**3.4 RATIONAL FUNCTIONS**

What is mathematics about? I think it’s really summed up in what I frequently tell my classes. That is that proofs really aren’t there to convince you that something is true—they’re there to show you why it is true. That’s what it’s all about—it’s to try to figure out how it’s all tied together.

Freeman Dyson

In this section we consider **rational functions**, which are defined as **quotients of polynomial functions**.

**Definition: rational functions**

Suppose \( p \) and \( q \) are polynomial functions, where \( q \) is not the zero function. Then the function \( f \) given by

\[
 f(x) = \frac{p(x)}{q(x)}
\]

is called a **rational function**, sometimes written \( f = \frac{p}{q} \). The domain of \( f \) consists of all real numbers for which \( q(x) \neq 0 \).

If \( p(x) \) and \( q(x) \) have no common factors, then we say that \( \frac{p}{q} \) is in **reduced form**.

For our discussion in this section, we assume that all rational functions are reduced unless we specify to the contrary. The most significant information for analyzing the behavior of a rational function is the set of zeros of the polynomial functions in the numerator and the denominator.

To get a feeling for the meaning of zeros in the denominator, we look at a variety of graphs. In Example 1 we consider pairs of graphs, each consisting of a polynomial and its reciprocal.

**Example 1: Functions and reciprocals**

Sketch the graphs of both functions, together with the horizontal lines \( y = 1 \) and \( y = -1 \), in the specified window. Then describe where the graphs of \( f \) and \( g \) meet, and where the graph of \( g \) goes off-scale (out of sight in the window).

(a) \( f(x) = x^2 - 2x \), \( g(x) = \frac{1}{x^2 - 2x} \). Decimal window

(b) \( f(x) = x^3 - 3x^2 \), \( g(x) = \frac{1}{x^3 - 3x^2} \). Change \( y \)-range to \([-5, 5]\)

**Solution**

(a) The graph of \( f \) is the solid parabola in Figure 23, and \( y = g(x) \) is the dotted curve. It appears that the graphs of \( f \) and \( g \) meet where \( f(x) = \pm 1 \). The graph of \( g \) appears to go off-scale at \( x = 0 \) and \( x = 2 \); that is, where \( f(x) = 0 \).

(b) In Figure 24a, again the graphs of \( f \) and \( g \) meet where \( f(x) = \pm 1 \). The graph of \( g \) appears to go off-scale at \( x = 0 \). It is less clear what happens to \( g \) near \( x = 3 \) (the other point where \( f(x) = 0 \)), so we look closer by reducing the \( x \)-range to \([2, 4]\) (Figure 24b). Now we still see the intersections where
f(x) = ±1, but the graph of g clearly goes off-scale as x nears 3. Your calculator may show a nearly vertical column of pixels near 3, this is a "false (calculator) asymptote." It occurs because the calculator connects widely separated y-values in adjacent columns, but it is not part of the graph of g.

**Rational functions and parentheses**

In graphing rational functions on a calculator, one of the most common errors involves the use of parentheses. We are so accustomed to using the fraction bar as a separator that we can get careless with our calculator. In Example 1(a),

\[ g(x) = \frac{1}{x^3 - 3x^2 - x + 2}. \]

On the calculator we could enter

\[ Y_1 = \frac{1}{x^3 - 2x} \text{ or } Y_2 = \frac{1}{x(x - 2)} \text{ or } Y_3 = \frac{1}{x^3(x - 2)} \]

but NOT \[ Y_4 = 1/3x^2 - 2x \text{ or } Y_5 = 1/x^3(x - 2). \]

Every calculator is programmed to handle operations differently. Try each of the above on your calculator and make sure you know how to get a graph that looks like Figure 23.

**Vertical Asymptotes**

The kind of “off-scale” behavior we observed in the calculator graphs in Example 1 deserves closer examination. When we plot \( y = 1/x \) in a decimal window (Figure 25), the graph goes off-scale in both directions, with the graph seeming to get closer and closer to the y-axis. If, however, we increase the y-range to \([-15, 15]\), the calculator graph climbs up the y-axis and then stops; the highest point is \((1/10, 10)\).

