$ (i.e., $G \subset \tilde{G} \subset \mathbf{O}(N)$, but $G \neq \tilde{G}$), $\#\tilde{G}(x) > k$.

The ground state solution of the following problem plays an important role for $(\mathcal{P}_\lambda)$

$$-\Delta w + w = w^p, w > 0, \lim_{|x|\to\infty} w(x) = 0, \ \text{in} \ \mathbf{R}^N. \tag{1.3}$$

By the results in [K] [KZ], we shall use w to denote the unique solution (up to translations) which satisfies $w(0) = \max_{x \in \mathbf{R}^N} w(x)$. Then it is known that

$$S_p := \frac{\int_{\mathbf{R}^N} |\nabla w|^2 + w^2}{(\int_{\mathbf{R}^N} |w|^{p+1})^{\frac{2}{p+1}}} = \inf_{u \in H^1(\mathbf{R}^N)\setminus\{0\}} \frac{\int_{\mathbf{R}^N} |\nabla u|^2 + u^2}{(\int_{\mathbf{R}^N} |u|^{p+1})^{\frac{2}{p+1}}}. \tag{1.4}$$

By the result in [GNN], w is radially symmetric and there exist constants $C_0 > 0$ and $\mu > 0$ such that

$$|w(x)| + |Dw(x)| \leq C_0 e^{-\mu|x|} \ \text{ for all } \ x \in \mathbf{R}^N. \tag{1.5}$$

Now we are ready to state our main results.

Theorem 1.1. *Suppose* **(S1)** *and* **(V1)** *hold. Then there exists $\lambda_0 > 0$ such that for $\lambda > \lambda_0$, $(\mathcal{P}_\lambda)$ possesses a solution v_λ satisfying*
(1°). v_λ is at least G-invariant, i.e., $v_\lambda(g^{-1}x) = v_\lambda(x)$ for all $x \in \mathbf{R}^N$ and $g \in G$.

(2°).

$$\lim_{\lambda\to\infty} \lambda^{\frac{-\beta_0(p+1)}{p-1}} \int_{\mathbf{R}^N} (|\nabla v_\lambda|^2 + \lambda V(x) v_\lambda^2) dx = k S_p^{\frac{p+1}{p-1}}, \tag{1.6}$$

where $\beta_0 = \frac{2N-(p+1)(N-2)}{2(p+1)} > 0$ *and* S_p *is defined in (1.4).*
(3°). v_λ *is a* k*-bump solution in the sense:* v_λ *has exactly* k *local maximum points over* $\mathbf{R}^N$ *which form a* G*-orbit:* $G(P_\lambda)$*, for some* $P_\lambda \in \mathbf{R}^l \times \{0\}$ *satisfying* $\#G(P_\lambda) = k$ *and* $\lim_{\lambda\to\infty} |P_\lambda| = 1$.
(4°). Defining $u_\lambda(x) = \lambda^{-\frac{1}{p-1}} v_\lambda(\lambda^{-\frac{1}{2}} x)$*, we have*
(1.7)

$$\lim_{\lambda\to\infty} \int_{\mathbf{R}^N} \left|\nabla\left(u_\lambda - \sum_{P\in G(P_\lambda)} w(x-\lambda^{\frac{1}{2}}P)\right)\right|^2 + V(\lambda^{-\frac{1}{2}}x)\left(u_\lambda - \sum_{P\in G(P_\lambda)} w(x-\lambda^{\frac{1}{2}}P)\right)^2 = 0.$$

(5°). If in addition to **(S1)** *and* **(V1)**, **(S2)** *holds, then* v_λ *is also at most* G*-invariant, i.e.,* v_λ *has exact* G *symmetry.*

As a corollary we present a result which gives more precise information about possible isotropy subgroups of the solutions. In the following corollary, l is as in the decomposition (1.1) and k is the number of bumps of the solutions. As indicated by l in some cases, the dimension N has to be greater than one. In the following, $\mathbf{T}, \mathbf{O}, \mathbf{I}$ are exceptional subgroups of $\mathbf{O}(3)$: the tetrahedral subgroup, the octahedral subgroup, and the icosahedral subgroup; $\mathbf{D}_k$ are the dihedral symmetry groups; $\mathbf{Z}_2(l) = \{Id, -Id\}$ are the reflection symmetry groups in $\mathbf{R}^l$; for $l \geq 3$, $\mathbf{T}_l$ are the hyper-tetrahedron groups and $\mathbf{B}_l$ are the hypercube groups in $\mathbf{R}^l$. (See details about these groups in Section 2.3.)

Corollary 1.1. *Suppose* $N \geq 2$ *and* **(V1)** *holds. Then for any positive integer* $k > 1$ *there is* $\lambda_k > 0$ *such that for all* $\lambda > \lambda_k$*,* $(\mathcal{P}_\lambda)$ *has a* k*-bump solution* v_λ *which satisfies (1°)-(4°) in Theorem 1.1. Furthermore, for* λ *sufficiently large there exist* k*-bump solutions of* $(\mathcal{P}_\lambda)$ *with isotropy subgroups* $\mathbf{T} \oplus \mathbf{O}(N-3)$ $(l = 3,\ k = 4)$, $(\mathbf{O} \oplus \mathbf{Z}_2(3)) \oplus \mathbf{O}(N-3)$ $(l = 3,\ k = 6)$, $(\mathbf{I} \oplus \mathbf{Z}_2(3)) \oplus \mathbf{O}(N-3)$ $(l = 3,\ k = 12)$, $\mathbf{D}_k \oplus \mathbf{O}(N-2)$ $(l = 2,\ k \geq 3)$, $\mathbf{Z}_2(1) \oplus \mathbf{O}(N-1)$ $(l = 1,\ k = 2)$, $\mathbf{B}_l \oplus \mathbf{O}(N-l)$ $(l \geq 3,\ k = 2l)$, $\mathbf{T}_l \oplus \mathbf{O}(N-l)$ $(l \geq 3,\ k = l+1)$.

Remark 1.2. Several remarks are in order here.
1. In the case of $N = 1$, we still get 2-bump solutions with isotropy subgroup $\mathbf{Z}_2$.
2. Solutions given in our results are distingushed not only by their 'energy' but also by their exact symmetry.
3. More general cases can be treated similarly and some extensions are stated in Section 3, where V needs not to be radially symmetric.

4. Regarding to the methods we present here, we want to remark that solutions given in this paper are found to be *local minimizers* of a related variational formulation and that in most cases these solutions can never be found as global minimizers [Wz2]. This local minimization procedure stems out of the study of some nonlinear Neumann problems ([Wz1] [MSW]).

This paper is organized as follows. Section 2 is devoted to the proof of Theorem 1.1. Section 3 contains extensions of our main results to cases where the potential function V satisfies more general conditions. In Section 4 we provide some technical results which are used in Section 2, including a generalization of P.L. Lions Concentration-Compactness Lemma which we reformulate into a more detailed version and give a self-contained proof.

2. Existence Theory and Symmetry of Solutions

This section is split into three subsections: in Section 2.1 we set up a local variational formulation and give some asymptotic estimates; in Section 2.2 we establish the existence theory and study the qualitative property of solutions, giving the proof of assertions (1°) - (4°) of Theorem 1.1; then in Section 2.3 we study the symmetry property of the solutions, proving assertion (5°) of Theorem 1.1 and Corollary 1.1.

2.1. *A Local Minimization Setup and Some Asymptotic Estimates*

Define

$$E_\lambda(u) = \int_{\mathbf{R}^N} (|\nabla u|^2 + \lambda V(x)u^2)dx \tag{2.1}$$

for $u \in M^G = \{u \in F^G \mid |u|_{p+1} = 1\}$, where

$$F := \{u \in H^1(\mathbf{R}^N) \mid \int_{\mathbf{R}^N} V(x)u^2dx < \infty \}, \tag{2.2}$$

and

$$F^G := \{u \in F \mid u(g^{-1}x) = u(x), a.e. \text{ in } \mathbf{R}^N, \text{ for all } g \in G \}. \tag{2.3}$$

By the symmetric criticality principle ([P]) positive critical points of $E_\lambda(u)$ on M^G correspond to solutions of $(\mathcal{P}_\lambda)$. Especially the global minimizers over M^G, if exist, are solutions (by rescaling). However we remark here that in most cases the global minimizers do not produce multi-bump solutions ([Wz2]).

In the following, we introduce a local minimization procedure to locate k-bump solutions. In this section, k is fixed as in **(S1)** and σ_0 is fixed as in **(V1)**. We define

$$\delta_k = \frac{1}{2k+1}. \tag{2.4}$$

Choose $d > 1$ and $\delta > 0$ such that

$$0 < \delta < 1 - \frac{1}{d} < \min\{\frac{\delta_k}{2}, \frac{\sigma_0}{2}\}. \tag{2.5}$$

d and δ will be fixed in the rest of the proof.

For $u \in M^G$ we define

$$\gamma(u) = \int_{\mathbf{R}^N} |u|^{p+1} \xi(|P_l x|) dx \tag{2.6}$$

where $P_l : \mathbf{R}^N \to \mathbf{R}^l \times \{0\}$ is the linear projection, and $\xi \in C^1(\mathbf{R}_+, \mathbf{R})$ is defined by

$$\xi(t) = \frac{t}{d}, \text{ for } t \in [0, d], \text{ and } \xi(t) = 1 \text{ for } t \geq d.$$

Note that $\gamma(u)$ is a continuous function assuming values in $(0, 1)$. Let us define

$$O_\delta = \{u \in M^G \ \mid \ \gamma(u) \in (\frac{1}{d} - \delta, \frac{1}{d} + \delta)\}, \tag{2.7}$$

an open subset in M^G. Finally we define

$$c_{\lambda,\delta} = \inf_{O_\delta} E_\lambda(u). \tag{2.8}$$

Solutions of k-bump type will be constructed as local minimizers in O_δ of the energy functional E_λ. We need an asymptotic estimate.

Proposition 2.1. Assume **(S1)** and **(V1)** and, with k given in **(S1)**, let us fix δ_k, δ and d in (2.4) and (2.5). Then

(a). $\lim_{\lambda\to\infty} \lambda^{-\beta_0} c_{\lambda,\delta} = k^{\frac{p-1}{p+1}} S_p$, where $\beta_0 = \frac{2N-(p+1)(N-2)}{2(p+1)} > 0$.

(b). Let $\lambda_n \to \infty$ and v_n be a sequence in $\overline{O_\delta}$ (the closure of O_δ) such that

$$\lim_{n\to\infty} \lambda_n^{-\beta_0} E_{\lambda_n}(v_n) = k^{\frac{p-1}{p+1}} S_p,$$

then there exist $x_n \in \mathbf{R}^l \times \{0\}$, satisfying $\lim_{n\to\infty} |x_n| = 1$ and $\#G(x_n) = k$, such that for any $\epsilon > 0$ there exists $R > 0$

$$\int_{B_{\frac{R}{\sqrt{\lambda_n}}}(x_n)} |v_n|^{p+1} \geq \frac{1}{k} - \epsilon, \tag{2.9}$$

here and later $B_r(x)$ denotes a ball centered at x with radius $r > 0$ in $\mathbf{R}^N$.

Remark 2.1. By (2.9), the sequence in Proposition 2.1 satisfies $\lim_{n\to\infty} \gamma(v_n) = \frac{1}{d}$.

The proof of this result depends upon another technical result (Proposition 4.1) which will be proved in Section 4. Before going into the proof of Proposition 2.1, we sketch in the following remark a upper bound of $\lim_{\lambda\to\infty}\lambda^{-\beta_0}c_{\lambda,\delta}$.

