(b) Based on your results for **(a)**, guess the minimum number of moves required if you start with an arbitrary number of n disks. (*Hint:* To help see a pattern, add 1 to the number of moves for $n = 1, 2, 3, 4$.)

(c) A legend claims that monks in a remote monastery are working to move a set of 64 disks, and that the world will end when they complete their sacred task. Moving one disk per second without error, 24 hours a day, 365 days a year, how long would it take to move all 64 disks?

55. ***Number of Handshakes*** Suppose there are n people at a party and that each person shakes hands with every other person exactly once. Let $f(n)$ denote the total number of handshakes. See Exercise 19, page 469. Show

$$f(n) = \frac{n(n-1)}{2} \qquad \text{for every positive integer } n.$$

56. Suppose n is an odd positive integer not divisible by 3. Show that $n^2 - 1$ is divisible by 24. (*Hint:* Consider the three consecutive integers $n - 1$, n, $n + 1$. Explain why the product $(n - 1)(n + 1)$ must be divisible by 3 and by 8.)

57. ***Finding Patterns*** If $a_n = \sqrt{24n + 1}$, **(a)** write out the first five terms of the sequence $\{a_n\}$. **(b)** What odd integers occur in $\{a_n\}$? **(c)** Explain why $\{a_n\}$ contains all primes greater than 3. (*Hint:* Use Exercise 56.)

8.6 THE BINOMIAL THEOREM

> We remake nature by the act of discovery, in the poem or in the theorem. And the great poem and the great theorem are new to every reader, and yet are his own experiences, because he himself recreates them. [And] in the instant when the mind seizes this for itself, in art or in science, the heart misses a beat.
>
> *J. Bronowski*

In this section we derive a general formula to calculate an expansion for $(a + b)^n$ for any positive integer power n, or to find any particular term in such an expansion. We begin by calculating the first few powers directly and then look for significant patterns. To go from one power of $(a + b)$ to the next, we simply multiply by $(a + b)$:

$$(a + b)^1 = a + b$$
$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$\begin{array}{lrr} (\times) & a + b & \\ \hline & a^4 + 3a^3b + 3a^2b^2 + ab^3 & \\ & a^3b + 3a^2b^2 + 3ab^3 + b^4 & \text{Add like terms} \\ \hline \end{array}$$

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$\begin{array}{lrr} (\times) & a + b & \\ \hline & a^5 + 4a^4b + 6a^3b^2 + 4a^2b^3 + ab^4 & \\ & a^4b + 4a^3b^2 + 6a^2b^3 + 4ab^4 + b^5 & \text{Add like terms} \\ \hline \end{array}$$

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

> *The thing that started it all was this silly newspaper puzzle that asked you to count up the total number of ways you could spell the words "Pyramid of Values" from a triangular array of letters. This led my friend and me to discover Pascal's triangle. This happened in grade 10 or 11.*
>
> Bill Gosper

When we look at these expansions of $(a + b)^n$ for $n = 1, 2, 3, 4$, and 5, several patterns become apparent.

1. There are $n + 1$ terms, from a^n to b^n.
2. Every term has essentially the same form: some coefficient times the product of a power of a times a power of b.
3. In each term the sum of the exponents on a and b is always n.
4. The powers (exponents) on a decrease, term by term, from n down to 0 where the last term is given by $b^n = a^0b^n$, and the exponents on b increase from 0 to n.

Knowing the form of the terms in the expansion and that the sum of the powers is always n, we will have the entire expansion when we know how to calculate the coefficients of the terms. If we display the coefficients from the computations above, we find precisely the numbers in the first few rows of Pascal's triangle:

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

Using the address notation for Pascal's triangle that we introduced in Section 8.4, the last row of coefficients in the triangle is $\binom{5}{0}, \binom{5}{1}, \binom{5}{2}, \binom{5}{3}, \binom{5}{4}, \binom{5}{5}$, and so we may write the expansion for $(a + b)^5$:

$$(a + b)^5 = \binom{5}{0}a^5b^0 + \binom{5}{1}a^4b^1 + \binom{5}{2}a^3b^2 + \binom{5}{3}a^2b^3 + \binom{5}{4}a^1b^4 + \binom{5}{5}a^0b^5$$

Each term exhibits the same form. For $n = 5$, each coefficient has the form $\binom{5}{r}$, where r is also the exponent on b. For each term the sum of the exponents on a and b is always 5, so that when we have b^r, we must also have a^{5-r}. Finally, since the first term has $r = 0$, the second term has $r = 1$, etc., the $(r + 1)$st term involves r.