Does the graph really stop at \((1/10, 10)\), or does it keep climbing? We could see more by zooming in near \((0, 10)\), but the calculator is not the best tool for answering the question. Looking at the equation \( y = 1/x \), it is easy to see that we can get a y-value of 100 (at \( x = 1/100 \)), or 1000, or ten million, by taking x-values small enough. There is no highest point. Taking positive x-values closer and closer to 0, the y-values keep growing without bound.

We use arrows to describe such behavior and write: as \( x \to 0^+ \), \( y \to \infty \) (or \( 1/x \to \infty \)), or more compactly, \( \lim_{x \to 0^+} \frac{1}{x} = \infty \).
Arrow notation

\[ x \rightarrow a^+ \text{ means that } x \text{ approaches } a \text{ from above; that is, } x \text{ takes on values near } a \text{, but greater than } a \text{ (such as } a + 0.01, a + 0.001, \ldots). \]

\[ x \rightarrow a^- \text{ means that } x \text{ approaches } a \text{ from below; that is, } x \text{ takes on values near } a \text{, but less than } a \text{ (such as } a - 0.01, a - 0.001, \ldots). \]

Similarly, \( x \rightarrow \infty \) or \( x \rightarrow -\infty \) means that \( x \) assumes larger and larger positive or negative values, respectively. The same notation is used to indicate functional behavior.

In calculus the concept of limit has a very important, precise meaning. Here we use the notation only for the intuitive notion embodied in our arrows. Looking back at Figure 23, we see the following:

\[ \lim_{x \to 0^+} g(x) = \infty, \quad \lim_{x \to 0^-} g(x) = -\infty, \]
\[ \lim_{x \to \infty} g(x) = -\infty, \quad \lim_{x \to -\infty} g(x) = \infty. \]

The vertical lines \( x = 0 \) (the \( y \)-axis) and \( x = 2 \) are called \textbf{vertical asymptotes} for the curve \( y = g(x) \). Without attempting a more precise definition, we say that a line is an \textbf{asymptote} for a curve if the distance between the curve and the line goes to zero as we move out along the line.

From the graphs in Example 1, it is clear that each reciprocal function \( 1/f(x) \) has a vertical asymptote at each zero of \( f(x) \), and furthermore, that the \( x \)-axis is a \textbf{horizontal asymptote} for each reciprocal function since \( g(x) \to 0 \) as \( x \to \infty \) and \( g(x) \to 0 \) as \( x \to -\infty \).

Asymptotes for rational functions can be vertical, horizontal, or oblique lines, as illustrated in Figure 26.

**FIGURE 26**

Shifts and reflections are also useful in graphing rational functions.

\[ y = \frac{1}{x - 1} \]
\[ y = \frac{2}{2 - x} \]
\[ y = \frac{1 + x}{x} \]

**Strategy:** Try to relate each function to \( f(x) = \frac{1}{x} \). In (a) \( f(x - 1) \) gives a horizontal translation. In (b) factor out \(-2\) to get \(-2 \cdot f(x - 2)\). In (c) \( y = \frac{1}{x} + 1 \), for a vertical translation.

**EXAMPLE 2** **Shifts and reflections**

Use the graph of \( f(x) = \frac{1}{x} \) to graph

\[ \text{(a) } y = \frac{1}{x - 1}, \quad \text{(b) } y = \frac{2}{2 - x}, \quad \text{(c) } y = \frac{1 + x}{x} \]

**Solution**

(a) Since \( f(x - 1) = \frac{1}{x - 1} \), graph \( y = \frac{1}{x - 1} \) by translating the graph of \( f \) one unit to the right. As a useful check, observe that \( y = \frac{1}{x - 1} \) has a vertical asymptote where the denominator is 0, at \( x = 1 \). The result of the translation is shown in Figure 27a.
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(b) If we factor out \(-1\) from the denominator, then \(\frac{2}{x-2} = \frac{2}{x-2}\), so \(\frac{2}{x-2} = -2f(x - 2)\). Translate the graph of \(f\) two units to the right, reflect it through the \(x\)-axis and stretch it vertically by a factor of 2. Plotting a few points gives the graph shown in Figure 27b.

