Section 3.1 begins with basic definitions and graphical concepts and gives an overview of key properties of polynomial functions. In Sections 3.2 and 3.3 we consider zeros in exact form, including some of the classical theorems, while learning something about approximations to zeros as well. The final section of the chapter builds on this material to define and discuss rational functions, or quotients of polynomials.

3.1 POLYNOMIAL FUNCTIONS

> I knew formulas for the quadratic and the cubic, and they said there was a subject called Galois theory, which was a general theory giving conditions under which any equation could be solved. That there could be such a thing was beyond my wildest comprehension!
>
> *Paul Cohen*

In earlier courses you learned that expressions such as

$$x^2 + 2x - 1, \qquad x^3 + 3x, \qquad -x^5 + 3x - 8.$$

are called polynomials. The following are not polynomials:

$$\frac{1-x}{x}, \qquad 2x^{-2} + 3x, \qquad |x| - 4, \qquad 4^x + 5.$$

We stated above that polynomials are functions built up as products of linear and quadratic functions. Unfortunately, polynomials seldom appear in real-world applications in factored form. Much of our work, in fact, will be devoted to finding the factors from which a given polynomial is constructed. Accordingly, we begin with the more standard definition.

Definition: polynomial function

A **polynomial function** of degree n is a function that can be written in the form

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \qquad \textbf{(1)}$$

where n is a nonnegative integer, $a_n \neq 0$, and $a_n, a_{n-1}, \ldots, a_1, a_0$ are numbers called coefficients. This course assumes that all coefficients are real numbers. The **leading term** is $a_n x^n$, the **leading coefficient** is a_n, and a_0 is the **constant term.** Equation 1 is the **standard form** for a polynomial function.

> *When I was thirteen, . . . I needed an emergency operation for appendicitis. I read two books in hospital. One was Jerome's* Three Men in a Boat, *and the other was Lancelot Hogben's* Mathematics for the Million. *Some of it I couldn't understand, but much of it I did. I remember coming across the idea of dividing one polynomial by another. I knew how to multiply them together, but I had never divided them before. So every time my father came to visit me in hospital he brought some more polynomials that he'd multiplied out.*
>
> Robin Wilson

It should be obvious from the definition that the domain of every polynomial function is the set of all real numbers. We already know about polynomial functions of degree 2 or less.

Degree 0: $f(x) = k,\ k \neq 0$ (constant function; the graph is a horizontal line).

Degree 1: $f(x) = ax + b$ (linear function; the graph is a line).

Degree 2: $f(x) = ax^2 + bx + c$ (quadratic function; the graph is a parabola).

For technical reasons, the zero polynomial function, $f(x) = 0$, is not assigned a degree.

Combining Polynomial Functions

It is appropriate to ask how the usual operations on functions apply to polynomial functions. What about sums, differences, products, quotients, or composition? All of these except quotients are also polynomials. The quotient of two polynomial functions is never a polynomial function unless the denominator is a constant function.

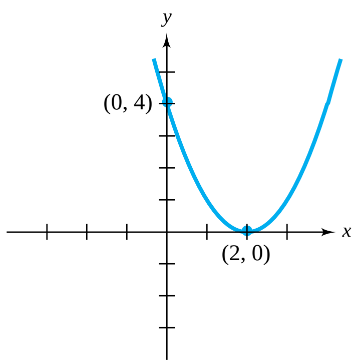
Products, Zeros, Roots, and Graphs

One consistent concern with polynomials, as in the work we did with many functions in Chapter 2, is locating their *zeros,* finding the *x-intercept* points of graphs, or finding *roots* of a polynomial equation. Every zero of a polynomial function is associated with a **factor** of the polynomial. The equivalence of these concepts for polynomials is summed up in the following box.