This leads to a general conjecture for the expansion of $(a + b)^n$ which we state as a theorem that can be proved using mathematical induction. (See the end of this section.)

Binomial theorem

Suppose n is any positive integer. The expansion of $(a + b)^n$ is given by

$$(a + b)^n = \binom{n}{0}a^nb^0 + \binom{n}{1}a^{n-1}b^1 + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + \binom{n}{n}a^0b^n \qquad \textbf{(1)}$$

where the $(r + 1)$st term is $\binom{n}{r}a^{n-r}b^r$, $0 \le r \le n$. In summation notation,

$$(a + b)^n = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r. \qquad \textbf{(2)}$$

At this point, we have established only that the form of our conjecture is valid for the first five values of n, and we have not completely justified our use of the name Pascal's triangle of **binomial coefficients.** Nonetheless, the multiplication of $(a + b)^4$ by $(a + b)$ to get the expansion for $(a + b)^5$ contains all the essential ideas of the proof.

We still lack a closed-form formula for the binomial coefficients. We know, for example, that the fourth term of the expansion of $(x + 2y)^{20}$ is $\binom{20}{3}x^{17}(2y)^3$, but we cannot complete the calculation without the binomial coefficient $\binom{20}{3}$. This would require writing at least the first few terms of 20 rows of Pascal's triangle.

Pascal himself posed and solved the problem of computing the entry at any given address within the triangle. He observed that to find $\binom{n}{r}$, we can take the product of all the numbers from 1 through r, and divide it into the product of the same number of integers, from n downward. This leads to the following formula.

Pascal's formula for binomial coefficients

Suppose n is a positive integer and r is an integer that satisfies $0 < r \le n$. The binomial coefficient $\binom{n}{r}$ is given by

$$\binom{n}{r} = \frac{n(n-1)\cdots(n-r+1)}{1 \cdot 2 \cdot 3 \cdots r} \quad \textbf{(3)}$$

We leave it to the reader to verify that the last factor in the numerator, $(n - r + 1)$, is the rth number counting down from n. This gives the same number of factors in the numerator as in the denominator.

▶EXAMPLE 1 ***Using Pascal's formula*** Find the first five binomial coefficients on the tenth row of Pascal's triangle, and then give the first five terms of the expansion of $(a + b)^{10}$.

Solution

Strategy: We know $\binom{10}{0} = 1$. Use Equation (3) to get the remaining coefficients.

Follow the strategy.

$$\binom{10}{1} = \frac{10}{1} = 10, \binom{10}{2} = \frac{10 \cdot 9}{1 \cdot 2} = 45, \binom{10}{3} = \frac{10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3} = 120, \text{ and}$$
$$\binom{10}{4} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4} = 210.$$

Therefore the first five terms in the expansions of $(a + b)^{10}$ are

$a^{10} + 10a^9b + 45a^8b^2 + 120a^7b^3 + 210a^6b^4$. ◀

There is another very common formula for binomial coefficients that uses factorials. Equation (3) has a factorial in the denominator, and we can get a factorial in the numerator if we multiply numerator and denominator by the product of the rest of the integers from $n - r$ down to 1:

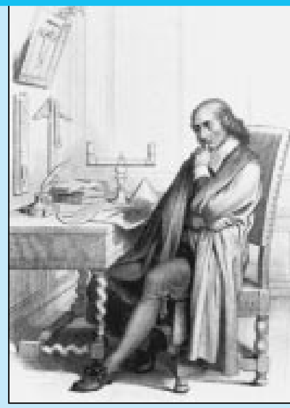
$$\binom{n}{r} = \frac{n(n-1)\ldots(n-r+1)}{1 \cdot 2 \cdot \ldots r}$$
$$= \frac{n(n-1)\ldots(n-r+1)}{r!} \cdot \frac{(n-r)\ldots 2 \cdot 1}{(n-r)\ldots 2 \cdot 1} = \frac{n!}{r!\,(n-r)!}.$$

HISTORICAL NOTE

BLAISE PASCAL

Pascal's triangle is named after Blaise Pascal, born in France in 1623. Pascal was an individual of incredible talent and breadth who made basic contributions in many areas of mathematics, but who died early after spending much of life embroiled in bitter philosophical and religious wrangling.