(c) To relate \(\frac{1 + x}{x}\) to \(f(x)\), rewrite \(\frac{1 + x}{x}\) as

\[
\frac{1 + x}{x} = \frac{1}{x} + \frac{x}{x} = \frac{1}{x} + 1 = f(x) + 1.
\]

Translate the graph of \(f\) one unit up, as shown in Figure 27c.

**Graphing Other Rational Functions**

All of the rational functions we have graphed thus far are either reciprocals or shifts of reciprocals of polynomial functions, where we have observed that there is a \textit{vertical asymptote at every zero of the denominator}. What is the significance of the zeros of the numerator? Suppose \(f(x) = \frac{p(x)}{q(x)}\) and \(f\) is in reduced form (so that \(p\) and \(q\) have no common zeros). Then if \(c\) is a zero of the numerator, we have

\[
f(c) = \frac{p(c)}{q(c)} = \frac{0}{q(c)} = 0.
\]

That is, \textit{the zeros of a rational function are the zeros of the numerator}. Collectively, we call the zeros of the numerator and the zeros of the denominator the set of \textit{cut points} for a rational function. As in Chapter 1, cut points identify the location of possible sign changes. In addition, for rational functions, cut points in the numerator identify \(x\)-intercept points, cut points in the denominator correspond to actual breaks in the graph, at each vertical asymptote. Summing up, we have the following.

**Vertical asymptotes and intercepts**

Suppose \(f(x) = \frac{p(x)}{q(x)}\) and that \(p\) and \(q\) have no common zeros.

Then there is a \textit{vertical asymptote} at every zero of the denominator, and there is an \textit{x-intercept point} at every zero of the numerator.
EXAMPLE 3  Graphing a rational function  Find the intercepts, cut points and asymptotes and sketch the graph of the rational function

\[ f(x) = \frac{x + 1}{4x(x - 1)}. \]

Solution

The zeros of the denominator are 0 and 1, so the vertical asymptotes are the lines \( x = 0 \) (the y-axis) and \( x = 1 \). The other cut point is the single zero of the numerator, \(-1\), and so we have only one \( x \)-intercept point, \((-1, 0)\). Since the y-axis is a vertical asymptote, there is no \( y \)-intercept. A calculator graph in a decimal window (Figure 28a) makes it appear that the graph ends near \((-1, 0)\), but of course we know that the domain of \( f \) consists of all real numbers except 0 and 1. We also know that the numerator, and hence \( f \), changes sign at \( x = -1 \). Tracing along the curve near \((-1, 0)\) or zooming in near the same point verifies that the graph crosses the \( x \)-axis there, and the \( x \)-axis is a horizontal asymptote, as in Figure 28b.

Horizontal and Slant Asymptotes

In all of the examples we have graphed thus far, the \( x \)-axis has been a horizontal asymptote, with the exception of one translated graph. The general principle that we list below is illustrated in all of these examples. What happens as \( x \to \infty \) or \( x \to -\infty \) depends on the degrees of the numerator and denominator.

**Horizontal and slant asymptotes**

\[ f(x) = \frac{ax^n + \cdots}{bx^m + \cdots} \]

If the degree of the denominator is larger \( (m < n) \),

then the \( x \)-axis is a **horizontal asymptote**.

If the degrees are equal \( (m = n) \),

then the line \( y = \frac{a}{b} \) is a **horizontal asymptote**.

Use long division to add additional information.

If the degrees differ by 1 \( (m = n + 1) \),

then there is a **slant (oblique) asymptote**.

Use long division to find an equation for the asymptote.