Remark 2.2. Using appropriate testing functions one can easily show that $\lim_{\lambda\to\infty}\lambda^{-\beta_0}c_{\lambda,\delta}\leq k^{\frac{p-1}{p+1}}S_p$. Let $\eta=\eta(t)$ be a smooth non-increasing function on $[0,\infty)$ such that $\eta(t)=1$, for $t\in[0,1]$, $\eta(t)=0$, for $t\geq 2$, and $|\eta'(t)|\leq 2$. Write $\eta^c(t):=1-\eta(t)$. Then taking $P_0\in(\mathbf{R}^l\times\{0\})$ such that $|P_0|=1$ and the $G-$orbit of P_0 containing exactly k points:$\{g_iP_0\mid i=1,...,k\}$, and defining

$$w_\lambda(x)=\sum_{i=1}^{k}\lambda^{\frac{N}{2(p+1)}}\eta(\frac{|x-g_iP_0|}{a})w(\lambda^{\frac{1}{2}}(x-g_iP_0))$$

where w is the ground state solution of (1.3), and $0<a<\frac{1}{2}\min\{|g_iP_0-g_jP_0|\mid i\neq j\}$ is fixed for the moment, we get $\frac{w_\lambda}{||w_\lambda||_{p+1}}\in M^G$. Also it is easy to check that

$$\gamma(\frac{w_\lambda}{||w_\lambda||_{p+1}})\to\frac{1}{d}$$

as $\lambda\to\infty$. Then using (1.4) and (1.5) a direct computation shows that there exists $\nu_0>0$ such that as $\lambda\to\infty$

$$||\nabla w_\lambda||^2_{L^2(R^N)}=\lambda^{\beta_0}\left(k\int_{\mathbf{R}^N}|\nabla w|^2dx+o(1)\right)+O(\lambda^{\frac{N+(p+1)}{p+1}}e^{-\mu\nu_0\lambda})$$

$$\int_{\mathbf{R}^N}\lambda V(x)|w_\lambda|^2dx\leq\lambda^{\beta_0}\left(k\max_{B_{2a}(P_0)}V(x)\int_{\mathbf{R}^N}w^2dx+o(1)\right)+O(\lambda^{\frac{N+(p+1)}{p+1}}e^{-\mu\nu_0\lambda})$$

and

$$||w_\lambda||^{p+1}_{L^{p+1}(R^N)}=\left(k\int_{\mathbf{R}^N}w^{p+1}dx\right)+o(1)+O(\lambda^{\frac{N}{2}}e^{-\frac{\mu\nu_0(p+1)\lambda}{2}}).$$

Then it follows from the last three formulas that

$$\lambda^{-\beta_0}E_\lambda(\frac{w_\lambda}{||w_\lambda||_{p+1}})\leq k^{\frac{p-1}{p+1}}\frac{\int_{\mathbf{R}^N}|\nabla w_\lambda|^2+\max_{B_{2a}(P_0)}V(x)\int_{\mathbf{R}^N}|w|^2}{||w||^2_{p+1}}+o(1),\quad\text{as}\quad\lambda\to\infty.$$

Since $a>0$ is arbitrary, we get $\lim_{\lambda\to\infty}\lambda^{-\beta_0}c_{\lambda,\delta}\leq k^{\frac{p-1}{p+1}}S_p$. □

Proof of Proposition 2.1. Let $v_n\in O_\delta$ be such that

$$A=\lim_{n\to\infty}\lambda_n^{-\beta_0}E_{\lambda_n}(v_n)\leq k^{\frac{p-1}{p+1}}S_p.$$

Define $u_n=\lambda_n^{-\frac{N}{2(p+1)}}v_n(\lambda_n^{-\frac{1}{2}}x)$. Then we have $||u_n||_{p+1}=1$ and

$$\int_{\mathbf{R}^N}|\nabla u_n|^2+V(\lambda_n^{-\frac{1}{2}}x)u_n^2=\lambda_n^{-\beta_0}E_{\lambda_n}(v_n).$$

Using the notation in Section 4, let $T_R = \{x \in \mathbf{R}^N || P_l x| < R\}$ and $T_R^c := R^N \setminus T_R$.

Claim 1. For any $R > 0$ fixed,

$$\limsup_{n\to\infty} \int_{T_R} |u_n|^{p+1} \leq \delta_k. \tag{2.10}$$

In fact we have

$$\begin{aligned}
\frac{1}{d} - \delta &\leq \liminf_{n\to\infty} \int_{\mathbf{R}^N} |v_n|^{p+1} \xi(|P_l x|) dx \\
&= \liminf_{n\to\infty} \int_{\mathbf{R}^N} |u_n|^{p+1} \xi(\lambda_n^{-\frac{1}{2}} |P_l x|) dx \\
&\leq \liminf_{n\to\infty} \left(\int_{T_R} |u_n|^{p+1} \xi(\lambda_n^{-\frac{1}{2}} |P_l x|) dx + \int_{|P_l x| \geq R} |u_n|^{p+1} \xi(\lambda_n^{-\frac{1}{2}} |P_l x|) dx \right) \\
&\leq \liminf_{n\to\infty} \frac{\lambda_n^{-\frac{1}{2}} R}{d} \int_{T_R} |u_n|^{p+1} dx + \liminf_{n\to\infty} \int_{|P_l x| \geq R} |u_n|^{p+1} dx \\
&= \liminf_{n\to\infty} \int_{T_R^c} |u_n|^{p+1} dx
\end{aligned}$$

which implies that $\limsup_{n\to\infty} \int_{T_R} |u_n|^{p+1} \leq 1 - (\frac{1}{d} - \delta) \leq \delta_k$.

Next we choose a sequence $R_{m+1} = 2R_m \to \infty$ as $m \to \infty$. Up to a subsequence we may assume $b_m = \lim_{n\to\infty} \int_{T_{R_m}} |u_n|^{p+1}$, and $b_0 = \lim_{m\to\infty} b_m$. Then one has $b_0 \leq \delta_k$ by (2.10). Then there exists $n_m \to \infty$ as $m \to \infty$ such that

$$\left| b_m - \int_{T_{R_m}} |u_{n_m}|^{p+1} \right| \leq \frac{1}{m}, \quad \left| b_{m+1} - \int_{T_{R_{m+1}}} |u_{n_m}|^{p+1} \right| \leq \frac{1}{m}.$$

For simplicity of notations, we recall this sequence $\{u_{n_m}\}_{m=1}^{\infty}$ as $\{u_n\}$ again. Define

$$w_n = \eta\left(\frac{|P_l x|}{R_n}\right) u_n(x)$$

and

$$h_n = \eta^c\left(\frac{|P_l x|}{R_n}\right) u_n(x).$$

Then $\lim_{n\to\infty} \int_{\mathbf{R}^N} |w_n|^{p+1} = b_0$ and $\lim_{n\to\infty} |h_n|^{p+1} = 1 - b_0$.

Claim 2. $b_0 = 0$.

A direct computation shows

$$\int_{\mathbf{R}^N} |\nabla u_n|^2 + V(\lambda_n^{-\frac{1}{2}} x) u_n^2 - \int_{\mathbf{R}^N} |\nabla w_n|^2 + w_n^2 - \int_{\mathbf{R}^N} |\nabla h_n|^2 + h_n^2 \geq -\frac{C_0}{R_n}$$

as $n \to \infty$, where C_0 is a constant independent n. Note that $h_n \in H^1_{0,G}(T^c_{R_n})$. Then by Prop. 4.1,

$$\begin{aligned} A &= \lim_{n\to\infty} \int_{\mathbf{R}^N} |\nabla u_n|^2 + V(\lambda_n^{-\frac{1}{2}} x) u_n^2 \\ &\geq \lim_{n\to\infty} \left(\int_{\mathbf{R}^N} (|\nabla w_n|^2 + w_n^2) + \int_{\mathbf{R}^N} (|\nabla h_n|^2 + h_n^2) - \frac{C_0}{R_n} \right) \\ &\geq \lim_{n\to\infty} \left(S_p |w_n|^2_{p+1} + S(R_n) |h_n|^2_{p+1} - \frac{C_0}{R_n} \right) \\ &= S_p (b_0)^{\frac{2}{p+1}} + k^{\frac{p-1}{p+1}} S_p (1-b_0)^{\frac{2}{p+1}}. \end{aligned}$$

Here Prop. 4.1 is used in the second inequality. This implies

$$k^{\frac{p-1}{p+1}} \geq (b_0)^{\frac{2}{p+1}} + k^{\frac{p-1}{p+1}} (1-b_0)^{\frac{2}{p+1}}.$$

Since $b_0 \leq \delta_k$ by Lemma 4.4, $b_0 = 0$. Then $\lim_{n\to\infty} |h_n|^{p+1} = 1$. Since $h_n \in H^1_{0,G}(T^c_{R_n})$ by Proposition 4.1 we conclude that there exist $y_n \in \mathbf{R}^l \times \{0\}$ with $\#G(y_n) = k$ and $|y_n| \to \infty$ as $n \to \infty$, such that for any $\epsilon > 0$ there exists $R > 0$

$$\lim_{n\to\infty} \int_{\bigcup_{y\in G(y_n)} B_R(y)} |u_n|^{p+1} \geq 1 - \epsilon. \tag{2.11}$$

Claim 3. $\limsup_{n\to\infty} |P_l^\perp y_n|$ is finite.

If this is not true, up to a subsequence we assume $\lim_{n\to\infty} |P_l^\perp y_n| = \infty$. Because u_n is $G-$invariant, there is $g \in G$ such that $|g P_l^\perp y_n - P_l^\perp y_n| \to \infty$ as $n \to \infty$. This is a contradiction with (2.11) for $|u_n|_{p+1} = 1$.

Claim 4. There is $C > 0$

$$\limsup_{n\to\infty} \operatorname{dist}(y_n, \lambda_n^{\frac{1}{2}}(A_{\sigma_0})) \leq C,$$

where $A_{\sigma_0} := \{x \in \mathbf{R}^N \mid 1 - \sigma_0 \leq |P_l x| \leq 1 + \sigma_0\}$.

If this is not the case, up to a subsequence

$$\lim_{n\to\infty} \operatorname{dist}(y_n, \lambda_n^{\frac{1}{2}}(A_{\sigma_0})) = \infty.$$

Then a simple computation shows either $\lim_{n\to\infty} \gamma(v_n) \geq 1+\sigma_0 > \frac{1}{d}+\delta$ or $\lim_{n\to\infty} \gamma(v_n) \leq 1 - \sigma_0 < \frac{1}{d} - \delta$, a contradiction.

From this claim we may assume that $\{y_n\} \subset \lambda_n^{\frac{1}{2}}(A_{\sigma_0})$.

Claim 5. For each $\sigma_0 > \sigma > 0$ there is $C_\sigma > 0$

$$\limsup_{n\to\infty} \operatorname{dist}(y_n, \lambda_n^{\frac{1}{2}}(A_\sigma)) \leq C_\sigma.$$

If this is not true, there is $\sigma \in (0, \sigma_0)$

$$\limsup_{n\to\infty} \operatorname{dist}(y_n, \lambda_n^{\frac{1}{2}}(A_\sigma)) = \infty. \tag{2.12}$$

By **(V1)**, if necessary decreasing σ we may assume that there is $a_\sigma > 0$

$$V(x) \geq 1 + a_\sigma, \quad \text{for any} \quad x \in (A_{2\sigma_0} \setminus A_\sigma) \cap \{x \mid |P_l^\perp x| \leq \sigma\}.$$

Taking $\epsilon_m \to 0$ as $m \to \infty$ we have $R_m \to \infty$ such that (2.11) holds. From Claims 3 and 4 and (2.12) above, there exists $n_m \to \infty$ such that

$$B_{2R_m}(y_{n_m}) \subset \lambda_{n_m}^{\frac{1}{2}} \left[(A_{2\sigma_0} \setminus A_\sigma) \cap \{x \mid |P_l^\perp x| \leq \sigma\}\right]$$

where $V(\lambda_{n_m}^{-\frac{1}{2}} x) \geq 1 + a_\sigma$, and

$$\int_{\bigcup_{y\in G(y_{n_m})} B_{2R_m}(y)} |u_{n_m}|^{p+1} \geq 1 - \epsilon_m.$$

For simplicity of notations, we call this subsequence $\{u_{n_m}\}$ as u_m and call y_{n_m} as y_m. Define

$$w_m(x) = \sum_{y\in G(y_m)} \eta(\frac{|x-y|}{R_m}) u_m(x), \quad h_m(x) = \prod_{y\in G(y_m)} \eta^c(\frac{|x-y|}{R_m}) u_m(x).$$

Then

$$\begin{aligned}
\int_{\mathbf{R}^N} |\nabla v_m|^2 + \lambda_m V(x) v_m^2 &= \int_{\mathbf{R}^N} |\nabla u_m|^2 + V(\lambda_m^{-\frac{1}{2}} x) u_m^2 \\
&\geq \int_{\mathbf{R}^N} |\nabla w_m|^2 + V(\lambda_m^{-\frac{1}{2}} x) w_m^2 + \int_{\mathbf{R}^N} |\nabla h_m|^2 + V(\lambda_m^{-\frac{1}{2}} x) h_m^2 - \frac{C_0}{R_m} \\
&\geq a_\sigma \int_{\mathbf{R}^N} w_m^2 + k^{\frac{p-1}{p+1}} S_p (1-\epsilon_m)^{\frac{2}{p+1}} - \frac{C_0}{R_m}
\end{aligned}$$

Sending m to infinity, we get

$$k^{\frac{p-1}{p+1}} S_p \geq a_\sigma \lim_{m\to\infty} \int_{\mathbf{R}^N} w_m^2 + k^{\frac{p-1}{p+1}} S_p,$$

a contradiction for $\lim_{m\to\infty} \int_{\mathbf{R}^N} w_m^2 \neq 0$ (otherwise by Lemma 4.3 $\lim_{m\to\infty} \int_{\mathbf{R}^N} w_m^{p+1} = 0$). Thus Claim 5 is proved.