For some reason, Pascal's father decided that his son should not be exposed to any mathematics. All mathematics books in the home were locked up and the subject was banned from discussion. We do not know if the appeal of the forbidden was at work, but young Pascal approached his father directly and asked what geometry was. His father's answer so fascinated the 12-year-old boy that be began exploring geometric relationships on his own. He apparently rediscovered much of Euclid completely on his own. When Pascal was introduced to conic sections (see Chapter 10) he quickly absorbed everything available; he submitted a paper on conic sections to the French academy when he was only 16 years of age.



Blaise Pascal made significant contributions to the study of mathematics before deciding to devote his life to religion.

At the age of 29, Pascal had a conversion experience that led to a vow to renounced mathematics for a life of religious contemplation. Before that time, however, in addition to his foundational work in geometry, he built a mechanical computing machine (in honor of which the structured computer language Pascal is named), explored relations among binomial coefficients so thoroughly that we call the array of binomial coefficients Pascal's triangle even though the array had been known, at least in part, several hundred years earlier, proved the binomial theorem, gave the first published proof by mathematical induction, and invented (with Fermat) the science of combinatorial analysis, probability, and mathematical statistics.

Before his death ten years later, Pascal spent only a few days on mathematics. During a night made sleepless by a toothache, he concentrated on some problems about the cycloid curve that had attracted many mathematicians of the period. The pain subsided, and, in gratitude, Pascal wrote up his work for posterity.

While Equation (3) does not give a formula for $\binom{n}{0}$, the formulation in terms of factorials does apply.

$$\binom{n}{0} = \frac{n!}{0!\,(n-0)!} = \frac{n!}{1 \cdot n!} = \frac{n!}{n!} = 1.$$

This gives an alternative to Pascal's formula.

Alternative formula for binomial coefficients

Suppose n is a positive integer and r an integer that satisfies $0 \leq r \leq n$. The binomial coefficient $\binom{n}{r}$ is given by

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!} \tag{4}$$

▶EXAMPLE 2 ***Symmetry in binomial coefficients*** Show that

$$\textbf{(a)}\ \binom{6}{2} = \binom{6}{4} \qquad \textbf{(b)}\ \binom{n}{r} = \binom{n}{n-r}.$$

Solution

Follow the strategy.

Strategy: Use Equation (4) to evaluate both sides of the given equations to show that the two sides of each equation are equal.

$$\textbf{(a)}\ \binom{6}{2} = \frac{6!}{2!\,(6-2)!} = \frac{6!}{2!\,4!} \quad \text{and} \quad \binom{6}{4} = \frac{6!}{4!\,(6-4)!} = \frac{6!}{4!\,2!}.$$

$$\textbf{(b)}\ \binom{n}{r} = \frac{n!}{r!\,(n-r)!} \quad \text{and}$$

$$\binom{n}{n-r} = \frac{n!}{(n-r)!\,[n-(n-r)]!} = \frac{n!}{(n-r)!\,r!}$$

Thus

$$\binom{n}{r} = \binom{n}{n-r}. \quad ◀$$

▶EXAMPLE 3 ***Adding binomial coefficients*** Show that $\binom{8}{3} + \binom{8}{4} = \binom{9}{4}$. Get a common denominator and add fractions, but do not evaluate any of the factorials or binomial coefficients.