Roots, zeros, factors, and intercepts

Let p be a polynomial function and suppose that a is any real number for which $p(a) = 0$. Then the following are equivalent statements:

a is a **root** of the equation $p(x) = 0$	a is a **zero** of the polynomial function p
$(x - a)$ is a **factor** of the polynomial $p(x)$	$(a, 0)$ is an ***x*-intercept point** of the graph of p



$f(x) = (x - 2)^2$

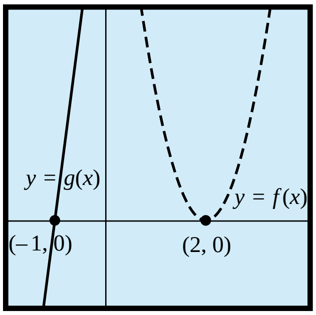
FIGURE 1

We now want to compare the graphs of some simple functions with what we get if we take their products. We know that the graph of $f(x) = (x - 2)^2$ is a core parabola shifted 2 units right. The graph of f touches the x-axis at only one point, (2, 0). See Figure 1. Since the equation $(x - 2)(x - 2) = 0$ has two solutions (by the zero product principle), we say that f has a **repeated zero** or a **zero of multiplicity two** at $x = 2$.

Just as a polynomial function can be built up as a product of linear and quadratic factors, its graph can be built up in a similar fashion. To take a simple example, consider $F(x) = (x + 1)(x - 2)^2$. The zeros of F are clearly 2 (repeated) and -1. When we take values of x near 2, the factor $x + 1$ is near 3, and so we might expect the graph of F to approximate the graph of $y = 3(x - 2)^2$. The same kind of reasoning suggests that the graph of F near -1 should be something like the graph of $y = (x + 1)(-3)^2 = 9(x + 1)$. That this reasoning is valid is borne out in Example 1. To look more closely at a particular point, we may wish to zoom in.

TECHNOLOGY TIP ◆ Zooming in on a point

On many calculators, when we press the ZOOM IN option, we get a cursor that we move to the desired location and ENTER. On HP calculators, to zoom in on some point other than the screen center, we must first redraw the graph with the point at the center of the display window by moving the cursor to the desired point and pressing CNTR from the ZOOM menu. Then, still in the ZOOM menu, press ZIN.

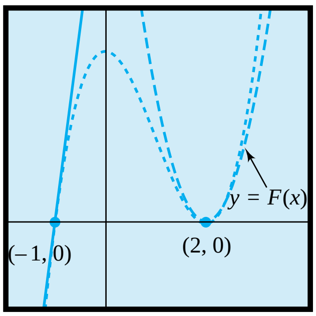


[−2, 4] by [−2, 5]

$f(x) = 3(x - 2)^2$

$g(x) = 9(x + 1)$

(a)



[−2, 4] by [−2, 5]

$F(x) = (x + 1)(x - 2)^2$

(b)

FIGURE 2

▶EXAMPLE 1 Products and zeros

(a) Graph $f(x) = 3(x - 2)^2$ and $g(x) = 9(x + 1)$ in the same window.

(b) Add the graph of the product, $F(x) = (x + 1)(x - 2)^2$. Zoom in on the point (2, 0) and on (−1, 0). Describe in words the behavior of the product function near its zeros.

Solution

(a) The graphs look like the diagram in Figure 2a.

(b) When we add the graph of F, we get the diagram in Figure 2b. When we zoom in on the point (2, 0), we see two curves that are barely distinguishable. Both graphs are tangent to the x-axis at the point (2, 0). We can see the tangent behavior more clearly if we zoom in again (or several times), but the graphs are so nearly identical near (2, 0) that we see only one.

Returning to the decimal window and zooming in on the point (−1, 0), we see a graph (just one) that looks like a fairly steep line. Tracing, we can tell that the graphs of g and F are not identical, but they are remarkably close.

The graph of the product function near each of its zeros appears to be very closely approximated by the graph of a constant times one of the factors of F, in particular, the factor of F which shares that zero. ◀

The kind of functional behavior we observed in Example 1 is typical of products, an observation we sum up in the following.