With Claim 5, we may take a sequence $\sigma_m \to 0$ as $m \to \infty$, and then there exist n_m such that

$$\lambda_{n_m}^{-\frac{1}{2}} C_{\sigma_m} \to 0.$$

Thus we may assume $y_m := y_{n_m}$ to satisfy that $y_m \in \lambda_m^{\frac{1}{2}} A_{\sigma_m}$ besides all other properties stated earlier.

Now, rescaling this subsequence back to v_m and calling $x_m = \lambda_m^{-\frac{1}{2}} y_m$, we get $x_m \in \mathbf{R}^l \times \{0\}$, $\#G(x_m) = k$, $x_m \in A_{\sigma_m}$ such that for any $\epsilon > 0$ there is $R > 0$

$$\int_{\bigcup_{z \in G(x_m)} B_{\frac{R}{\sqrt{\lambda_m}}}(z)} |v_m|^{p+1} \geq 1 - \epsilon.$$

$x_m \in A_{\sigma_m}$ implies $|x_m| \to 1$. □

Using Proposition 2.1, we have the following.

Proposition 2.2. For any $\epsilon > 0$ there exist $R = R(\epsilon) > 0$ and $\lambda(\epsilon) > 0$, such that for all $\lambda > \lambda(\epsilon) > 0$, whenever $\{v_n\}$ is a minimizing sequence of E_λ in O_δ for $c_{\lambda,\delta}$ then

$$\liminf_{n\to\infty} \int_{\{x \mid \lambda^{\frac{1}{2}}(1-\frac{\delta}{4}) \leq |P_l x| \leq \lambda^{\frac{1}{2}}(1+\frac{\delta}{4})\} \cap \{x \mid |P_l^\perp x| < R\}} |u_n|^{p+1} \geq (1-\epsilon), \tag{2.13}$$

where $u_n(x) = \lambda^{-\frac{N}{2(p+1)}} v_n(\lambda^{-\frac{1}{2}} x)$ and $P_l^\perp = Id_{R^N} - P_l$.

Proof. If this is not the case, there exist $\epsilon_0 > 0$, $R_m \to \infty$, $\lambda_m \to \infty$ and minimizing sequences $\{v_{m,n}\}_{n=1}^\infty$ of E_{λ_m} in O_δ such that when writing $u_{m,n}(x) = \lambda_m^{-\frac{N}{2(p+1)}} v_{m,n}(\lambda_m^{-\frac{1}{2}} x)$

$$\liminf_{n\to\infty} \int_{\{x \mid \lambda_m^{\frac{1}{2}}(1-\frac{\delta}{4}) \leq |P_l x| \leq \lambda_m^{\frac{1}{2}}(1+\frac{\delta}{4})\} \cap \{x \mid |P_l^\perp x| < R_m\}} |u_{m,n}|^{p+1} \leq 1 - \epsilon_0.$$

Then by Prop. 2.1 (a) there exist $n_m \to \infty$ as $m \to \infty$ such that

$$\liminf_{m\to\infty} \int_{\{x \mid \lambda_m^{\frac{1}{2}}(1-\frac{\delta}{4}) \leq |P_l x| \leq \lambda_m^{\frac{1}{2}}(1+\frac{\delta}{4})\} \cap \{x \mid |P_l^\perp x| < R_m\}} |u_{m,n_m}|^{p+1} \leq 1 - \epsilon_0, \tag{2.14}$$

and

$$\lim_{m\to\infty} \lambda_m^{-\beta_0} E_{\lambda_m}(v_{m,n_m}) = k^{\frac{p-1}{p+1}} S_p.$$

Call this subsequence u_m. Applying Proposition 2.1 to the corresponding v_m there exist $x_m \in \mathbf{R}^l \times \{0\}$ with $|x_m| \to 1$ and $\#G(x_m) = k$ such that for $\epsilon_0 > 0$ above there exists $R_0 > 0$

$$\int_{B_{\frac{R_0}{\sqrt{\lambda_m}}}(x_m)} |v_m|^{p+1} \geq \frac{1}{k} - \frac{\epsilon_0}{2k}.$$

Calling $y_m = \lambda_m^{\frac{1}{2}} x_m$ we have $y_m \in \mathbf{R}^l \times \{0\}$ and $\#G(y_m) = k$ and

$$\lim_{m\to\infty} \int_{\bigcup_{y \in G(y_m)} B_{R_0}(y)} |u_m|^{p+1} \geq 1 - \frac{\epsilon_0}{2} > 1 - \epsilon_0.$$

Since $|x_m| \to 1$ as $m \to \infty$, we may assume $y_m \in \{x \mid \lambda_m^{\frac{1}{2}}(1-\frac{\delta}{8}) \le |P_l x| \le \lambda_m^{\frac{1}{2}}(1+\frac{\delta}{8})\}$. Hence we have a contradiction to (2.14) because for m large

$$\bigcup_{y \in G(y_m)} B_{R_0}(y) \subset \{x \mid \lambda_m^{\frac{1}{2}}(1-\frac{\delta}{4}) \le |P_l x| \le \lambda_m^{\frac{1}{2}}(1+\frac{\delta}{4})\} \cap \{x \mid |P_l^{\perp} x| < R_m\}.$$

□

2.2. *Existence and Qualitative Properties of Solutions*

Proposition 2.3. There is $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$, if $\{v_n\} \in O_\delta$ is a minimizing sequence of E_λ for $c_{\lambda,\delta}$ then there exists $v \in O_\delta$, $v_n \to v$ in $L^{p+1}(\mathbf{R}^N)$ as $n \to \infty$. In fact, $c_{\lambda,\delta}$ is achieved at *an interior point* in O_δ.

Proof. First from Proposition 2.1, there is $\lambda_1 > 0$ such that for $\lambda > \lambda_1$, $\lambda^{-\beta_0} c_{\lambda,\delta} \le (2k)^{\frac{p-1}{p+1}} S_p$. Take $0 < \epsilon_0 < \frac{\delta}{4}$. Then there is $\lambda_2 > 0$ such that for $\lambda > \lambda_2$ Proposition 2.2 holds with $\epsilon = \epsilon_0$. Let $\lambda' = \max\{\lambda_1, \lambda_2\}$. For a fixed $\lambda > \lambda'$ and a minimizing sequence $\{v_n\} \in O_\delta$ of E_λ for $c_{\lambda,\delta}$, we shall first show that $u_n(x) = \lambda^{-\frac{N}{2(p+1)}} v_n(\lambda^{-\frac{1}{2}} x)$ converges to some u in $L^{p+1}(\mathbf{R}^N)$.

Let $R_{m+1} = 2R_m \to \infty$ as $m \to \infty$. Define up to a subsequence

$$\lim_{n\to\infty} \int_{B_{R_m}(0)} |u_n|^{p+1} = b_m.$$

It suffices to show $\lim_{m\to\infty} b_m = 1$, which implies that $u_n \to u$ in $L^{p+1}(R^N)$ for some $u \in L^{p+1}(R^N)$. Assume $\lim_{m\to\infty} b_m = b < 1$. Then by Proposition 2.2, $b \ge 1 - \epsilon_0$. We choose $n_m \to \infty$ such that

$$\left| b_m - \int_{B_{R_m}(0)} |u_{n_m}|^{p+1} \right| \le \frac{1}{m} \quad \text{and} \quad \left| b_{m+1} - \int_{B_{R_{m+1}}(0)} |u_{n_m}|^{p+1} \right| \le \frac{1}{m}.$$

Define

$$w_m(x) = \eta\left(\frac{|x|}{R_m}\right) u_{n_m}(x), \quad \text{and} \quad h_m(x) = \eta^c\left(\frac{|x|}{R_m}\right) u_{n_m}(x).$$

Here η and η^c are as used in Remark 2.2. Then

$$\lim_{m\to\infty} \int_{\mathbf{R}^N} |w_m|^{p+1} = b, \qquad \lim_{m\to\infty} \int_{\mathbf{R}^N} |h_m|^{p+1} = 1 - b.$$

Rescaling w_m back to $\tilde{w}_m(x) = \lambda^{\frac{N}{2(p+1)}} w_m(\lambda^{\frac{1}{2}} x)$, we have

Claim. For m large, $\frac{\tilde{w}_m}{|\tilde{w}_m|_{p+1}} \in O_\delta$.

Before proving this claim we first finish the proof of $b = 1$. In fact,

$$
\begin{aligned}
&\lambda^{-\beta_0} c_{\lambda,\delta} \\
=&\lim_{m\to\infty} \int_{\mathbf{R}^N} |\nabla u_{n_m}|^2 + V(\lambda^{-\frac{1}{2}}x)u_{n_m}^2 \\
\geq&\limsup_{m\to\infty} \left(\int_{\mathbf{R}^N} |\nabla w_m|^2 + V(\lambda^{-\frac{1}{2}}x)w_m^2 + \int_{\mathbf{R}^N} |\nabla h_m|^2 + V(\lambda^{-\frac{1}{2}}x)h_m^2 - \frac{C_0}{R_m} \right) \\
=&\limsup_{m\to\infty} \left(\frac{\int_{\mathbf{R}^N} |\nabla w_m|^2 + V(\lambda^{-\frac{1}{2}}x)w_m^2}{|w_m|_{p+1}^2} |w_m|_{p+1}^2 + \int_{\mathbf{R}^N} |\nabla h_m|^2 + V(\lambda^{-\frac{1}{2}}x)h_m^2 - \frac{C_0}{R_m} \right) \\
=&\limsup_{m\to\infty} \left(\lambda^{-\beta_0} \frac{\int_{\mathbf{R}^N} |\nabla \tilde{w}_m|^2 + \lambda V(x)\tilde{w}_m^2}{|\tilde{w}_m|_{p+1}^2} |w_m|_{p+1}^2 + \int_{\mathbf{R}^N} |\nabla h_m|^2 + V(\lambda^{-\frac{1}{2}}x)h_m^2 - \frac{C_0}{R_m} \right) \\
\geq&\lambda^{-\beta_0} c_{\lambda,\delta} b^{\frac{2}{p+1}} + S_p(1-b)^{\frac{2}{p+1}}.
\end{aligned}
$$

Note we have used the claim in the last step of the above estimate. Thus we have

$$
\frac{\lambda^{-\beta_0} c_{\lambda,\delta}}{S_p} \geq \frac{\lambda^{-\beta_0} c_{\lambda,\delta}}{S_p} b^{\frac{2}{p+1}} + (1-b)^{\frac{2}{p+1}}.
$$

By Lemma 4.4, we have $b = 1$ because $1 - b < \epsilon_0 < \delta_k$ and $\frac{\lambda^{-\beta_0} c_{\lambda,\delta}}{S_p} \leq (2k)^{\frac{p-1}{p+1}}$. Hence using the claim, we have proved $b = 1$.