Solution

Use Equation (3) to get $\binom{8}{3}$ and $\binom{8}{4}$, get common denominators, then add.

$$\binom{8}{3} + \binom{8}{4} = \frac{8\cdot 7\cdot 6}{1\cdot 2\cdot 3} + \frac{8\cdot 7\cdot 6\cdot 5}{1\cdot 2\cdot 3\cdot 4} = \frac{(8\cdot 7\cdot 6)\cdot 4}{(1\cdot 2\cdot 3)\cdot 4} + \frac{8\cdot 7\cdot 6\cdot 5}{1\cdot 2\cdot 3\cdot 4}$$

$$= \frac{8\cdot 7\cdot 6(4+5)}{1\cdot 2\cdot 3\cdot 4} = \frac{9\cdot 8\cdot 7\cdot 6}{1\cdot 2\cdot 3\cdot 4} = \binom{9}{4}.$$

Thus

$$\binom{8}{3} + \binom{8}{4} = \binom{9}{4}. \quad ◀$$

Example 3 focuses more on the process than the particular result, hence the instruction to add fractions without evaluating. When we write out the binomial coefficients as fractions, we can identify the extra factors we need to get a common denominator and then add. In Example 2, we proved that $\binom{n}{r} = \binom{n}{n-r}$ giving a symmetry property for the nth row of Pascal's triangle. Example 3 illustrates the essential steps to prove the following additivity property (see Exercise 61):

$$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$$

Symmetry and additivity properties

Binomial coefficients have the following properties:

Symmetry $\binom{n}{r} = \binom{n}{n-r}$ **(5)**

Additivity $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$ **(6)**

Notice that we used the additivity property from Equation (6) in Section 8.4 to get the $(n + 1)$st row from the nth row in Pascal's triangle. This justifies our claim that the entries in Pascal's triangle are binomial coefficients.

TECHNOLOGY TIP ◆ Evaluating binomial coefficients

Most graphing calculators have the capacity to evaluate binomial coefficients directly, but we need to know where to look for the needed key. Most calculators use the notation nCr (meaning "number of combinations taken r at a time," language from probability), and the key is located in a probability (PRB, PROB) submenu under the MATH menu. To evaluate, say $\binom{20}{4}$, the process is as follows.

All TI-calculators: Having entered 20 on your screen, press MATH PRB nCr, which puts nCr on the screen. Then type 4 so you have 20nCr4. When you enter, the display should read 4845.

All Casio calculators: Having entered 20 on your screen, press MATH PRB nCr, which puts C on the screen. Then type 4 so you have 20C4. When you execute, the display should read 4845.

HP-38: Press MATH, go down to PROB, highlight COMB, OK. Then, on the command line you want COMB(20,4). Enter to evaluate.

HP-48: Put 20 and 4 on the stack. As with many HP-48 operations, COMB (for "combinations") works with two numbers. MTH NXT PROB COMB returns 4845.

▶EXAMPLE 4 Binomial Theorem Use the binomial theorem to write out the first five terms of the binomial expansion of $(x + 2y^2)^{20}$ and simplify.

Solution

Use Equation (1) with $a = x$, $b = 2y^2$, and $n = 20$. The first five terms of $(x + 2y^2)^{20}$ are

$$x^{20} + \binom{20}{1}x^{19}(2y^2) + \binom{20}{2}x^{18}(2y^2)^2 + \binom{20}{3}x^{17}(2y^2)^3 + \binom{20}{4}x^{16}(2y^2)^4.$$

Before simplifying, find the binomial coefficients, using either Equation (3) or the Technology Tip.

$$\binom{20}{1} = \frac{20}{1} = 20 \qquad \binom{20}{2} = \frac{20 \cdot 19}{1 \cdot 2} = 190$$

$$\binom{20}{3} = \frac{20 \cdot 19 \cdot 18}{1 \cdot 2 \cdot 3} = 1140 \qquad \binom{20}{4} = \frac{20 \cdot 19 \cdot 18 \cdot 17}{1 \cdot 2 \cdot 3 \cdot 4} = 4845$$

Therefore, the first five terms of $(x + 2y^2)^{20}$ are

$$x^{20} + 20 \cdot 2x^{19}y^2 + 190 \cdot 4x^{18}y^4 + 1140 \cdot 8x^{17}y^6 + 4845 \cdot 16x^{16}y^8, \text{ or}$$

$$x^{20} + 40x^{19}y^2 + 760x^{18}y^4 + 9120x^{17}y^6 + 77520x^{16}y^8. \quad ◀$$

►EXAMPLE 5 ***Finding a middle term*** In the expansion of $(2x^2 - \frac{1}{x})^{10}$, find the middle term.