Graphs of products near zeros

Let $F(x) = f(x)g(x)$ and suppose that a is a zero of F, where $f(a) = 0$ and $g(a) \neq 0$. Then near $(a, 0)$,

the graph of the product function F looks very much like the graph of $y = Af(x)$, where A is the constant given by $A = g(a)$.

▶EXAMPLE 2 Products, zeros, and graphs

(a) Express the function $F(x) = 2x^3 - x^4$ in factored form and identify all zeros of F with their multiplicities.

(b) For each zero a of F, find a constant A such that the graph of F near $(a, 0)$ is approximated by the graph of the form $y = Af(x)$. Check by graphing.

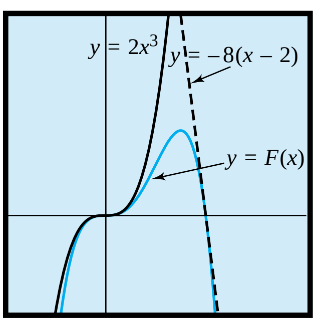
Solution

(a) If we factor out x^3, we can write $F(x) = x^3(2 - x)$. By the zero product principle, the zeros of F are 0 (of multiplicity 3) and 2.

(b) Near $x = 0$, the other factor, $(2 - x)$, is near 2, so we would expect $F(x)$ to be approximated by $y = 2x^3$ near (0, 0).

For the other zero, when x is close to 2, x^3 is close to 8. F should be very nearly equal to $y = 8(2 - x) = -8(x - 2)$.

The graphs of $y = 2x^3$, $y = -8(x - 2)$, and $y = F(x)$ are all shown in Figure 3. ◀



[−2, 4] by [−2, 4]

$F(x) = x^3(2 - x)$

FIGURE 3

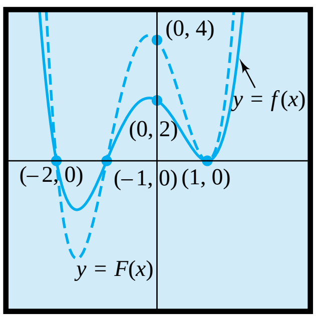
From the polynomial functions in the first two examples, it appears that we should be able to graph such functions by piecing together combinations of shifted multiples of the functions x, x^2, x^3, and so on. We have not considered the possibility of a quadratic factor with no real zeros. It turns out that there is also a close connection between the graph of a function having such a factor and the graph of a constant multiple of that factor, but we will not pursue the connection in this text. We invite the curious reader, however, to explore the graph of a function such as $f(x) = (x + 3)(x^2 + 1)$. The graph of $y = x^2 + 1$ is a parabola with vertex where $x = 0$. Compare the graph of f with the pieces $y = 3(x^2 + 1)$ and $y = 10(x + 3)$.

For each nonrepeated zero, there is a single factor, and an x-intercept point where the graph crosses the x-axis in essentially *linear* fashion. We associate a double zero, a zero with multiplicity two, with a point where the graph is tangent to the x-axis. Zeros of greater multiplicity correspond locally to translations of the graphs of $y = x^3$, $y = x^4$, and so on. We can use this observation to build product functions with any desired set of zeros. An equation for a product function can be written in factored form, or the factors can be multiplied out to obtain what is called the **expanded form.**

►EXAMPLE 3 Polynomials with specified zeros

(a) Write an equation for a polynomial function f having zeros -1, -2, and 1 as a zero of multiplicity two.

(b) Write an equation for a polynomial function F, with the same zeros as f, whose graph contains (0, 4).