EXAMPLE 4  Equal degrees  Graph the rational function

\[ f(x) = \frac{x^2 - 4}{x^2 + 2}. \]

Solution

We note first that \( f(-x) = f(x) \), so \( f \) is an even function; the graph is symmetric about the \( y \)-axis. Setting the numerator equal to 0, we find that its zeros are \( \pm 2 \), so the \( x \)-intercept points are \((2, 0)\) and \((-2, 0)\). The denominator has no real zeros, so the graph has no vertical asymptotes. Since \( f(0) = -2 \), the \( y \)-intercept point is \((0, -2)\).
The degree of both numerator and denominator is 2, so \( m = n \). Thus the line \( y = \frac{x}{2} = \frac{1}{1} = 1 \) is a horizontal asymptote. The graph is shown in Figure 29. Note that without the boxed information above, it would be difficult to tell from a calculator graph that the graph really does flatten out along the line \( y = 1 \), although increasing the \( x \)-range makes that conclusion very plausible.

\[ g(x) = \frac{x^2 + 2x - 7}{x^2 - 2x - 3}. \]

**Solution**

The zeros of the denominator are easy to find because the denominator factors: \((x - 3)(x + 1)\). There are zeros, and hence vertical asymptotes at \( x = -1 \) and \( x = 3 \). The numerator has zeros at \( x = 1 \pm 2\sqrt{2} \), so we have two \( x \)-intercept points. \( g(0) = \frac{1}{4} \), so the \( y \)-intercept point is \((0, \frac{1}{4})\).

The degree of both numerator and denominator is 2, so \( m = n \). Thus the line \( y = \frac{x}{2} = \frac{1}{1} = 1 \) is a horizontal asymptote. If we use long division, as suggested in the box above, we have

\[
g(x) = 1 + \frac{4x - 4}{x^2 - 2x - 3} = 1 + \frac{4(x - 1)}{(x - 3)(x + 1)}.
\]

In this form, it is easy to see that \( g(x) = 1 \) when \( x = 1 \), so the graph crosses the horizontal asymptote at the point \((1, 1)\). Graphing in a decimal window doesn’t show much of the asymptotic behavior. To see a little more, we zoom out by a factor of 2 to get the graph shown in Figure 30.

**Intersections of Graphs and Asymptotes**

Because graphs get so close to asymptotes it is sometimes difficult to look at a graph and tell whether or not it crosses an asymptote. Calculator graphs are particularly difficult to read in some areas where we most need detail. In Example 3 the graph crosses the horizontal asymptote at the \( x \)-intercept point \((-1, 0)\) and in Example 5 the graph crosses the horizontal asymptote as well. When numerator and denominator have equal degrees, long division gives us a form from which we can use algebraic techniques to find intersections. On the other hand, no graph can cross a vertical asymptote. If \( x = c \) is a vertical asymptote for a rational function \( f \), then there would have to be a point \((c, f(c))\) on the graph, but vertical asymptotes occur at zeros of the denominator, where by definition, \( f \) is undefined.

**Slant Asymptotes**

From the box above, slant (oblique) asymptotes occur when the degree of the numerator is 1 greater than the degree of the denominator, as in the next example.

\[ h(x) = \frac{x^2 + 2x}{2x + 2}. \]
Solution

Both numerator and denominator can be factored: \( \frac{x(x + 2)}{2(x + 1)} \). Thus the cut points are 0, -2, and -1. The first two give us \( x \)-intercept points (0, 0) and (-2, 0); the zero in the denominator indicates a vertical asymptote at \( x = -1 \). A calculator graph in \([-4, 4] \times [-4, 4]\) shows the graph near the vertical asymptote (Figure 31a), but it isn’t clear what happens to the graph as we move further to the right or the left. If we zoom out, essentially looking at the graph from further away, say in the window \([-10, 10] \times [-6, 6]\), the graph looks very much like a line, except near the vertical asymptote. What line does it approach? The slant asymptote.

The numerator has degree 2 and denominator degree 1, so there is a slant asymptote. Dividing \( x + 1 \) into \( x^2 + 2x \) using long division, we get

\[
h(x) = \frac{1}{2}(x + 1) - \frac{1}{2(x + 1)}.
\]

The slant asymptote is the line \( y = (x + 1)/2 \). Adding that line to the graph in the \([-10, 10] \times [-6, 6]\) window shows that the two graphs are indistinguishable except in the region near the vertical asymptote.