Solution

There are $10 + 1$ or 11 terms in the expansion of a tenth power, so the middle term is the sixth (five before and five after). The sixth term is given by $r = 5$.

$$\binom{10}{5}(2x^2)^5\left(-\frac{1}{x}\right)^5 = 252(32x^{10})\left(-\frac{1}{x}\right)^5 = -8064x^5$$

The middle term is $-8064x^5$. ◄

►EXAMPLE 6 ***Finding a specified term*** In the expansion of $(2x^2 - \frac{1}{x})^{10}$, find the term whose simplified form involves $\frac{1}{x}$.

Solution

Strategy: First find the general term, then simplify. Finally, find the value of r that gives -1 as the exponent of x.

Follow the strategy. The general term given in Equation (2) is

$$\binom{10}{r}(2x^2)^{10-r}\left(-\frac{1}{x}\right)^r = \binom{10}{r}2^{10-r}x^{20-2r}(-1)^r x^{-r}$$
$$= \binom{10}{r}(-1)^r 2^{10-r}x^{20-3r}.$$

For the term that involves $\frac{1}{x}$ or x^{-1}, find the value of r for which the exponent on x is -1: $20 - 3r = -1$, or $r = 7$. The desired term is given by

$$\binom{10}{7}(2x^2)^3\left(-\frac{1}{x}\right)^7 = -\frac{120(8)x^6}{x^7} = -\frac{960}{x}.$$ ◄

Proof of the Binomial Theorem

We can use mathematical induction to prove that Equation (1) holds for every positive integer n.

(a) For $n = 1$, Equation (1) is $(a + b)^1 = \binom{1}{0}a^1b^0 + \binom{1}{1}a^0b^1 = a + b$, so Equation (1) is valid when n is 1.

(b) Hypothesis:
$$(a + b)^k = \binom{k}{0}a^k + \binom{k}{1}a^{k-1}b + \cdots + \binom{k}{r}a^{k-r}b^r + \cdots + \binom{k}{k}b^k \quad \textbf{(7)}$$

Conclusion:
$$(a + b)^{k+1} = \binom{k+1}{0}a^{k+1} + \binom{k+1}{1}a^k b + \cdots + \binom{k+1}{r}a^{k+1-r}b^r + \cdots + \binom{k+1}{k+1}b^{k+1} \quad \textbf{(8)}$$

Since $(a + b)^{k+1} = (a + b)^k(a + b) = (a + b)^k a + (a + b)^k b$, multiply the right side of Equation (7) by a, then by b, and add, combining like terms. It is also helpful to replace $\binom{k}{0}$ by $\binom{k+1}{0}$ and $\binom{k}{k}$ by $\binom{k+1}{k+1}$, since all are equal to 1.

$$\begin{aligned}(a + b)^k(a + b) = \binom{k+1}{0}a^{k+1} &+ \left[\binom{k}{0} + \binom{k}{1}\right]a^k b \\ &+ \left[\binom{k}{1} + \binom{k}{2}\right] a^{k-1}b^2 + \cdots \\ &+ \left[\binom{k}{r-1} + \binom{k}{r}\right]a^{k+1-r}b^r + \cdots \\ &+ \binom{k+1}{k+1}b^{k+1}.\end{aligned}$$

Apply the additive property given in Equation (6) to the expressions in brackets to get Equation (8), as desired. Therefore, by the Principle of Mathematical Induction, Equation (1) is valid for every position integer n.

EXERCISES 8.6

Check Your Understanding

Exercises 1–6 True or False. Give reasons.

1. For every positive integer n, $(3n)! = (3!)(n!)$.
2. There are ten terms in the expression of $(1 + x)^{10}$.
3. The middle term of the expansion of $(x + \frac{1}{x})^8$ is 70.
4. The expansion of $(x^2 + 2x + 1)^8$ is the same as the expansion of $(x + 1)^{16}$.
5. $\binom{8}{1} + \binom{8}{2} - \binom{9}{2} = 0$.
6. For every positive integer x, $\left(\sqrt{x} + \dfrac{1}{x}\right)^4 = x^2 + \dfrac{1}{x^4}$.