[−3, 3] by [−5, 5]

FIGURE 4
$f(x) = (x - 1)^2(x + 1)(x + 2)$
$F(x) = 2(x - 1)^2(x + 1)(x + 2)$

Solution

(a) Without specifying some additional point, there is not a unique polynomial function with the given zeros, so we build the simplest. For the repeated zero 1, 1, we need a factor $(x - 1)^2$, and we also need linear factors $x + 1$ and $x + 2$. We can write an equation for f as

$$f(x) = (x - 1)^2(x + 1)(x + 2) = x^4 + x^3 - 3x^2 - x + 2.$$

The graph is the solid curve in Figure 4.

(b) Tracing along the graph of f, we see that the y-intercept point is (0, 2), a fact that is also obvious from the expanded form, since $f(0) = 2$. For a function F with the same zeros as f such that $F(0) = 4$, we want to dilate the graph of f vertically by a factor of 2. Thus $F(x) = 2f(x) = 2(x - 1)^2(x + 1)(x + 2)$, or in expanded form, $F(x) = 2x^4 + 2x^3 - 6x^2 - 2x + 4$. Its graph is the dotted curve in Figure 4. ◄

TECHNOLOGY TIP ◆ *Checking algebra*

For most purposes, expanded form is not necessary, but a graphing calculator can be used to check our algebra even if it does not handle symbolic forms. To see if our expanded form of F in Example 3 is correct, we can graph both $2(x - 1)^2(x + 1)(x + 2)$ and the expanded form, $2x^4 + 2x^3 - 6x^2 - 2x + 4$, in the same screen. If the graphs show any differences, then we obviously need to check our multiplication again.

In calculus courses, techniques are developed to find maximum and minimum values of a function. Important as these techniques are, a graphing calculator can be used to get excellent approximations for such values. It is handy to have some terminology and definitions. We assume that the graph of f contains no isolated points.

Definition: local extrema and turning points

Suppose c is in the domain D of a function f.

If $f(x) \geq f(c)$ for all x in D in some open interval containing c, then $f(c)$ is called a **local minimum** of f.

If $f(x) \leq f(c)$ for all x in D in some open interval containing c, then $f(c)$ is called a **local maximum** of f.

Local maxima and minima are called **local (or relative) extrema.** If the above inequalities hold for every x in D, then $f(c)$ is called an **absolute minimum (or maximum).**

If $f(c)$ is a local extremum, then the point $(c, f(c))$ is called a **turning point** of the graph.

When we want to find zeros and local extrema of polynomials, the choice of viewing windows is critical, as illustrated in the next example.

►EXAMPLE 4 ***Windows and graphs*** Draw graphs of $y = x^3 - 3x^2 + 2x - 10$ to locate all zeros and local extrema.