As with horizontal asymptotes, we want to know whether the graph of \( y = h(x) \) intersects the oblique asymptote. Such an intersection would come from a solution to the equation \( h(x) = (x + 1)/2 \), which clearly has no solution. The graph is shown in Figure 31b.

Rational Functions Not Reduced

A rational function \( f(x) = \frac{p(x)}{q(x)} \) is not in reduced form if there are common factors in the numerator and denominator. To handle common zeros, remember that if \( p(x) \) and \( q(x) \) have the same zero, say \( x = c \), then since \( \frac{p}{q} \) is not defined, \( c \) is not in the domain of the function. In such a case the graph has a single point removed. Consider the function

\[
f(x) = \frac{x^2 - 1}{x - 1}.
\]
Factor the numerator.

\[ f(x) = \frac{(x - 1)(x + 1)}{x - 1}. \]

This is identical to the function \( g(x) = x + 1 \) except when \( x = 1 \). When \( x = 1 \), \( g(1) = 2 \), but \( f(1) \) is not defined. Therefore, the graph of \( y = f(x) \) is the same as the graph of \( y = x + 1 \) with the point \( (1, 2) \) removed (see Figure 32).

**TECHNOLOGY TIP**

**Missing points**

In graphing nonreduced rational functions, we may or may not be able to see that a point is missing. Such a gap is visible on a calculator graph if and only if there is a pixel with \( x \)-coordinate corresponding to value where the function has the form \( \frac{0}{x} \).

Thus for the example considered above, \( f(x) = \frac{(x^2 - 1)(x - 1)}{x^2 - x + 1} \), in any window where there is a pixel for \( x = 1 \), such as a decimal window, the graph makes the missing point quite apparent. If you then change the \( x \)-range by almost any small amount, you will no longer be able to see where the point is missing; the graph looks just like the line \( y = x + 1 \).

**EXERCISES 3.4**

**Check Your Understanding**

Use a graph whenever you think it will be helpful. True or False. Give reasons.

1. If the graph of \( y = \frac{1}{x} \) is translated two units left, then the resulting graph will be that of \( y = \frac{1}{x - 2} \).

2. If the graph of \( y = \frac{1}{x^2 + 2} \) is translated down one unit, then the resulting graph will be that of \( y = -\frac{x - 1}{x + 2} \).

3. If the graph of \( y = \frac{1}{x^2 + 2} \) is translated up one unit, then the resulting graph will be that of \( y = \frac{x + 1}{x + 2} \).

4. The line \( y = \frac{x}{2} \) is an asymptote to the graph of \( y = \frac{x + 1}{2x + 1} \).

5. The horizontal line \( y = -2 \) is an asymptote to the graph of \( y = \frac{1 - 2x^2}{5 + 2x + x^2} \).

6. The graph of \( y = -\frac{x - 2}{x^2 - x + 2} \) has no vertical asymptotes.

**Exercises 7–8** Suppose \( f(x) = \frac{3x^2 + 1}{x^2 + 1} \).

7. There is no value of \( x \) for which \( f(x) = 3 \).

8. For every real number \( x \), \( f(x) \) is in the interval \([-1, 3]\).

9. If \( f(x) = \frac{x^2 + 100}{x} \), then the graph of \( f \) has a local minimum point in the third quadrant.

10. The graph of \( f(x) = \frac{2x^4 - 3x^2 + 500}{x^4 + 8x + 50} \) has one zero and no vertical asymptotes.

**Develop Mastery**

1. The line \( x = 1 \) is a vertical asymptote for the function \( f(x) = \frac{x^3}{x^3 - 2x + 1} \).

   (a) To see what happens to the graph of \( f \) as \( x \to 1^- \), evaluate \( f \) at \( x = 0.8, 0.9, 0.99, \) etc. For what values of \( x \) is \( f(x) \) greater than 100? Greater than 1000? (b) Repeat part (a) as \( x \to 1^+ \).

2. The line \( y = 1 \) is a horizontal asymptote for the function \( f(x) = \frac{x^2 + 2x}{x^2 + 1} \).
Graph Rational Functions

Exercises 10–13 Related Graphs Graph \( f(x) = \frac{2x^2 + 1}{x^2 + 1} \) and the given function \( g \) simultaneously. (a) Determine how the graphs are related and verify algebraically. (b) Find the coordinates of any intersection points (1 decimal place).