Exercises 7–10 Fill in the blank so that the resulting statement is true.

7. After simplifying the expansion of $(x^2 - \frac{1}{x})^5$, the coefficient of x^4 is ______.
8. In the expansion of $\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^6$, the middle term is ______.
9. $\binom{8}{3} - \binom{8}{2} =$ ______.
10. The number of terms in the expansion of $(x^2 + 4x + 4)^{12}$ is ______.

Develop Mastery

Exercises 1–14 Evaluate and simplify. Use Equations (3)–(6). Then verify by calculator.

1. **(a)** $\binom{9}{3}$ **(b)** $\binom{9}{6}$
2. **(a)** $\binom{14}{3}$ **(b)** $\binom{14}{11}$
3. **(a)** $\binom{8}{5}$ **(b)** $\binom{8}{3}$
4. **(a)** $\binom{100}{98}$ **(b)** $\binom{100}{2}$
5. **(a)** $\binom{20}{2} + \binom{20}{3}$ **(b)** $\binom{21}{3}$
6. **(a)** $\binom{7}{3} + \binom{7}{4}$ **(b)** $\binom{8}{4}$
7. **(a)** $\frac{5}{6} \cdot \binom{10}{5}$ **(b)** $\binom{10}{6}$
8. **(a)** $\frac{9}{4} \cdot \binom{12}{3}$ **(b)** $\binom{12}{4}$
9. **(a)** $\binom{10}{6} \cdot \binom{6}{3}$ **(b)** $\binom{10}{7} \cdot \binom{7}{3}$
10. **(a)** $\binom{12}{10} \cdot \binom{10}{4}$ **(b)** $\binom{8}{5} \cdot \binom{5}{3}$
11. **(a)** $\dfrac{10!}{7!}$ **(b)** $\dfrac{10!}{7!\,3!}$
12. **(a)** $8! + 2!$ **(b)** $10!$
13. **(a)** $\dfrac{6! + 4!}{3!}$ **(b)** $\dfrac{8! - 5!}{3!}$
14. **(a)** $6! - 3!$ **(b)** $(6 - 3)!$

Exercises 15–18 ***Calculator Evaluation*** *Use the Technology Tip to evaluate the expression.*

15. **(a)** $\binom{24}{8}$ **(b)** $\binom{37}{3} + \binom{37}{5}$
16. **(a)** $\binom{31}{5}$ **(b)** $\binom{16}{4} - \binom{12}{10}$
17. **(a)** $\binom{12}{8} \cdot \binom{20}{3}$ **(b)** $\binom{25}{7} \div \binom{25}{4}$
18. **(a)** $\binom{32}{3} \cdot \binom{31}{29}$ **(b)** $\binom{31}{8} \div \binom{31}{3}$

Exercises 19–24 Evaluate and simplify.

19. $\binom{n}{n-1}$ **20.** $\binom{n}{n-2}$ **21.** $\binom{n+1}{n-1}$

22. $\dfrac{(n+1)!}{(n-1)!}$ **23.** $\dfrac{\binom{n}{k+1}}{\binom{n}{k}}$ **24.** $\dfrac{\binom{n+1}{r}}{\binom{n}{r-1}}$

Exercises 25–30 ***Binomial Theorem*** *Use the binomial theorem formula to expand the expression, then simplify your result.*

25. $(x-1)^5$ **26.** $(x-3y)^4$

27. $\left(\frac{1}{x}-2y^2\right)^4$ **28.** $\left(x^2+\frac{2}{x}\right)^6$

29. $\left(3x+\frac{1}{x^2}\right)^5$ **30.** $(x-1)^7$

Exercises 31–34 ***Expansion*** *Use the formula in Equation (2).* ***(a)*** *Write the expansion in sigma form.* ***(b)*** *Expand and simplify.*