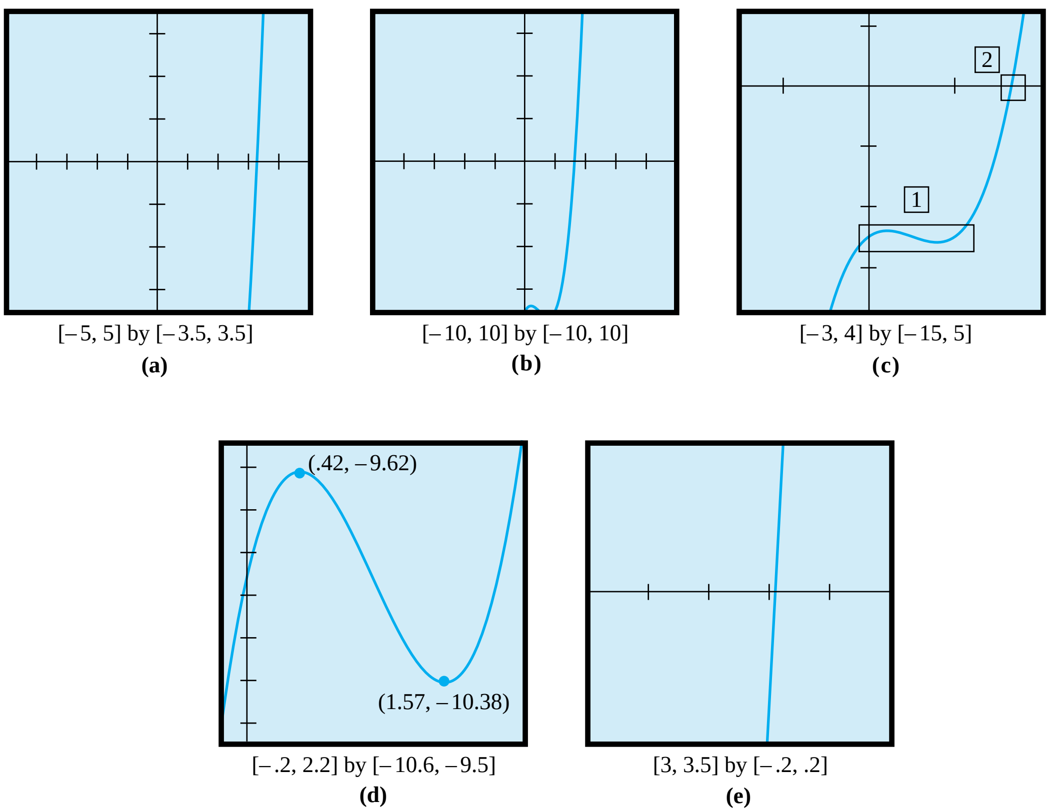


FIGURE 5
$y = x^3 - 3x^2 + 2x - 10$

Solution

If we begin with a decimal window (Figure 5a), we can see an x-intercept near 3, but very little else of interest. We clearly need a larger window.

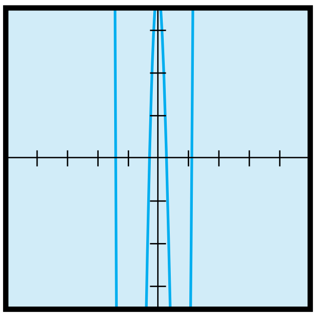
Setting a window of $[-10, 10] \times [-10, 10]$, it appears that something is happening near the y-intercept point $(0, -10)$, but the graph does not yet show enough detail to allow us even to know what we should be interested in. See Figure 5b.

To see better what is happening near $(0, -10)$, we set a window of $[-3, 4] \times [-15, 5]$ and get the graph in Figure 5c. We can see two "humps," as well as the x-intercept point, but there is too much compression in the y-direction to get much detail. Accordingly, we zoom in to look at the points of interest more closely.

When we zoom into a box like the one labeled 1 in Figure 5c, we exaggerate the vertical dimensions and we can trace to get a pretty good estimate of the local maximum near $(0.42, -9.62)$ and the local minimum near $(1.57, -10.38)$. See Figure 5d.

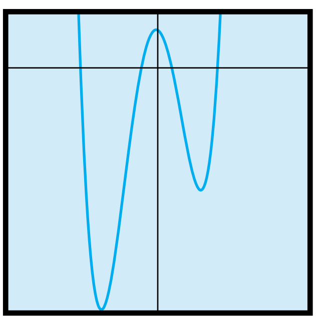
Returning to Figure 5c, if we zoom into a box like the one labeled 2, we can trace to find that $y = 0$ when x is about 3.31. See Figure 5e. ◀

There are some obvious questions about what we have done in Example 4. How do we know we have located all the zeros and local extrema? At this point we have no real justification for claiming to have completed the example. Part of our task in this section is to look at enough graphs of polynomial functions to make some reasonable guesses about "typical" polynomial graphs. In the next section we get a number of theorems to justify our observations. In particular, we will learn that the graph of a cubic polynomial such as the one in Example 1 can have at most two "humps" or turning points, so that there can be no more local extrema, and the graph can never turn back to the x-axis.