To prove the claim it suffices to show that

$$
\begin{aligned}
\frac{1}{d} - \delta &< \liminf_{m\to\infty} \int_{\mathbf{R}^N} \left| \frac{w_m}{|w_m|_{p+1}} \right|^{p+1} \xi(\lambda^{-\frac{1}{2}}|P_l x|)dx \\
&\leq \limsup_{m\to\infty} \int_{\mathbf{R}^N} \left| \frac{w_m}{|w_m|_{p+1}} \right|^{p+1} \xi(\lambda^{-\frac{1}{2}}|P_l x|)dx < \frac{1}{d} + \delta.
\end{aligned}
$$

Indeed, by Proposition 2.2, we have for ϵ_0 chosen above there exists $R_0 > 0$ such that (2.13) holds for our subsequence $u_m = u_{n_m}$ here. Then because $\epsilon_0 < \frac{\delta}{4}$ and $b \geq 1 - \epsilon_0$, using (2.13) one has

$$
\begin{aligned}
&\liminf_{m\to\infty} \int_{\mathbf{R}^N} \left| \frac{w_m}{|w_m|_{p+1}} \right|^{p+1} \xi(\lambda^{-\frac{1}{2}}|P_l x|)dx \\
\geq&\liminf_{m\to\infty} \frac{1}{|w_m|_{p+1}^{p+1}} \int_{\{\lambda^{\frac{1}{2}}(1-\frac{\delta}{4}) \leq |P_l x| \leq \lambda^{\frac{1}{2}}(1+\frac{\delta}{4})\} \cap \{|P_l^\perp x| \leq R_0\}} |u_m|^{p+1} \frac{\lambda^{-\frac{1}{2}}|P_l x|}{d} dx \\
\geq&\frac{1}{b} \cdot \frac{1}{d}(1 - \frac{\delta}{4})(1 - \epsilon_0) \\
>&\frac{1}{d} - \delta.
\end{aligned}
$$

Similarly, using (2.13)

$$
\begin{aligned}
&\limsup_{m\to\infty}\int_{\mathbf{R}^N}\left|\frac{w_m}{|w_m|_{p+1}}\right|^{p+1}\xi(\lambda^{-\frac{1}{2}}|P_lx|)dx\\
\le&\limsup_{m\to\infty}\frac{1}{|w_m|_{p+1}^{p+1}}\left(\int_{\{|P_lx|\le\lambda^{\frac{1}{2}}(1+\frac{\delta}{4})\}}|u_m|^{p+1}\frac{\lambda^{-\frac{1}{2}}|P_lx|}{d}dx+\int_{\{|P_lx|\ge\lambda^{\frac{1}{2}}(1+\frac{\delta}{4})\}}|u_m|^{p+1}dx\right)\\
\le&\frac{1}{b}\cdot\frac{1}{d}\left(1+\frac{\delta}{4}+d\epsilon_0\right)\\
<&\frac{1}{d}+\delta.
\end{aligned}
$$

Thus rescaling u_n back to v_n we have proved that there is $v_\lambda\in L^{p+1}(\mathbf{R}^N)$ such that $v_n\to v_\lambda$ in $L^{p+1}(\mathbf{R}^N)$. Obviously we have $v_\lambda\in\overline{O_\delta}$ since $\gamma(v)$ is continuous. We may also assume v_n converges weakly to v_λ in F. Because E_λ is weakly lower semi-continuous in F we get $E_\lambda(v_\lambda)\le c_{\lambda,\delta}$, i.e., $E_\lambda(v_\lambda)=c_{\lambda,\delta}$. By Proposition 2.1 again (and Remark 2.1) there exists $\lambda_0\ge\lambda'$ such that for $\lambda>\lambda_0$, v_λ is an interior point of O_δ. □

Proof of Theorem 1.1.(Part I) By Proposition 2.3, for $\lambda>\lambda_0$, $c_{\lambda,\delta}$ is achieved in O_δ by some w_λ (an interior point). Then $v_\lambda(x)=[E_\lambda(w_\lambda)]^{\frac{1}{p-1}}w_\lambda(x)$ solves $(\mathcal{P}_\lambda)$. We may assume w_λ to be nonnegative (for they are local minimizers and $E_\lambda(u)=E_\lambda(|u|)$) and therefore by the maximum principle we get that v_λ are solutions of $(\mathcal{P}_\lambda)$. We shall show that for λ large v_λ possess the properties stated in Theorem 1.1. Assertion (1°) is satisfied because $w_\lambda\in M^G$. To see assertion (2°), we apply Proposition 2.1 to get

$$\lim_{\lambda\to\infty}\lambda^{-\beta_0}c_{\lambda,\delta}=k^{\frac{p-1}{p+1}}S_p.$$

A direct computation shows

$$\lim_{\lambda\to\infty}\lambda^{\frac{-\beta_0(p+1)}{p-1}}\int_{\mathbf{R}^N}|\nabla v_\lambda|^2+\lambda V(x)v_\lambda^2=kS_p^{\frac{p+1}{p-1}}.$$

To show assertions (3°) and (4°), we shall use some ideas in [NT1-2] [Wz1] in which peakedness of solutions in bounded domains are considered. Let us consider a sequence $\lambda_n\to\infty$ as $n\to\infty$ and $u_n(x)=\lambda_n^{-\frac{1}{p-1}}v_n(\lambda_n^{-\frac{1}{2}}x)$. Then by $(\mathcal{P}_\lambda)$, $\lim_{n\to\infty}|u_n|_{p+1}^{p+1}=kS_p^{\frac{p+1}{p-1}}$. By Proposition 2.1, there exist $y_n\in\mathbf{R}^l\times\{0\}$ satisfying $\#G(y_n)=k$ and $\lim_{n\to\infty}\lambda_n^{-\frac{1}{2}}|y_n|=1$ such that for any $\epsilon>0$ there exists $R>0$

$$\int_{\bigcup_{y\in G(y_n)}B_R(y)}|u_n|^{p+1}\ge kS_p^{\frac{p+1}{p-1}}-\epsilon.$$

Taking $\epsilon_m \to 0$, we have $R_m \to \infty$ and a subsequence $n_m \to \infty$ as $m \to \infty$ such that

$$B_{R_m}(y') \cap B_{R_m}(y'') = \emptyset, \quad \forall\, y', y'' \in G(y_{n_m}), \quad y' \neq y'', \tag{2.15}$$

and

$$\int_{\bigcup_{y \in G(y_{n_m})} B_{R_m}(y)} |u_{n_m}|^{p+1} \geq k S_p^{\frac{p+1}{p-1}} - \epsilon_m. \tag{2.16}$$

Therefore

$$\lim_{m \to \infty} \int_{\mathbf{R}^N \setminus \bigcup_{y \in G(y_{n_m})} B_{R_m}(y)} |u_{n_m}|^{p+1} = 0. \tag{2.17}$$

We call u_{n_m} to be u_m for simplicity. Note that u_m are solutions of the following problem

$$-\Delta u_m + V(\lambda_m^{-\frac{1}{2}} x) u_m = u_m^p, u_m > 0, \quad \text{in } \mathbf{R}^N. \tag{2.18}$$

Let x_m be a local maximum point of u_m. By the maximum principle and equation $(\mathcal{P}_\lambda)$ we have $u_m(x_m) > 1 = \min_{x \in R^N} V(x)$. By the elliptic theory ([GT]) and (2.17) we conclude that there exist $m_0 > 0$ and $R_0 > 0$ such that for $m > m_0$, u_m must achieve any local maximum point inside $\bigcup_{y \in G(y_m)} B_{R_0}(y)$. I.e., if x_m is a local maximum point of u_m it holds

$$\operatorname{dist}(x_m, G(y_m)) \leq R_0.$$

We want to show that u_m has exactly k local maximum points in $\bigcup_{y \in G(y_m)} B_{R_0}(y)$. It suffices to show the following claim.

Claim. u_m has a unique local maximum point in $B_{R_0}(y_m)$.

Now suppose there are two local maximum points of u_m in $B_{R_0}(y_m)$, say, at $\overline{x}_m$ and $\tilde{x}_m$. Consider

$$h_m(x) = u_m(x + \overline{x}_m) \quad \text{for } x \in B_{2R_m}(0),$$

where $R_m \to \infty$ are from (2.15). Then by the elliptic theory ([GT]) we have up to a subsequence

$$h_m \to h \ \text{ in } C^2_{loc}(\mathbf{R}^N) \tag{2.19}$$

and by (2.18)

$$-\Delta h_m + V(\lambda_m^{-\frac{1}{2}}(x + \overline{x}_m)) h_m = h_m^p \quad \text{in} \quad B_{2R_m}(0).$$

Since for m large $B_{R_m}(y_m) \subset B_{2R_m}(\overline{x}_m)$, and by (2.16), $h \neq 0$. Sending $m \to \infty$, we get

$$-\Delta h + V(y_0) h = h^p \quad \text{in} \quad \mathbf{R}^N,$$

where $\lim_{m\to\infty} \lambda_m^{-\frac{1}{2}} \overline{x}_m = y_0 \in \mathbf{R}^l \times \{0\}$ with $|y_0| = 1$. We also get by (2.15) and (2.19)

$$
\begin{aligned}
(2.20)\qquad & \int_{\mathbf{R}^N} |\nabla h|^2 + h^2 \\
\leq & \liminf_{m\to\infty} \int_{B_{R_m}(0)} |\nabla h_m|^2 + h_m^2 \\
\leq & \limsup_{m\to\infty} \int_{B_{R_m}(0)} |\nabla h_m|^2 + V(\lambda_m^{-\frac{1}{2}}(x + \overline{x}_m)) h_m^2 \\
= & \limsup_{m\to\infty} \int_{B_{R_m}(\overline{x}_m)} |\nabla u_m|^2 + V(\lambda_m^{-\frac{1}{2}} x) u_m^2 \\
\leq & \frac{1}{k} \limsup_{m\to\infty} \int_{\mathbf{R}^N} |\nabla u_m|^2 + V(\lambda_m^{-\frac{1}{2}} x) u_m^2 \\
= & S_p^{\frac{p+1}{p-1}}.
\end{aligned}
$$

I.e., $h \in H^1(\mathbf{R}^N)$. Comparing with (1.3) and (1.4) we see from (2.20) that h is a ground state solution of (1.3). By the uniqueness result of [K] [KZ] there exists $z \in \mathbf{R}^N$ such that $h(x) = w(x+z)$, where w is the ground state solution in (1.3). Therefore $\int_{\mathbf{R}^N} |\nabla h|^2 + h^2 = S_p^{\frac{p+1}{p-1}}$. Since $\nabla h_m(0) = 0$ for all m, we get $\nabla w(0+z) = 0$ which implies $z = 0$ for w has only one critical point which is 0. Now there are two possibilities: (a) $|\tilde{x}_m - \overline{x}_m| \geq a_0 > 0$, as $m \to \infty$ for some a_0. (b) $|\tilde{x}_m - \overline{x}_m| \to 0$ as $m \to \infty$.

For case (a), we assume $\tilde{x}_m - \overline{x}_m \to x_0 \neq 0$ as $m \to \infty$. We have

$$\nabla h_m(\tilde{x}_m - \overline{x}_m) = \nabla u_m(\tilde{x}_m - \overline{x}_m + \overline{x}_m) = 0.$$

By (2.19) this implies $\nabla w(x_0) = 0$, a contradiction. In case (b), we can find z_m which lies on the line segment joining 0 and $\tilde{x}_m - \overline{x}_m$ such that $\frac{\partial^2}{\partial \nu_m^2} h_m(z_m) \geq 0$ (there is a local minimum point along that direction), where $\nu_m = \frac{\tilde{x}_m - \overline{x}_m}{|\tilde{x}_m - \overline{x}_m|}$. Sending $m \to \infty$, up to a subsequence we have $\nu_m \to \nu_0$(a unit vector), $z_m \to 0$, such that $\frac{\partial^2}{\partial \nu_0^2} w(0) \geq 0$, a contradiction again because for any unit vector ν_0, $\frac{\partial^2}{\partial \nu_0^2} w(0) < 0$.

Thus we have proved the claim, i.e., u_m achieves its local maximum points only at k points: $G(x_m)$, which are contained in $\bigcup_{y \in G(y_m)} B_{R_0}(y)$. Calling $P_m = \lambda_m^{-\frac{1}{2}} x_m$, we get $|P_m| \to 1$ as $m \to \infty$. (3°) is proved now. Finally, (2.19) and (2.20), together with the fact that $\lim_{m\to\infty} \int_{\mathbf{R}^N} |\nabla u_m|^2 + V(\lambda_m^{-\frac{1}{2}} x) u_m^2 = k S_p^{\frac{p+1}{p-1}}$, show that

$$\lim_{m\to\infty} \int_{\mathbf{R}^N} \left| \nabla \left(u_m - \sum_{y \in G(x_m)} w(x-y) \right) \right|^2 + V(\lambda_m^{-\frac{1}{2}} x) \left(u_m - \sum_{y \in G(x_m)} w(x-y) \right)^2 = 0.$$

This proves (4°). □

2.3. *Symmetry Properties of Solutions*

The solutions of $(\mathcal{P}_\lambda)$ given in Theorem 1.1 are constructed in G-invariant subspaces and therefore they are at least G-invariant. A question we shall address in this subsection is: what are the exact symmetry of these solutions? More generally we may ask the question whether we can find solutions of $(\mathcal{P}_\lambda)$ with prescribed symmetry? Partial answers are given by assertion (5°) of Theorem 1.1.