10. \( g(x) = \frac{4x^2 + 2}{x^2 + 1} \)
11. \( g(x) = -\frac{2}{x^2 + 1} \)
12. \( g(x) = \frac{2x^2 + 4x + 2}{x^2 + 2x + 2} \)
13. \( g(x) = \frac{2x^2 - 8x + 8}{x^2 - 4x + 5} \)

Exercises 14–21 Graph Rational Functions Sketch a graph of \( f \), identifying asymptotes and intercepts.

14. \( f(x) = \frac{1}{x - 1} \)
15. \( f(x) = \frac{2}{x + 2} \)
16. \( f(x) = \frac{x}{x - 1} \)
17. \( f(x) = \frac{2x - 3}{x + 2} \)
18. \( f(x) = \frac{2x^2}{(x - 1)^2} \)
19. \( f(x) = \frac{x^2}{x^2 - 4} \)
20. \( f(x) = \frac{x + 2}{x^2 - 3x - 4} \)
21. \( f(x) = \frac{x}{x^2 - 2x + 1} \)

Exercises 22–28 Intercepts, Domain, Range Draw a graph and use it to find (a) the x-intercept points, (b) the domain of \( f \), (c) the range of \( f \).

22. \( f(x) = \frac{2}{x + 2} \)
23. \( f(x) = \frac{2}{x^2 + 1} \)
24. \( f(x) = \frac{2x}{x^2 + 1} \)
25. \( f(x) = \frac{x^2 + 2}{x^2 + 1} \)
26. \( f(x) = \frac{4}{x^2 - 3x - 4} \)
27. \( f(x) = \frac{x - 2}{x^2 - 3x - 4x} \)
28. \( f(x) = \frac{2x - 4}{x^2 - 3x - 4x} \)

Exercises 29–32 Increasing, Decreasing Determine the intervals on which \( f \) is (a) increasing, (b) decreasing.

29. \( f(x) = \frac{x^2 + 4x + 3}{2x + 4} \)
30. \( f(x) = \frac{x^2 - 2x - 3}{x^2 - x - 2} \)
31. \( f(x) = \frac{x - 1}{x^2 - x - 2} \)
32. \( f(x) = \frac{4}{x^2 + 2x - 3} \)

Exercises 33–36 Solution Set Find the solution set algebraically.

33. \( \frac{2}{x} - \frac{4}{x - 2} = 6 = 0 \)
34. \( \frac{x^2 - 6x + 5}{x^3 + 3} = 3 \)
35. \( \frac{x^3 - 4x}{x^2 + 1} = \frac{3}{2} \)
36. \( \frac{2x}{x + 1} + \frac{4}{2x + 1} = \frac{14}{2x^2 + 3x + 1} \)

Exercises 37–40 Solution Set (a) Find the solution set algebraically. (b) Use a graph to support your answer.

37. \( \frac{1}{x + 1} > 2 \)
38. \( \frac{x - 3}{x + 2} < 0 \)
39. \( \frac{2x^2 + x - 3}{x^2 + 1} \leq 0 \)
40. \( \frac{x^2 - 3x - 4}{x^2 + 2x + 1} \geq 0 \)

Exercises 41–44 Not Reduced Sketch a graph of \( f \). (Hint: First express the function in reduced form; keep in mind the domain.)

41. \( f(x) = \frac{x}{x^2 + 2x} \)
42. \( f(x) = \frac{x^3}{x^2 - 2x} \)
43. \( f(x) = \frac{x^2 - 2x + 1}{x^2 + 2x - 3} \)
44. \( f(x) = \frac{x^2 + 2x - 3}{x^2 + x - 2} \)

Exercises 45–48 Find an equation for the horizontal asymptote and find the coordinates of the points (if any) where the graph of \( y = f(x) \) intersects the horizontal asymptote.