$[-10, 10]$ by $[-20, 20]$

(a)



$[-10, 10]$ by $[-130, 25]$

(b)

FIGURE 6

$p(x) = (x^2 - 1)(x^2 + x - 20)$

▶EXAMPLE 5 Graphs, factors, and zeros

Let $p(x) = (x^2 - 1)(x^2 + x - 20) = x^4 + x^3 - 21x^2 - x + 20$.

(a) Find a window in which you can see four zeros and three turning points on the graph of $y = p(x)$.

(b) Use the factored form of $p(x)$ to find all zeros.

Solution

(a) In a decimal window, we see nothing but essentially vertical lines. Increasing our ranges to $[-10, 10] \times [-20, 20]$ is a little better. We can at least see four x-intercepts, and what appears to be a turning point near $[0, 20]$. See Figure 6a. Tracing in both directions, we can read y-coordinates below -120, so we try a y-range of $[-130, 25]$. The graph in Figure 6b shows four zeros and three turning points. There are, of course, many windows that would work as well.

(b) From the factored form, we can use the zero-product principle to assert that $p(x) = 0$ only when $x^2 - 1 = 0$ or $x^2 + x - 20 = 0$. Each of these equations is a quadratic that factors readily, so $p(x) = (x - 1)(x + 1)(x + 5)(x - 4)$. By the zero-product principle, we get one zero from each factor. The zeros are $-5, -1, 1$, and 4. ◀

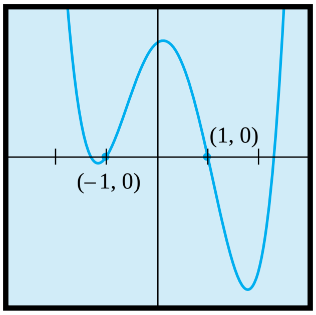
TECHNOLOGY TIP ◆ *Scaling and autoscaling*

In Example 5, the graph of p in the $[-10, 10] \times [-20, 20]$ window went "off-scale," dipping down out of our view.

Tracing displays function values as y-coordinates even for points that do not appear in the window. This allows us to estimate the y-range needed to see features that aren't visible in a particular window, as we did in the example.

Another feature available on many graphing calculators is called Autoscale or Zscale. Having set the x-range, when we use Autoscale, the calculator computes function values for the entire x-range and makes the y-range big enough to show all computed y-values.

This can be handy at times, but with many functions, including polynomials because of their steep end behavior, the resulting graph has so much vertical compression that interesting behavior is "squashed" out of sight. From the $[-10, 10] \times [-20, 20]$ window in Example 5, try autoscaling to see what happens.



[−3, 3] by [−4, 4]

FIGURE 7
$p(x) = (x^2 - 1)(x^2 - x - 3)$

►EXAMPLE 6 ***Graphs, factors, and zeros*** Repeat Example 5 for the function $p(x) = (x^2 - 1)(x^2 - x - 3) = x^4 - x^3 - 4x^2 + x + 3$.

Solution

(a) Now a decimal window is almost good enough, but one turning point is off screen. Figure 7 shows a graph in $[-3, 3] \times [-4, 4]$.

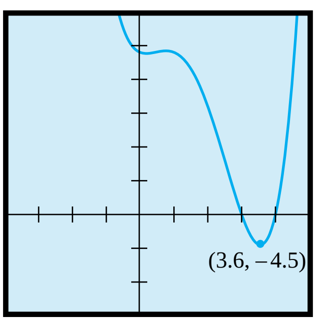
(b) From the factored form, we again have zeros at -1 and 1, from $x^2 - 1 = 0$. Solving $x^2 - x - 3 = 0$, however, requires the quadratic formula to find the two remaining zeros: $x = \frac{1 \pm \sqrt{13}}{2}$. ◄

►EXAMPLE 7 ***Finding turning points*** Let $p(x) = x^4 - 6x^3 + 7x^2 - 2x + 24$.

(a) Find a window in which you can see three turning points and two real zeros.

(b) Find the coordinates of the lowest turning point to one decimal place.