We denote the isotropy subgroup of u by Σ_u, i.e.,

$$\Sigma_u := \{g \in \mathbf{O}(N) \mid u(g^{-1}x) = u(x), a.e., \text{in } \mathbf{R}^N\}. \tag{2.21}$$

Proof of Theorem 1.1.(Part II) We know by assertion (1°) of Theorem 1.1, $G \subset \Sigma_{v_\lambda}$. We want to show $G = \Sigma_{v_\lambda}$. Assume this is not true, i.e., Σ_{v_λ} is a nontrivial supergroup of G. Then since v_λ is Σ_{v_λ} invariant, the set of maximum points is Σ_{v_λ}-invariant as well. By assertion (3°) of Theorem 1.1, this set is given by $G(P_\lambda)$ and hence $\Sigma_{v_\lambda}(P_\lambda)$ is also a set of k points. But this is impossible by **(S2)** for Σ_{v_λ} is a nontrivial supergroup of G. □

While **(S2)** involves in checking G-orbits of some points on $\mathbf{R}^l \times \{0\}$ with $|x| = 1$, the following assumption **(S3)** is easier to be checked because it involves in only checking some G_1-orbits.

(S3). Let $G = G_1 \times G_2$ satisfy **(S1)**. Assume further

(a) For all $x \in \mathbf{R}^l \times \{0\}$ with $|x| = 1$ and $\#G_1(x) = k$ it holds

$(a)_i$ $\text{span}\{G_1(x)\} = \mathbf{R}^l \times \{0\}$ and

$(a)_{ii}$ $\#\tilde{G}_1(x) > k$ for all nontrivial supergroups $\tilde{G}_1$ of G_1 in $\mathbf{O}(l)$ (i.e., $G_1 \subset \tilde{G}_1 \subset \mathbf{O}(l), \tilde{G}_1 \neq G_1$).

(b) $G_2 = \mathbf{O}(N - l)$.

Proposition 2.4. **(S3)** implies **(S2)**.

Remark 2.3. We refer readers to [MSW] for a proof of this implication. Though **(S3)** is stronger than **(S2)** we shall see there are a lot of interesting examples where **(S3)** is satisfied.

Next we shall present some examples that satisfy **(S3)**. We shall denote the tetrahedral, octahedral and icosahedral subgroups of $\mathbf{O}(3)$ by $\mathbf{T}$, $\mathbf{O}$ and $\mathbf{I}$, respectively. Furthermore, $\mathbf{Z}_2(l) = \{Id, -Id\}$ are the reflection subgroups in $\mathbf{O}(l)$, $\mathbf{D}_k$ denote the dihedral subgroups of $\mathbf{O}(2)$, for $l \geq 3$, $\mathbf{B}_l$ denote the l-dimensional hypercube subgroups in $\mathbf{O}(l)$, hence $\mathbf{B}_3 = \mathbf{O} \oplus \mathbf{Z}_2(3)$, and $\mathbf{T}_l$ denote the l-dimensional hyper-tetrahedron groups in $\mathbf{O}(l)$.

Example 2.1. (Exceptional subgroups of $\mathbf{O}(3)$). Let $l = 3$. Consider $G_1 = \mathbf{B}_3$ the symmetry group of the cube in $\mathbf{R}^3$ (including reflections) and $G_2 = \mathbf{O}(N - 3)$. It can be verified that $k = 6$ is the length of the minimal G_1-orbit, $G_1 e_1$, $e_1 = (1, 0, 0)$, which is unique and spans $\mathbf{R}^3$.

We next use the icosahedral group. Consider $G_1 = \mathbf{I} \oplus \mathbf{Z}_2(3)$ and $G_2 = \mathbf{O}(N-3)$ with $l = 3$. Here the minimal G_1-orbit has length $k = 12$ (vertices of the icosahedron). It is also unique and spans $\mathbf{R}^3$.

The last exceptional subgroup of $\mathbf{O}(3)$ is the tetrahedral subgroup. Let $G_1 = \mathbf{T}$ and $G_2 = \mathbf{O}(N-3)$ with $l = 3$. There are two different minimal orbits of length $k = 4$.

Example 2.2. (Cube symmetry). For any $l \geq 3$, we consider $G_1 = \mathbf{B}_l$ and $G_2 = \mathbf{O}(N-l)$. The minimal $\mathbf{B}_l$-orbit contains $k = 2l$ elements and is unique (orbit of the midpoints of the faces of the cube).

Example 2.3. (Regular simplex symmetry). For any $l \geq 3$, we consider $G_1 = \mathbf{T}_l$ which leaves invariant the l-dimensional regular simplex (tetrahedron) and $G_2 = \mathbf{O}(N-l)$. There are two minimal $\mathbf{T}_l$-orbits which contain $k = l + 1$ elements each. Here, **(S3)** is satisfied as verified in [CW].

Example 2.4. (Reflection symmetry). Let $l = 1$ and $G_1 = \mathbf{Z}_2(1)$ and $G_2 = \mathbf{O}(N-1)$. Here, **(S3)** is satisfied with $k = 2$.

To close the list of examples we consider the dihedral groups acting naturally on $\mathbf{R}^2$.

Example 2.5. (Dihedral symmetry) Set $l = 2$ and $G_1 = \mathbf{D}_k$ for $k \geq 3$ and $G_2 = \mathbf{O}(N-2)$. Here the minimal orbit length is k (with two minimal orbits for odd k and a unique minimal orbit for even k).

Proof of Corollary 1.1. Summarizing the examples above and applying Theorem 1.1, we then have proved Corollary 1.1 which gives a list of isotropy subgroups which occur for solutions u_λ and the number of their bumps. □

Remark 2.4. Assumption **(S2)** gives a sufficient condition to obtain solutions having *exact symmetry*. An interesting question is whether **(S2)** is also necessary for solutions (at least for local minimizers) to have exact symmetry. Is one symmetry more preferable to another for local minimizers to take? Evidences exist indicating **(S2)** might be necessary for solutions as local minimizers carrying discrete bumps. It seems that the powerful moving plane method ([GNN]) and the Schwartz symmetrization method would not apply here.

Remark 2.5. We finish with an example in which **(S2)** is not satisfied. Consider $N = 2$ and $G_1 = G = \mathbf{Z}_k$ with the $\frac{2\pi}{k}$ rotations. Then it is easy to see (**S2**) is not satisfied. In this case we can still apply Theorem 1.1 to get a solution satisfying (1°) – (4°). However, we believe the symmetry group of this solution is $\mathbf{D}_k$, which is larger than $\mathbf{Z}_k$.

3. Extensions of the Main Results

Minor modifications would enable us to treat more general problems where V needs not to be radially symmetric. Let $V \in C^1(\mathbf{R}^N, \mathbf{R})$ and let us define the isotropy subgroup of V:

$$\Sigma_V := \{g \in \mathbf{O}(N) \mid V(g^{-1}x) = V(x), \text{ for all } x \in \mathbf{R}^N\}.$$

With the decomposition (1.1) we define

$$\Sigma_V^{(1)} := \{g \in \mathbf{O}(l) \mid V(g^{-1}x) = V(x), \text{ for all } x \in \mathbf{R}^N\},$$

and

$$\Sigma_V^{(2)} := \{g \in \mathbf{O}(N-l) \mid V(g^{-1}x) = V(x), \text{ for all } x \in \mathbf{R}^N\}.$$

It is obvious that when V is radially symmetric $\Sigma_V = \mathbf{O}(N)$, $\Sigma_V^{(1)} = \mathbf{O}(l)$ and $\Sigma_V^{(2)} = \mathbf{O}(N-l)$.

We assume

(S1') Let $G \subset \Sigma_V$ and $G = G_1 \times G_2$ with G_1 and G_2 satisfying **(S1)**, i.e., $\mathrm{Fix}_{G_2} = \mathbf{R}^l \times \{0\}$, and

$$k := \min\{\#G(x) \mid x \in \mathbf{R}^l \times \{0\}, |x| = 1\} < \infty.$$

(V1') Suppose $\inf_{\mathbf{R}^N} V(x) = 1$, and suppose that V achieves its infimum at a point $x_0 \in \mathbf{R}^l \times \{0\}$ with $|x_0| = 1$ and $\#G(x_0) = k$, and that there exists $\sigma_0 > 0$ such that $V(x) > 1$ for all $x \in N_{2\sigma_0}(\Sigma_V(x_0)) \setminus \Sigma_V(x_0)$, here $N_{2\sigma_0}(\cdot)$ is a $2\sigma_0$ neighborhood.

(S2') For all $x \in \mathbf{R}^l \times \{0\}$ with $|x| = 1$ and $\#G(x) = k$ and all nontrivial supergroups $\tilde{G}$ of G in Σ_V (i.e., $G \subset \tilde{G} \subset \Sigma_V$, but $G \neq \tilde{G}$), $\#\tilde{G}(x) > k$.

Theorem 3.1. *Suppose* **(S1')** *and* **(V1')** *hold. Then there exists $\lambda_0 > 0$ such that for $\lambda > \lambda_0$, ($\mathcal{P}_\lambda$) possesses a solution v_λ which satisfies assertions (1°), (2°), (3°) and (4°) of Theorem 1.1, and*

(5°). If in addition to **(S1')** *and* **(V1')**, **(S2')** *holds, then v_λ is also at most G-invariant, i.e., v_λ has exact G symmetry.*

We may also state a corresponding assumption to **(S3)** which implies **(S2')**.

(S3') Let $G = G_1 \times G_2$ satisfy **(S1')**. Assume further

(a) For all $x \in \mathbf{R}^l \times \{0\}$ with $|x| = 1$ and $\#G_1(x) = k$ it holds

$(a)_i$ $\mathrm{span}\{G_1(x)\} = \mathbf{R}^l \times \{0\}$ and

$(a)_{ii}$ $\#\tilde{G}_1(x) > k$ for all nontrivial supergroups $\tilde{G}_1$ of G_1 in $\Sigma_V^{(1)}$ (i.e., $G_1 \subset \tilde{G}_1 \subset \Sigma_V^{(1)}, \tilde{G}_1 \neq G_1$).

(b) $G_2 = \Sigma_V^{(2)}$.

4. Some technical results

In order to prove Proposition 2.1, we need a Concentration-Compactness Lemma due to P.L.Lions ([Lp1], [Sm]). In fact, we need a more detailed version of this lemma ([Lp2]). We shall reformulate it as Lemma 4.1 and give it a different and self-contained proof. Then using this lemma we give in subsection 4.2 another technical result which may be of independent interest and which was used in the proof of Proposition 2.1.

4.1. *A Concentration-Compactness Lemma*

Let μ_n be a sequence of nonnegative measures on $\mathbf{R}^N$. For $s \in \mathbf{N}$ and $r > 0$ we define a family of concentration functions:

$$Q_{n,s}(r) := \sup_{y_i \in \mathbf{R}^N} \int_{\bigcup_{i=1}^{s} B_r(y_i)} \mu_n. \tag{4.1}$$

Lemma 4.1. Suppose that μ_n is a sequence of measures on $\mathbf{R}^N$: $\mu_n \geq 0$, $\int_{\mathbf{R}^N} \mu_n dx = 1$. Then there exist a subsequence of μ_n (still denoted by μ_n) such that $\lim_{n\to\infty} Q_{n,s}(m)$ exists for all $s \in \mathbf{N}$ and $m \in \mathbf{N}$ and such that

(i). $\{\alpha_i\}_1^\infty$, defined by

$$\alpha_1 = \lim_{m\to\infty} \lim_{n\to\infty} Q_{n,1}(m) \text{ and for } s \geq 2,\ \ \alpha_s = \lim_{m\to\infty} \lim_{n\to\infty} Q_{n,s}(m) - \sum_{i=1}^{s-1} \alpha_i, \tag{4.2}$$

is a nonnegative, nonincreasing sequence satisfying

$$\lim_{s\to\infty} \sum_{i=1}^{s} \alpha_i \leq 1. \tag{4.3}$$

(ii). For each $\alpha_i > 0$ there is an associated sequence $\{x_{n,i}\} \subset \mathbf{R}^N$ satisfying that

$$\liminf_{n\to\infty} |x_{n,i} - x_{n,j}| \to \infty, \quad n \to \infty, \quad \forall\, i \neq j, \tag{4.4}$$

such that given $s \geq 1$ with $\alpha_s > 0$, for any $\epsilon > 0$ there exists $R > 0$ such that for all $r \geq R$ and all $r' \geq R$

$$\limsup_{n\to\infty} \sum_{i=1}^{s} \left| \alpha_i - \int_{B_r(x_{n,i})} \mu_n \right| + \left| (1 - \sum_{i=1}^{s} \alpha_i) - \int_{\mathbf{R}^N \setminus \bigcup_{i=1}^{s} B_{r'}(x_{n,i})} \mu_n \right| < \epsilon. \tag{4.5}$$

Remark 4.1. Lemma 4.1 is a generalization of P.L. Lions Concentration Compactness Lemma in [Lp1] (see also [Sm]) and is a reformulation of a result in [Lp2]. Here we give the precise definition of those $\{\alpha_i\}_{i=1}^\infty$ and we give a self-contained proof below which is different from [Lp2]. Note that $\alpha_1 = 1$ corresponds to the compactness in [Lp1] and $\alpha_1 = 0$ corresponds to the vanishing there.