31. $(2-x)^5$ **32.** $\left(2x+\frac{y}{2}\right)^5$

33. $\left(x^2+\frac{2}{x}\right)^5$ **34.** $(x^2-2)^6$

Exercises 35–38 ***Number of Terms*** ***(a)*** *How many terms are there in the expansion of the given expression?* ***(b)*** *If the answer in* ***(a)*** *is odd, then find the middle term. If it is even, find the two middle terms.*

35. $(x^2-3)^8$ **36.** $\left(x^2-\frac{1}{x}\right)^{15}$

37. $(1+\sqrt{x})^5$ **38.** $(x+2\sqrt{x})^{10}$

Exercises 39–40 Find the first three terms in the expansion of

39. $\left(x+\frac{1}{x}\right)^{20}$ **40.** $\left(x-\frac{3}{x}\right)^{25}$

Exercises 41–44 ***Find Specified Term*** *If the expression is expanded using Equation (1), find the indicated term and simplify.*

41. $\left(x^3-\frac{2}{x}\right)^5$; third term

42. $\left(\frac{x}{2}-2y\right)^{12}$; tenth term

43. $\left(2x-\frac{y}{2}\right)^{10}$; fourth term

44. $(x^{-1}+2x)^8$; fourth term

Exercises 45–52 ***Specified Term*** *If the expression is expanded and each term is simplified, find the coefficient of the term that contains the given power of x. See Example 6.*

45. $\left(x^3-\frac{2}{x}\right)^4$; x^4 **46.** $\left(2x-\frac{1}{3}\right)^{10}$; x^7

47. $(x^2+2)^{11}$; x^8 **48.** $\left(x^2-\frac{2}{x}\right)^{10}$; x^8

49. $\left(x^3-\frac{1}{x}\right)^{15}$; x^{25} **50.** $\left(x^2-\frac{3}{x}\right)^{12}$; x^9

51. $(x^2-2x+1)^3$; x^4 **52.** $(x^2+4x+4)^3$; x^2

Exercises 53–60 ***Solve Equation*** *Find all positive integers n that satisfy the equation.*

53. $(2n)! = 2(n!)$ **54.** $(3n)! = (3!)(n!)$

55. $2(n-2)! = n!$ **56.** $(3n)! = 3(n+1)!$

57. $\binom{n}{3}=\binom{n}{5}$ **58.** $\binom{n}{3}+\binom{n}{4}=\binom{8}{4}$

59. $\binom{n}{2}=15$ **60.** $\binom{n}{2}=28$

61. **(a)** Show that $\binom{10}{6}+\binom{10}{7}=\binom{11}{7}$ by carrying out the following steps. Using Equation (3), express each term of $\binom{10}{6}+\binom{10}{7}$ as a fraction with factorials; then, without expanding, get a common denominator and express the result as a fraction involving factorials. By Equation (3), show that the result is equal to $\binom{11}{7}$. See Example 3.
(b) Following a pattern similar to that described in part **(a)**, prove the additivity property for the binomial coefficients

$$\binom{n}{r}+\binom{n}{r+1}=\binom{n+1}{r+1}.$$

62. By expanding the left- and right-hand sides, verify that

$$\binom{n}{k+1}=\frac{n-k}{k+1}\cdot\binom{n}{k}.$$

63. ***Explore***
Let $S_n = 1\cdot 1! + 2\cdot 2! + \cdots + n\cdot n! + 1.$
$S_1 = 1\cdot 1! + 1 = 2 = 2!$ and
$S_2 = 1\cdot 1! + 2\cdot 2! + 1 = 6 = 3!$
(a) Evaluate S_3, S_4, and S_5 and look for a pattern. On the basis of your data, guess the value of S_8. Verify your guess by evaluating S_8 directly.
(b) Guess a formula for S_n and use mathematical induction to prove that your formula is correct.

64. ***Explore*** Suppose $f_n(x) = (x+\frac{1}{x})^{2n}$ and let a_n be the middle term of the expansion of $f_n(x)$.
(a) Find a_1, a_2, a_3, and a_4.
(b) Guess a formula for the general term a_n. Is $a_n = \dfrac{(2n)!}{n!\,n!}$?