45. \( f(x) = \frac{x^2 - 6}{x^2 - 2x} \)
46. \( f(x) = \frac{2x^2 - 2}{x^2 - 3x + 2} \)
47. \( f(x) = \frac{x^2 - x}{x^2 + x + 2} \)
48. \( f(x) = \frac{x^2 - x}{x^2 - x - 2} \)
Exercises 49–52  Oblique Asymptotes  The graph of f has an oblique asymptote. Find an equation for the asymptote and find the coordinates of the points (if any) where the graph of y = f(x) intersects the asymptote. Graph the function.

49. \( f(x) = \frac{x^2 - 3}{x - 1} \)  
50. \( f(x) = \frac{x^2 + x - 2}{x + 1} \)  
51. \( f(x) = \frac{2x^2 + 3x + 1}{x} \)  
52. \( f(x) = \frac{x^2 + 3x - 2}{x + 1} \)

Exercises 53–54  Minima  Find the minimum value of f algebraically and check graphically. What value of x gives the local minimum point(s) on the graph of f?

53. \( f(x) = \frac{x^4 + x^2 + 1}{x^2} \)  
   Hint: \( f(x) = x^2 + 1 + \frac{1}{x^2} = \left( x - \frac{1}{x} \right)^2 + 3 \)
54. \( f(x) = \frac{x^4 + 2x^2 + 4}{x^2} \)  
   Hint: See Exercise 53.

Exercises 55–56  Your Choice  Give a formula for a rational function whose graph satisfies the given conditions. Check with a graph.

55. x-intercept point (2, 0), vertical asymptote \( x = -1 \), horizontal asymptote \( y = 2 \).
56. x-intercept points \( (-2, 0), \) vertical asymptote \( x = 1 \), horizontal asymptote \( y = 2 \).

Exercises 57–58  Local Maxima  Find the coordinates of the local maximum point(s) on the graph of f.

57. \( f(x) = \frac{x^2 - 7x + 16}{x - 3} \)  
58. \( f(x) = \frac{x^4 + x + 1}{x} \)

Exercises 59–60  Local Minima  Find the coordinates of the local minimum point(s) on the graph of f.

59. \( f(x) = \frac{x^2 + 5x + 7}{x + 2} \)  
60. \( f(x) = \frac{x^3 - x + 4}{x} \)

Exercises 61–68  Match Functions  Match the graph with the appropriate function from the following list. Check by graphing.

(a) \( f(x) = \frac{x + 1}{x - 1} \)  
(b) \( f(x) = \frac{1}{x - 1} \)  
(c) \( f(x) = \frac{x + 1}{(x - 1)^2} \)  
(d) \( f(x) = \frac{x^2}{x^2 + 1} \)  
(e) \( f(x) = \frac{1}{1 - x} \)  
(f) \( f(x) = \frac{x^3}{x^2 - x - 2} \)  
(g) \( f(x) = \frac{x}{x^2 - x - 2} \)  
(h) \( f(x) = \frac{x^3}{x^2 - x - 2} \)

Exercises 69–72  Solution Set  Functions g and h are given and function f is defined by \( f(x) = \frac{g(x)}{h(x)} \). Find the solution set for \( f(x) < 0 \) algebraically. Draw a graph to support your answer. (Hint: First show that \( h(x) > 0 \) for every value of \( x \) (draw a graph).) Why is the solution set for \( f(x) < 0 \) the same as the solution set for \( g(x) < 0 \)?