Proof. The proof is divided into four steps.

Step A. First for $Q_{n,s}(r)$ in (4.1) using a diagonal process of extracting subsequences we may get a subsequence of μ_n such that $\lim_{n\to\infty} Q_{n,s}(m)$ exists for all $s \in \mathbf{N}$ and $m \in \mathbf{N}$, and such that $\{\alpha_i\}_1^\infty$ is well defined and is a nonnegative nonincreasing sequence and $\sum_{i=1}^s \alpha_i \le 1$ for all $s \ge 1$. (i) is proved. In the following we show this subsequence of μ_n (still denoted by μ_n) satisfies the properties of the lemma. In Step B and C, we first prove given $s \ge 1$ with $\alpha_s > 0$ for any $\epsilon > 0$ there exist $R > 0$ and ϵ-dependent sequences $\{x_{n,i}\}, i = 1, ..., s$, such that (4.4) and (4.5) hold. In Step D, we show we may choose $\{x_{n,i}\}$ independent of ϵ.

Step B. Fix $s \ge 1$ with $\alpha_s > 0$. For given $\epsilon > 0$ (without loss of generality we assume $\epsilon < \alpha_s$) by definition there exist $m_1 > 0$ such that for $m \ge m_1$

$$\left| \lim_{n\to\infty} \left(\sup_{y_i \in \mathbf{R}^N} \int_{\bigcup_{i=1}^s B_m(y_i)} \mu_n \right) - \sum_{i=1}^s \alpha_i \right| < \frac{\epsilon}{8}.$$

Then there exist sequences $\{x_{n,i}\} \subset \mathbf{R}^N$ for $i = 1, ..., s$ such that

$$\limsup_{n\to\infty} \left| \int_{\bigcup_{i=1}^s B_{m_1}(x_{n,i})} \mu_n - \sum_{i=1}^s \alpha_i \right| < \frac{\epsilon}{4}. \tag{4.6}$$

We claim

$$\liminf_{n\to\infty} |x_{n,i} - x_{n,j}| = \infty, \quad \forall\, i \ne j. \tag{4.7}$$

If not, for simplicity, we assume $i = 1$ and $j = s$ and for a subsequence $\lim_{n\to\infty} |x_{n,1} - x_{n,s}| \le C$ for some constant $C > 0$. Choose $m \in \mathbf{N}$ with $m \ge C + 1$. Then by (4.6)

$$\begin{aligned} \sum_{i=1}^s \alpha_i - \frac{\epsilon}{4} &\le \liminf_{n\to\infty} \int_{\bigcup_{i=1}^s B_{m_1}(x_{n,i})} \mu_n \\ &\le \limsup_{n\to\infty} \int_{\bigcup_{i=1}^{s-1} B_{m_1+m}(x_{n,i})} \mu_n \le \lim_{n\to\infty} Q_{n,s-1}(m_1 + m) \le \sum_{i=1}^{s-1} \alpha_i, \end{aligned}$$

a contradiction for $\epsilon < \alpha_s$. Thus (4.7) is proved. Next we claim

$$\lim_{r\to\infty} \liminf_{n\to\infty} \int_{\mathbf{R}^N \setminus \bigcup_{i=1}^s B_r(x_{n,i})} \mu_n \ge 1 - \sum_{i=1}^s \alpha_i - \frac{\epsilon}{8}. \tag{4.8}$$

If (4.8) is not true, there exist $m \in \mathbf{N}$ and a subsequence of μ_n such that

$$\lim_{n\to\infty} \int_{\mathbf{R}^N \setminus \bigcup_{i=1}^s B_m(x_{n,i})} \mu_n < 1 - \sum_{i=1}^s \alpha_i - \frac{\epsilon}{8}.$$

This gives

$$\lim_{n\to\infty}\int_{\bigcup_{i=1}^{s} B_m(x_{n,i})}\mu_n = 1-\lim_{n\to\infty}\int_{\mathbf{R}^N\setminus\bigcup_{i=1}^{s} B_m(x_{n,i})}\mu_n > \sum_{i=1}^{s}\alpha_i+\frac{\epsilon}{8},$$

a contradiction for

$$\lim_{n\to\infty}\int_{\bigcup_{i=1}^{s} B_m(x_{n,i})}\mu_n \le \lim_{n\to\infty} Q_{n,s}(m)\le \sum_{i=1}^{s}\alpha_i.$$

Thus (4.8) is proved. By (4.8) there exists $m_2\in\mathbf{N}$ with $m_2\ge m_1$, for all $r'\ge m_2$,

$$\liminf_{n\to\infty}\int_{\mathbf{R}^N\setminus\bigcup_{i=1}^{s} B_{r'}(x_{n,i})}\mu_n \ge 1-\sum_{i=1}^{s}\alpha_i-\frac{\epsilon}{4}.$$

On the other hand,

$$\begin{aligned}\limsup_{n\to\infty}\int_{\mathbf{R}^N\setminus\bigcup_{i=1}^{s} B_{r'}(x_{n,i})}\mu_n-(1-\sum_{i=1}^{s}\alpha_i) &= \sum_{i=1}^{s}\alpha_i-\liminf_{n\to\infty}\int_{\bigcup_{i=1}^{s} B_{r'}(x_{n,i})}\mu_n\\ &\le \sum_{i=1}^{s}\alpha_i-\liminf_{n\to\infty}\int_{\bigcup_{i=1}^{s} B_{m_1}(x_{n,i})}\mu_n\le\frac{\epsilon}{4},\end{aligned}$$

where we used (4.6) in the last step. Thus we have proved that for any $r'\ge m_2$

$$\limsup_{n\to\infty}\left|(1-\sum_{i=1}^{s}\alpha_i)-\int_{\mathbf{R}^N\setminus\bigcup_{i=1}^{s} B_{r'}(x_{n,i})}\mu_n\right|<\frac{\epsilon}{4}. \tag{4.9}$$

Next by (4.6), (4.7) and (4.9), for any $r\ge m_2$

(4.10)

$$\begin{aligned}&\limsup_{n\to\infty}\left|\sum_{i=1}^{s}\alpha_i-\sum_{i=1}^{s}\int_{B_r(x_{n,i})}\mu_n\right|\\ =&\limsup_{n\to\infty}\left|\sum_{i=1}^{s}\alpha_i-\sum_{i=1}^{s}\int_{B_{m_1}(x_{n,i})}\mu_n-\sum_{i=1}^{s}\int_{B_r(x_{n,i})\setminus B_{m_1}(x_{n,i})}\mu_n\right|\\ \le&\frac{\epsilon}{4}+\limsup_{n\to\infty}\left(1-\int_{\bigcup_{i=1}^{s} B_{m_1}(x_{n,i})}\mu_n-\int_{\mathbf{R}^N\setminus\bigcup_{i=1}^{s} B_r(x_{n,i})}\mu_n\right)\\ \le&\frac{\epsilon}{4}+\limsup_{n\to\infty}\left|\sum_{i=1}^{s}\alpha_i-\int_{\bigcup_{i=1}^{s} B_{m_1}(x_{n,i})}\mu_n\right|+\limsup_{n\to\infty}\left|(1-\sum_{i=1}^{s}\alpha_i)-\int_{\mathbf{R}^N\setminus\bigcup_{i=1}^{s} B_r(x_{n,i})}\mu_n\right|\\ \le&\frac{3\epsilon}{4}.\end{aligned}$$

Taking $R \geq m_2$, by (4.9) and (4.10) we have proved that for any $r \geq R$ and $r' \geq R$

$$\limsup_{n\to\infty} \left|\sum_{i=1}^{s} \alpha_i - \sum_{i=1}^{s} \int_{B_r(x_{n,i})} \mu_n\right| + \left|(1 - \sum_{i=1}^{s} \alpha_i) - \int_{\mathbf{R}^N \setminus \bigcup_{i=1}^{s} B_{r'}(x_{n,i})} \mu_n\right| < \epsilon.$$

Step C. Next, we show that by rearranging indices we can have

$$\limsup_{n\to\infty} \sum_{i=1}^{s} \left|\alpha_i - \int_{B_r(x_{n,i})} \mu_n\right| + \left|(1 - \sum_{i=1}^{s} \alpha_i) - \int_{\mathbf{R}^N \setminus \bigcup_{i=1}^{s} B_{r'}(x_{n,i})} \mu_n\right| < \epsilon. \tag{4.11}$$

We shall do a mathematical induction on s. It is obvious for $s = 1$. Suppose at the $s - 1$ step, for $\epsilon > 0$ there exist $R_1 > 0$ and $\{x_{n,i}\} \subset \mathbf{R}^N$ for $i = 1, ..., s-1$, such that

$$\liminf_{n\to\infty} |x_{n,i} - x_{n,j}| = \infty, \quad \forall\, i \neq j. \tag{4.12}$$

and that for all $r \geq R_1$ and all $r' \geq R_1$

$$\limsup_{n\to\infty} \sum_{i=1}^{s-1} \left|\alpha_i - \int_{B_r(x_{n,i})} \mu_n\right| + \left|(1 - \sum_{i=1}^{s-1} \alpha_i) - \int_{\mathbf{R}^N \setminus \bigcup_{i=1}^{s-1} B_{r'}(x_{n,i})} \mu_n\right| < \frac{\epsilon}{4}. \tag{4.13}$$

At the s step, by what we have proved, there exist $R_2 > 0$ (we may assume $R_2 \geq R_1$) and $\{y_{n,i}\} \subset \mathbf{R}^N$ for $i = 1, ..., s$ such that

$$\liminf_{n\to\infty} |y_{n,i} - y_{n,j}| = \infty, \quad i \neq j,$$

and that for all $r \geq R_2$ and all $r' \geq R_2$

$$\limsup_{n\to\infty} \left|\sum_{i=1}^{s} \alpha_i - \int_{\bigcup_{i=1}^{s} B_r(y_{n,i})} \mu_n\right| + \left|(1 - \sum_{i=1}^{s} \alpha_i) - \int_{\mathbf{R}^N \setminus \bigcup_{i=1}^{s} B_{r'}(y_{n,i})} \mu_n\right| < \frac{\epsilon}{8}. \tag{4.14}$$

Define

$I_1 := \{i \in \{1, ..., s\} \mid \exists j \in \{1, ..., s-1\}$, such that $\liminf_{n\to\infty} |y_{n,i} - x_{n,j}| \leq C_i$ for some $C_i > 0\}$,

and $I_2 := \{1, 2, ..., s\} \setminus I_1$.

We claim $I_2 \neq \emptyset$. If $I_2 = \emptyset$, there exists a subsequence of μ_n (still denoted by μ_n) and $C > 0$ so that for each $i \in \{1, ..., s\}$ there is $j \in \{1, ..., s-1\}$

$$\lim_{n\to\infty} |y_{n,i} - x_{n,j}| \leq C.$$

Choose $m \in \mathbf{N}$ with $m \geq R_2 + C + 1$. Then by (4.14)

$$\sum_{i=1}^{s} \alpha_i - \frac{\epsilon}{8} \leq \liminf_{n\to\infty} \int_{\bigcup_{i=1}^{s} B_{R_2}(y_{n,i})} \mu_n \leq \limsup_{n\to\infty} \int_{\bigcup_{i=1}^{s-1} B_m(x_{n,i})} \mu_n \leq \sum_{i=1}^{s-1} \alpha_i,$$

a contradiction for $\epsilon < \alpha_s$. Next we claim there exists $i_0 \in I_2$ such that for all $r \geq R_2$

$$\limsup_{n\to\infty} \left| \alpha_s - \int_{B_r(y_{n,i_0})} \mu_n \right| \leq \frac{\epsilon}{4}. \tag{4.15}$$

First for all $i \in I_2$ by (4.12)

$$\begin{aligned} &\limsup_{n\to\infty} \int_{B_r(y_{n,i})} \mu_n \\ =&\limsup_{n\to\infty} \left(\int_{B_r(y_{n,i}) \bigcup \bigcup_{j=1}^{s-1} B_{R_1}(x_{n,j})} \mu_n - \int_{\bigcup_{j=1}^{s-1} B_{R_1}(x_{n,j})} \mu_n \right) \\ \leq&\limsup_{n\to\infty} \int_{B_r(y_{n,i}) \bigcup \bigcup_{j=1}^{s-1} B_{R_1}(x_{n,j})} \mu_n - \liminf_{n\to\infty} \int_{\bigcup_{j=1}^{s-1} B_{R_1}(x_{n,j})} \mu_n \\ \leq&\sum_{i=1}^{s} \alpha_i - (\sum_{i=1}^{s-1} \alpha_i - \frac{\epsilon}{4}) \\ \leq&\alpha_s + \frac{\epsilon}{4}. \end{aligned} \tag{4.16}$$

On the other hand, we claim there exists $i_0 \in I_2$ such that

$$\liminf_{n\to\infty} \int_{B_{R_2}(y_{n,i_0})} \mu_n \geq \alpha_s - \frac{\epsilon}{4}. \tag{4.17}$$

If (4.17) is not true we assume for all $i \in I_2$, $\liminf_{n\to\infty} \int_{B_{R_2}(y_{n,i})} \mu_n < \alpha_s - \frac{\epsilon}{4}$. Then we have by (4.14)

$$\begin{aligned} \sum_{i=1}^{s} \alpha_i - \frac{\epsilon}{8} \leq& \liminf_{n\to\infty} \int_{\bigcup_{i=1}^{s} B_{R_2}(y_{n,i})} \mu_n = \liminf_{n\to\infty} \left(\int_{\bigcup_{i\in I_1} B_{R_2}(y_{n,i})} \mu_n + \int_{\bigcup_{i\in I_2} B_{R_2}(y_{n,i})} \mu_n \right) \\ \leq& \sum_{i=1}^{|I_1|} \alpha_i + |I_2|(\alpha_s - \frac{\epsilon}{4}) \leq \sum_{i=1}^{s} \alpha_i - \frac{|I_2|\epsilon}{4} \end{aligned}$$

a contradiction. Thus using (4.16) and (4.17) for this i_0 we have proved claim (4.15).