Solution



[−4, 5] by [−15, 30]

FIGURE 8
$p(x) = x^4 - 6x^3 + 7x^2 - 2x + 24$

(a) After some experimentation, we get the calculator graph of Figure 8 in the $[-4, 5] \times [-15, 30]$ window. The two turning points near the y-intercept point are not very pronounced, but there are clearly three turning points on the graph. We could set a window in which the turning points near the y-intercept are more visible, but then we would not see the lowest. The graph has only two x-intercept points.

(b) Tracing along the curve and zooming in as needed, we find that the lowest turning point is near $(3.6, -4.5)$. ◄

Not all graphs of polynomial functions have turning points. The most obvious case is the set of all polynomials of degree one or less, whose graphs are straight lines. The graph of $y = x^3$, which we have met before, levels out to run tangent to the x-axis at the origin; the function is always increasing, and so there are no turning points. The graph of the cubic function $f(x) = x^3 + 2x$ does not even level out. See Figure 9.

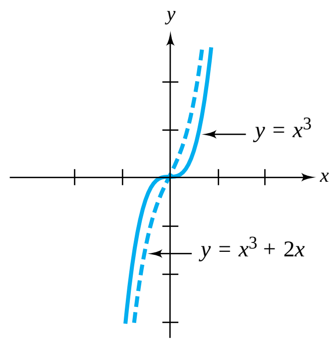


FIGURE 9

▶EXAMPLE 8 ***Finding intersections*** Let $p(x) = x^3 - 2x^2 + 5x + 3$ and $g(x) = 2\sqrt{x + 4}$. Graph both f and g in a window that shows the intersection of the curves, and locate the coordinates of the intersection to one decimal place.

Solution

After some experimentation, it appears that the graph of p has no turning points and that the intersection shown in $[-5, 5] \times [-1, 10]$ (see Figure 10a) is the only intersection of the two curves. Zooming in as needed on the point of intersection, we read the coordinates as approximately (0.2, 4.1). ◀

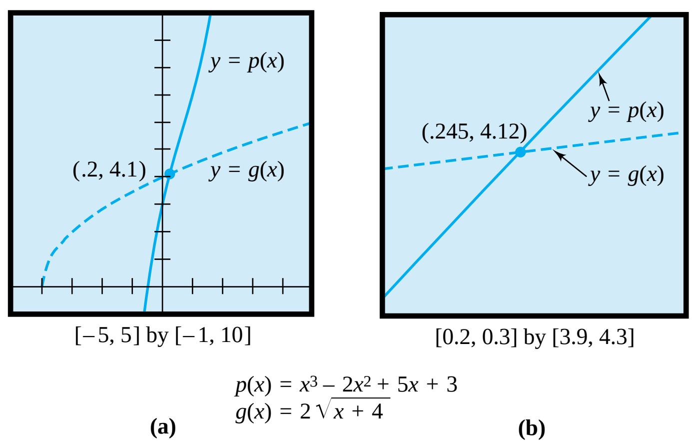


FIGURE 10

TECHNOLOGY TIP ◆ ***Trapping an intersection***

Rather than simply zooming in or drawing a box, some people prefer a process that lets us keep track of the window size and thus the accuracy. We can trace in Figure 10a and find the intersection is between $x = .2$ and $x = .3$, and between $y = 3.9$ and $y = 4.3$. Setting these numbers as range values, we get a picture something like Figure 10b, in which we can trace, knowing that the pixel increment in the new window is about $\dfrac{.3 - .2}{\#\text{ pixel cols.}}$ ($\approx .001$).