69. \( g(x) = x^3 - 2x - 3, \quad h(x) = x^3 - 2x + 3 \)
70. \( g(x) = x^3 - 2x^2 - 8x, \quad h(x) = x^3 - 3x^2 + 4 \)
71. \( g(x) = x^3 + 2x^2 - x - 2, \quad h(x) = x^3 + 2x + 2 \)
72. \( g(x) = 1 - x^3, \quad h(x) = 1 + x^2 \)
73. For \( f(x) = \frac{2x^3 + 3x^2 + x - 2}{x^2} \),
   (a) show that \( y = 2x + 3 \) is an oblique asymptote.
   (b) Draw graphs of \( y = f(x) \) and \( y = 2x + 3 \) simultaneously and see that the graph of \( y = f(x) \) is approaching the asymptote as \( x \to \infty \). Is it approaching from above or from below?
   (c) Does the graph of \( y = f(x) \) intersect the asymptote? If it does, find the point of intersection algebraically.
   (d) Find some values of \( x \) for which the difference of the two \( y \) values is less than 0.01.
74. Solve the same problem as in Exercise 72 except use \( f(x) = \frac{x^3 - 3x^2 + 2x - 1}{x^2 + 1} \). First find an equation for the oblique asymptote.
75. Show that if \( g \) and \( h \) are polynomial functions with no common zeros and the degree of \( h \) is 3, then the function \( f(x) = \frac{g(x)}{h(x)} \) (lowest terms) must have at least one vertical asymptote.
76. (a) Function \( f(x) = \frac{x - 3}{|x| + 2} \) is not a rational function, why?
   (b) Draw a graph and see that the graph \( f \) has two horizontal asymptotes.
   (c) By considering two cases, \( x \geq 0 \), and \( x < 0 \), find equations for the two horizontal asymptotes.

Enquiries 77–78. Intercepts and Asymptotes. Does the graph of \( f \) have (a) x-intercept points? (b) Any vertical asymptotes? (Hint: Draw graphs of the numerator and denominator separately.)
77. \( f(x) = \frac{x^2 - 2x + 5}{x^2 + 3x - 4} \)
78. \( f(x) = \frac{x^3 + x - 1}{x^2 - 3x + 4} \)
79. Of all rectangles with an area of 160 square inches, what are the dimensions of the one having the smallest perimeter?
80. Solve the problem in Exercise 79 for a rectangle of area 240 square inches.

CHAPTER 3 REVIEW

Test Your Understanding

True or False. Give reasons.
1. \( F(x) = x^2 + x^{-1} + 1 \) is a polynomial function of degree –2.

2. The equation \( x^3 + x + 1 = 0 \) has no positive roots.
3. The equation \( x^3 + x - 1 = 0 \) has no negative roots.
4. The equation \( x^3 + x^2 - 1 = 0 \) has no positive roots.

81. A cylindrical can is to contain 48 ounces (87 cubic inches) of apple juice. If the can is to use the least amount of tin, what should the radius and the height be?
82. Suppose \( x \) ounces of pure acid are added to 50 ounces of 40% solution of acid. Let \( u \) denote the concentration (percent) of the resulting solution.
   (a) Express \( u \) as a function of \( x \).
   (b) Why cannot \( u \) be 100 or greater?
   (c) What is the domain of this function?
   (d) How many ounces of acid must be added to get a 65% solution? Support your answer by drawing a graph and finding the value of \( x \) that gives 65 for \( u \).
83. Find the positive number (2 decimal places) such that the sum of its square and its reciprocal is a minimum.
84. A rectangular box is to have a base whose length is twice its width, and whose volume is 2460 cubic inches. Of all such boxes, what are the dimensions (1 decimal place) of the one that will require the least amount of material if the box has (a) a top? (b) no top?
85. Solve the problem in Exercise 84 if the length of the base is three times its width.

Exercises 86–89. Applied Minima
86. A rectangular printed page is to have margins 2 inches wide at the top and bottom and 1 inch wide on each of the two sides. If the page is to have 60 square inches of printed material,
   (a) What is the minimum possible area of the page?
   (b) What are the dimensions of the page?
87. A factory has a fixed daily overhead cost of $600. If it produces \( x \) units daily, then the cost for labor and materials is \( 3x \) dollars. The daily cost of equipment maintenance is \( \frac{x^2}{240000} \) dollars.
   (a) Find a function giving the total daily cost, \( c(x) \), when \( x \) units are produced.
   (b) How many units should be produced each day to minimize the cost per unit \( \left( \text{minimize} \frac{c(x)}{x} \right) \) ? (Hint: \( x > 10000 \).
88. The x-axis, y-axis and any line with negative slope passing through the point \( P(3, 5) \) determine a triangle. Of all such triangles, determine the line for which the area of the triangle is a minimum.
89. Solve the problem in Exercise 88 if \( P \) is the point (5, 4).