Now at the s step, replacing $s-1$ sequences of $\{\{y_{n,i}\}\}_{i=1}^{s}$ by $\{\{x_{n,i}\}\}_{i=1}^{s-1}$ with $\{y_{n,i_0}\}$ retained we get s sequences $\{z_{n,i}\}$ for $i = 1, ..., s$ with $z_{n,i} = x_{n,i}$ for $i = 1, ..., s-1$ and $z_{n,s} = y_{n,i_0}$ so that (by (4.12) and $i_0 \in I_2$)

$$\liminf_{n\to\infty} |z_{n,i} - z_{n,j}| = \infty, \ \ \forall\, i \neq j,$$

and that for all $r \geq R_2$ and all $r' \geq R_2$ (by (4.13) and (4.15))

$$\limsup_{n\to\infty} \sum_{i=1}^{s} \left| \alpha_i - \int_{B_r(z_{n,i})} \mu_n \right| + \left| (1 - \sum_{i=1}^{s} \alpha_i) - \int_{\mathbf{R}^N \setminus \bigcup_{i=1}^{s} B_{r'}(z_{n,i})} \mu_n \right| \leq \epsilon.$$

Step D. Finally, we show $\{x_{n,i}\}$ can be taken independent of $\epsilon > 0$ such that (4.4) and (4.5) hold. Since $\sum_i \alpha_i \leq 1$, we may group $\alpha_i's$ into groups which contain equal values. I.e., there exist β_i, a positive and strictly decreasing sequence and $s_i \in \mathbf{N}$ such that $\alpha_j = \beta_1$, for $j = 1, ..., s_1$ and $\alpha_j = \beta_i$ for $j = \sum_{l=1}^{i-1} s_l < j \leq \sum_{l=1}^{i} s_l$. We shall do a mathematical induction on β_i. For $i = 1$, let $0 < \epsilon_1 < \frac{\beta_2 - \beta_1}{2}$. Then by Step C there exist $\{x_{n,i}\}$ for $i = 1, ..., s_1$, $R_1 > 0$ such that (4.4) and (4.5) hold for ϵ_1 and R_1. Next, for any $0 < \epsilon' < \epsilon_1$, also by Step C, there exist $\{y_{n,i}\}$ $i = 1, ..., s_1$, $R' > 0$ (without loss of generality we may assume $R' \geq R_1$) such that (4.4) and (4.5) hold for ϵ' and R'. We claim that there exist $C > 0$ (after a possible rearrangement of indices)

$$\limsup_{n\to\infty} |x_{n,i} - y_{n,i}| \leq C.$$

In fact, if this is not the case, we may assume that there exists i_0 such that

$$\limsup_{n\to\infty} |x_{n,i} - y_{n,i_0}| = \infty, \tag{4.18}$$

for all $i = 1, ..., s_1$. Then by definition, (4.5) and (4.18)

$$s_1\beta_1 + \beta_2 \geq \liminf_{n\to\infty} \int_{\bigcup_{i=1}^{s_1} B_{R'}(x_{n,i}) \bigcup B_{R'}(y_{n,i_0})} \mu_n \geq s_1\beta_1 - \epsilon_1 + \beta_1 - \epsilon'$$

a contradiction. Thus for $0 < \epsilon' < \epsilon_1$, we may still use $\{x_{n,i}\}$ with a new $\overline{R}' = R' + C + 1$ to have (4.4) and (4.5) hold.

Suppose for $k \geq 1$, with $\beta_k > 0$, there exist $\{x_{n,i}\}$ for $i = 1, ..., \sum_{j=1}^{k} s_j$, for any $\epsilon > 0$, there exists $R > 0$ such that (4.4) and (4.5) hold. If $\beta_{k+1} > 0$, we take $0 < \epsilon_1 < \frac{\beta_{k+1} - \beta_{k+2}}{2}$. Then by Step C again for this $\epsilon_1 > 0$ we first have $\{y_{n,i}\}$, $i = 1, ..., \sum_{j=1}^{k+1} s_j$, and $R_1 > 0$ such that (4.4) and (4.5) hold. Then a similar argument to the above shows that after a possible rearrangement of indices, we have $C > 0$ such that for all $1 \leq i \leq \sum_{j=1}^{k} s_j$

$$\limsup_{n\to\infty} |x_{n,i} - y_{n,i}| \leq C.$$

Therefore for $1 \leq i \leq \sum_{j=1}^{k} s_j$ we may replace all these $\{y_{n,i}\}$ by $\{x_{n,i}\}$ and define for $(\sum_{j=1}^{k} s_j) + 1 \leq i \leq \sum_{j=1}^{k+1} s_j$, $x_{n,i} = y_{n,i}$. Next, for any $0 < \epsilon' < \epsilon_1$, by Step C again, there exist $\{z_{n,i}\}$ for $1 \leq i \leq \sum_{j=1}^{k+1} s_j$, $R' > 0$(assume $R' \geq R_1$) such that (4.4) and (4.5) hold.

Similarly we may replace $\{z_{n,i}\}$ for $i = 1, ..., \sum_{j=1}^{k} s_j$ by $\{x_{n,i}\}$. And we claim that there is $C > 0$ such that for $i = (\sum_{j=1}^{k} s_j) + 1, ..., \sum_{j=1}^{k+1} s_j$

$$\limsup_{n\to\infty} |x_{n,i} - z_{n,i}| \le C.$$

If this is not the case, there exists i_0 with $(\sum_{j=1}^{k} s_j) + 1 \le i_0 \le \sum_{j=1}^{k+1} s_j$ such that

$$\limsup_{n\to\infty} |x_{n,i} - z_{n,i_0}| = \infty.$$

This implies by the definition and (4.5)

$$\sum_{i=1}^{k+1} s_i\beta_i + \beta_{k+2} \ge \liminf \int_{\bigcup_{i=1}^{\bar{s}} B_{R'}(x_{n,i}) \bigcup B_{R'}(z_{n,i_0})} \mu_n \ge \sum_{i=1}^{k+1} s_i\beta_i - \epsilon_1 + \beta_{k+1} - \epsilon'$$

where $\bar{s} = \sum_{i=1}^{k+1} s_i$. This is a contradiction. Therefore we can replace $\{z_{n,i}\}$ by $\{x_{n,i}\}$ for $i = (\sum_{j=1}^{k} s_j) + 1, ..., \sum_{j=1}^{k+1} s_j$, and for any $0 < \epsilon' < \epsilon_1$ we may take $\overline{R}' = R' + C + 1$ so that (4.4) and (4.5) hold. The induction is finished. The proof is complete. □

In Lemma 4.1, in general $\sum_{i=1}^{\infty} \alpha_i < 1$ can happen. However, we have the following.

Lemma 4.2. Let $(u_n) \subset H^1(\mathbf{R}^N)$ be uniformly bounded such that $\int_{\mathbf{R}^N} |u_n|^{p+1} dx = 1$ with $2 < p + 1 \le 2^* := \frac{2N}{N-2}$. Then when applying Lemma 4.1 to $\mu_n = |u_n|^{p+1}$ we have $\sum_{i=1}^{\infty} \alpha_i = 1$.

To prove Lemma 4.2, we need the following.

Lemma 4.3. Let (u_n) be bounded in $H^1(\mathbf{R}^N)$. If for some $q \in [2, 2^*]$ and $r > 0$

$$\sup_{x\in\mathbf{R}^N} \int_{B_r(x)} |u_n|^q \, dx \to 0 \quad \text{as } n \to \infty$$

then $u_n \to 0$ in $L^\theta(\mathbf{R}^N)$ for any $2 < \theta < 2^*$. If $q = 2^*$ then we also have $u_n \to 0$ in $L^{2^*}(\mathbf{R}^N)$.

For a proof of the first part of this lemma see [Lp1] and for the second part see [RWW] (and [Wz1] for similar ideas).

Proof of Lemma 4.2. Applying Lemma 4.1 to $\mu_n = |u_n|^{p+1}$, we have a sequence $\{\alpha_i\}$ having the properties in Lemma 4.1. There are two cases: (a). $\alpha_i > 0$ for all $i \ge 1$. (b). $\alpha_i = 0$ starting from some $i > 1$. Case (b) is simpler and can be treated in similar way as we treat case (a) below. Hence we assume $\alpha_i > 0$ for all $i \ge 1$. If $\sum_{i=1}^{s} \alpha_i \to \overline{\alpha} < 1$ as $s \to \infty$ we get a contradiction as follows. Take a sequence $\epsilon_s \to 0$ as $s \to \infty$. For each s then there exist $r_s \ge s$ such that

$$\limsup_{n\to\infty} \sum_{i=1}^{s} \left| \alpha_i - \int_{B_{r_s}(x_{n,i})} \mu_n \right| + \left| (1 - \sum_{i=1}^{s} \alpha_i) - \int_{\mathbf{R}^N \setminus \bigcup_{i=1}^{s} B_{r_s}(x_{n,i})} \mu_n \right| < \epsilon_s. \tag{4.19}$$

We now fix a $r>0$ and let $\overline{r}_s=\max\{r,r_s\}$. Then for each s there exists $n_s>0$ such that

$$\sup_{y_i\in R^N}\int_{\bigcup_{i=1}^{s+1}B_{\overline{r}_s}(y_i)}\mu_{n_s}\le\limsup_{n\to\infty}\sup_{y_i\in R^N}\int_{\bigcup_{i=1}^{s+1}B_{\overline{r}_s}(y_i)}\mu_n+\epsilon_s\le\sum_{i=1}^{s+1}\alpha_i+\epsilon_s, \tag{4.20}$$

and

$$|x_{n_s,i}-x_{n_s,j}|\ge 4r_s,\ \ \forall\ \ i\neq j,\ \ i,j=1,2,...,s. \tag{4.21}$$

We define $w_s=\prod_{i=1}^s\eta^c\left(\frac{|x-x_{n_s,i}|}{r_s}\right)u_{n_s}$. Here η and η^c are the cut-off functions used in Remark 2.2. Then we have $w_s\in H^1(\mathbf{R}^N)$ and $||w_s||_{H^1(\mathbf{R}^N)}\le 2||u_{n_s}||_{H^1(\mathbf{R}^N)}$ because of $r_s\ge s$ and (4.21). Also we have by (4.20)

$$\begin{aligned}
&\sup_{y\in\mathbf{R}^N}\int_{B_r(y)}|w_s|^{p+1}\\
\le&\sup_{y\in\mathbf{R}^N}\int_{B_r(y)\cap(\mathbf{R}^N\setminus\bigcup_{i=1}^s B_{r_s}(x_{n_s,i}))}\mu_{n_s}+\int_{\bigcup_{i=1}^s B_{r_s}(x_{n_s,i})}\mu_{n_s}dx-\sum_{i=1}^s\alpha_i\\
&\qquad\qquad+\sum_{i=1}^s\alpha_i-\int_{\bigcup_{i=1}^s B_{r_s}(x_{n_s,i})}\mu_{n_s}\\
\le&\sup_{y\in\mathbf{R}^N}\int_{B_r(y)\bigcup\bigcup_{i=1}^s B_{r_s}(x_{n_s,i})}\mu_{n_s}dx-\sum_{i=1}^s\alpha_i+\epsilon_s\\
\le&\sup_{y_i\in\mathbf{R}^N}\int_{\bigcup_{i=1}^{s+1}B_{\overline{r}_s}(y_i)}\mu_{n_s}dx-\sum_{i=1}^s\alpha_i+\epsilon_s\\
\le&\sum_{i=1}^{s+1}\alpha_i+\epsilon_s-\sum_{i=1}^s\alpha_i+\epsilon_s\\
=&\alpha_{s+1}+2\epsilon_s\to 0,\ \ \text{as}\ \ s\to\infty.
\end{aligned}$$

By Lemma 4.3 we get

$$\int_{R^N}|w_s|^{p+1}\to 0$$

as $s\to\infty$. On the other hand, by (4.19)

$$\limsup_{s\to\infty}\left|(1-\sum_{i=1}^s\alpha_i)-\int_{\mathbf{R}^N\setminus\bigcup_{i=1}^s B_{r_s}(x_{n,i})}\mu_n\right|<\epsilon_s.$$

This implies $\lim_{s\to\infty}\int_{\mathbf{R}^N}|w_s|^{p+1}\ge\lim_{s\to\infty}(1-\sum_{i=1}^s\alpha_i)=1-\overline{\alpha}>0$, a contradiction. Thus $\sum_{i=1}^s\alpha_i\to 1$ as $s\to\infty$. □

Remark 4.2. Another variant of Lemma 4.1 is for a G-invariant sequence μ_n in the sense that $\mu_n(g^{-1}x)=\mu_n(x), a.e.,$ in $\mathbf{R}^N$ for all $g\in G$, where $G\subset\mathbf{O}(N)$ a closed subgroup acting on $\mathbf{R}^N$. In this case, as we proved in [CW] (Lemma 2.5 in [CW]) we may assume that for each $s \geq 1$ with $\alpha_s > \alpha_{s+1}$, $G\{x_{n,1}, ..., x_{n,s}\} = \{x_{n,1}, ..., x_{n,s}\}$, i.e., $(x_{n,i})$ appear in G-orbits, and we may assume that $\#G(x_{n,i}) = k_i$ is a constant integer for all i with $\alpha_i > 0$.

We finish this subsection with an elmentary lemma which will be used later.

Lemma 4.4. Let $k \geq 1$ be fixed. Define for $t \in [1, (2k)^{\frac{p-1}{p+1}}]$ and $\alpha \in [0, 1]$

$$f_t(\alpha) = t(1-\alpha)^{\frac{2}{p+1}} + \alpha^{\frac{2}{p+1}}.$$

Then for all $t \in [1, (2k)^{\frac{p-1}{p+1}}]$, we have $f_t(\alpha) > f_t(0)$ for $\alpha \in (0, \frac{1}{2k+1}]$.

Proof. For each $t \in [1, (2k)^{\frac{p-1}{p+1}}]$, f_t achieves its unique maximum in $[0, 1]$ at $\alpha_t = \frac{1}{1+t^{\frac{p+1}{p-1}}} \geq \frac{1}{2k+1}$.

4.2. *A Technical Result*

Let $1 \leq l \leq N$ be an integer. Let $G_1 \subset \mathbf{O}(l)$ satisfy the assumption on G_1 in Section 1, i.e., $k := \min\{\#G_1(x) \mid x \in \mathbf{R}^l \times \{0\} \ \ |x| = 1\} < \infty$. Let $G_2 = \{e\}$ be the trivial group, and $G = G_1 \times G_2$ act canonically on $\mathbf{R}^l \times \mathbf{R}^{N-l} = \mathbf{R}^N$. Define for $R \geq 0$

$$S(R) := \inf_{H^1_{0,G}(T^c_R)} \frac{\int_{T^c_R} |\nabla u|^2 + u^2}{(\int_{T^c_R} |u|^{p+1})^{\frac{2}{p+1}}} \tag{4.22}$$

where $T_R = \{x \in \mathbf{R}^N | |P_l x| < R\}$ and $T^c_R := \mathbf{R}^N \setminus T_R$, and

$$H^1_{0,G}(T^c_R) = \{u \in H^1_0(T^c_R) \mid u(g^{-1}x) = u(x), \ a.e., \text{for all} \ \ g \in G\}.$$

It is easy to see that $S(R)$ is nondecreasing in R.

Proposition 4.1.

(a). $\lim_{R\to\infty} S(R) = k^{\frac{p-1}{p+1}} S_p$, where S_p is defined in (1.4).

(b). Let $R_n \to \infty$ and $u_n \in H^1_{0,G}(T^c_{R_n})$ be such that $||u_n||_{p+1} = 1$ and $\int_{T^c_{R_n}} |\nabla u_n|^2 + u_n^2 \to k^{\frac{p-1}{p+1}} S_p$ as $n \to \infty$. Then there exist $x_n \in T^c_{R_n}$ satisfying $\#G(x_n) = k$ such that for any $\epsilon > 0$ there exists $R > 0$

$$\int_{B_R(x_n)} |u_n|^{p+1} \geq \frac{1}{k} - \epsilon. \tag{4.23}$$

Proof of Proposition 4.1. It is easy to see by appropriate test functions that $\lim_{R\to\infty} S(R) \leq k^{\frac{p-1}{p+1}} S_p$ (See similar argument in Remark 2.2). Suppose $\lim_{R\to\infty} S(R) = A$. For $k = 1$ one can see immediately that $S(R) = S_p$ for all $R \geq 0$. So we assume here $k \geq 2$.

Take a sequence $R_n \to \infty$ and $u_n \in H^1_{0,G}(T^c_{R_n})$ such that $\int_{\mathbf{R}^N} |u_n|^{p+1} = 1$ and $\int_{T^c_{R_n}} |\nabla u_n|^2 + u_n^2 \to A$ as $n \to \infty$. Applying Lemma 4.2 to $\mu_n = |u_n|^{p+1}$ we have (α_i)

satisfying $\sum_{i=1}^{s}\alpha_i \to 1$ and $(x_{n,i})$ satisfying (4.4) such that (4.5) holds. We shall show $\alpha_1 = \cdots = \alpha_k = \frac{1}{k}$ and $\alpha_i = 0$ for $i > k$. Because of the symmetry $\alpha_1 < 1$. Let $\overline{\alpha}_s = 1 - \sum_{i=1}^{s}\alpha_i$. We choose $s_0 > s_0 + 1$ such that $\overline{\alpha}_{s_0} < \frac{1}{2k+1}$. Taking $\epsilon > 0$ we have $R > 0$ such that for all $r \geq R$ and all $r' \geq R$

$$\limsup_{n\to\infty}\sum_{i=1}^{s_0}\left|\alpha_i - \int_{B_r(x_{n,i})}\mu_n\right| + \left|(1-\sum_{i=1}^{s_0}\alpha_i) - \int_{\mathbf{R}^N\setminus\bigcup_{i=1}^{s}B_{r'}(x_{n,i})}\mu_n\right| < \epsilon.$$

Take $r' = 2r > \epsilon^{-1}$. By Remark 4.2, we may assume that $G\{x_{n,1},...,x_{n,s_0}\} = \{x_{n,1},...,x_{n,s_0}\}$. Then there are $\{g_{j,i}\}\subset G$ for $j = 1,...k_i$ with $k_i = \#G(x_{n,i})$ such that

$$\min\{|g_{j,i}x_{n,i} - g_{m,i}x_{n,i}| \mid \text{for } j \neq m\} \to \infty, \quad \text{as } n\to\infty. \tag{4.24}$$

We may also relabel α_i's for $i = 1,...,s_0$ to be β_i for $i = 1,...,s_1$ with β_i repeating k_i times and s_1 satisfying $\sum_{i=1}^{s_1}k_i = s_0$. Thus $\sum_{i=1}^{s_1}k_i\beta_i = \sum_{i=1}^{s_0}\alpha_i$. With this notation, we may write $\{x_{n,1},...,x_{n,s_0}\} = \{G(y_{n,1}),...,G(y_{n,s_1})\}$.

Let $\eta = \eta(t)$ be the cut-off function used in Remark 2.2. We define, for $i = 1,\dots,s_1$ and $j = 1,...,k_i$,

$$w_{n;i,j}(x) = \eta\left(\frac{|x - g_{j,i}y_{n,i}|}{r}\right)u_n(x)$$

and

$$v_n(x) = \prod_{i=1}^{s_1}\prod_{j=1}^{k_i}\eta^c\left(\frac{|x - g_{j,i}y_{n,i}|}{r}\right)\cdot u_n(x).$$

Then $w_{n;i,j}, v_n$ all belong to $H^1(\mathbf{R}^N)$. By (4.24), a straight forward computation shows

$$\int_{T_{R_n}^c}(|\nabla u_n|^2 + u_n^2)dx - \sum_{i=1}^{s_1}\sum_{j=1}^{k_i}\int_{T_{R_n}^c}(|\nabla w_{n;i,j}|^2 + w_{n;i,j}^2)dx - \int_{T_{R_n}^c}(|\nabla v_n|^2 + v_n^2)dx \geq -\frac{C_0}{r}$$

where C_0 is a constant independent of n. Then we have

$$\begin{aligned}
A =& \lim_{n\to\infty}\int_{T_{R_n}^c}(|\nabla u_n|^2 + u_n^2)dx \\
\geq& \limsup_{n\to\infty}\left(\sum_{i=1}^{s_1}\sum_{i=j}^{k_i}\int_{T_{R_n}^c}(|\nabla w_{n;i,j}|^2 + w_{n;i,j}^2)dx + \int_{T_{R_n}^c}(|\nabla v_n|^2 + v_n^2)dx\right) - \frac{C_0}{r} \\
\geq& S_p\left(\sum_{i=1}^{s_1}\sum_{i=j}^{k_i}||w_{n;i,j}||_{p+1}^2 + (\overline{\alpha}_{s_0} - \epsilon)^{\frac{2}{p+1}}\right) - \frac{C_0}{r} \\
\geq& S_p\left(\sum_{i=1}^{s_1}k_i(\beta_i - \epsilon)^{\frac{2}{p+1}} + (\overline{\alpha}_{s_0} - \epsilon)^{\frac{2}{p+1}}\right) - \frac{C_0}{r} \\
=& S_p\left(\sum_{i=1}^{s_1}k_i^{\frac{p-1}{p+1}}(k_i\beta_i - k_i\epsilon)^{\frac{2}{p+1}} + (\overline{\alpha}_{s_0} - \epsilon)^{\frac{2}{p+1}}\right) - \frac{C_0}{r}
\end{aligned}$$

Since we have $A \le k^{\frac{p-1}{p+1}} S_p$, we get

$$k^{\frac{p-1}{p+1}} \ge \frac{A}{S_p} \ge k^{\frac{p-1}{p+1}} \sum_{i=1}^{s_1} (k_i\beta_i - k_i\epsilon)^{\frac{2}{p+1}} + (\overline{\alpha}_{s_0} - \epsilon)^{\frac{2}{p+1}} - \frac{C_0}{r}.$$

Now we send ϵ to zero (then r tends to infinity as $r \ge \epsilon^{-1}$). We get

$$k^{\frac{p-1}{p+1}} \ge \frac{A}{S_p} \ge k^{\frac{p-1}{p+1}} \sum_{i=1}^{s_1} (k_i\beta_i)^{\frac{2}{p+1}} + (\overline{\alpha}_{s_0})^{\frac{2}{p+1}} \tag{4.25}$$

which is a contradiction if $\overline{\alpha}_{s_0} \ne 0$ for $\overline{\alpha_0} < \frac{1}{2k+1}$ and Lemma 4.4. Thus one has $\overline{\alpha}_{s_0} = 0$. Then (4.25) shows $A = k^{\frac{p-1}{p+1}} S_p$ and $s_1 = 1$. Hence we conclude that $k_1 = k$ and $\beta_1 = \frac{1}{k}$. Now for each $\epsilon > 0$ there exist $r > 0$ and $g_i \in G$ for $i = 1, ..., k$ such that as $n \to \infty$, $|g_i x_n - g_j x_n| \to \infty$ for $i \ne j$ and

$$\sum_{i=1}^{k} \left| \frac{1}{k} - \int_{B_r(g_i x_n)} \mu_{n,i} \right| < \epsilon.$$

Next, repeating the above arguments for the sequence given in the assertion (b) we finish the proof of Proposition 4.1. □

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