Smoothness and End Behavior

All the graphs of polynomial functions we have looked at so far are *smooth,* with no jumps, breaks, or corners. You will learn in calculus that these properties follow from the fact that polynomial functions are *continuous* and *differentiable.* For now, we simply accept these properties about polynomial graphs. Furthermore, graphs of polynomial functions (of degree greater than 1) continue to rise or fall very steeply as we move along the graph to the right or left. We are asking what happens as x becomes large and positive or large and negative (for which we use the notation $x \to \infty$ or $x \to -\infty$). This **end behavior** depends solely on the degree of the polynomial and the sign of the leading coefficient. With a positive leading coefficient, we always have $f(x) \to \infty$ as $x \to \infty$. In most cases, we can look at the equation defining the polynomial and see what the end behavior will be.

To get a better feeling for some of the variety of graphs of polynomial functions, numbers of turning points and zeros, and so on, we have a table for several polynomial functions. We do not show graphs here. Rather, we ask you to graph each one and verify for yourself the observations we record in the table. You may select whatever window will be most helpful. Most of the pertinent information can be seen in a window such as $[-4, 5] \times [-10, 15]$, but adjust the window as needed. The arrows suggest the end behavior by indicating the direction in which the graph is heading.

Polynomial Function	*End*		*Degree*	*Real Zeros*	*Turning Points*
$y = x^5 - 3x^4 - 5x^3 + 15x^2 + 4x - 12$	↙	↗	5	5	4
$y = -x^3 + 2x^2 + 3x - 1$	↖	↘	3	3	2
$y = -x^4 + 5x^2 - 4$	↙	↘	4	4	3
$y = x^6 - 2x^4 - 3x^2 - 5x + 8$	↖	↗	6	2	3
$y = -x^6 + 3x^5 + 5x^4 - 15x^3 - 3x^2 + 12x - 5$	↙	↘	6	4	5

On the basis of our discussions thus far, we can make some observations that we will substantiate in the next section.

Suppose f is a polynomial function of degree n, where $n \geq 1$.

1. The number of **real zeros** (counting multiplicities) is either n or some even number less than n (such as $n - 2$, $n - 4$, etc.).
2. The number of **turning points** or **local extrema** is $n - 1$ or some even number less than $n - 1$ (such as $n - 3$, etc.).
3. **End behavior:**

If n is *even,* and the leading coefficient is

positive, as $x \to \pm\infty$, then $y \to \infty$; ↖. . .↗

negative, as $x \to \pm\infty$, then $y \to -\infty$. ↙· · ·↘

If n is *odd* and the leading coefficient is

positive, as $x \to -\infty$, $y \to -\infty$ and as $x \to \infty$, $y \to \infty$; ↙· · ·↗

negative, as $x \to -\infty$, $y \to \infty$ and as $x \to \infty$, $y \to -\infty$. ↖. . .↘

EXERCISES 3.1

Check Your Understanding

True or False. Give reasons. Draw a graph whenever you think it might be helpful.

1. If k is any positive number, then the graph of $y = 1 - kx^3$ contains no points in Quadrant III.
2. The graph of $f(x) = x^3 + x^2 - 2x + 3$ has two turning points.
3. If c is a zero of f, then $(0, c)$ is a point on the graph of f.
4. Every real zero of $f(x) = (1 + x^2)(x^2 - x - 2)$ is also a zero of $g(x) = x^2 - x - 2$.
5. The graph of $f(x) = x^3 - 3x^2 - 7x - 5$ contains points in all four quadrants.
6. The degree of $f(x) = x^3 + x(1 - x^2)$ is 3.
7. The function $f(x) = x^3 - 3x^2 - 7x + 3$ has one negative zero and two positive zeros.
8. There is no fourth degree polynomial function whose graph has exactly two turning points.
9. Using the window $[-5, 5] \times [-40, 40]$ we can conclude that $f(x) = x^3 - x^2 + 5x + 4$ has a positive zero.
10. For $f(x) = x^3 - 18x^2 + 24x + 125$, using the window $[-8, 24] \times [-1300, 400]$ we can conclude that all zeros of f are between -3 and 20.

Develop Mastery

Exercises 1–4 Determine whether or not f is a polynomial function. If it is, give its degree.

1. $f(x) = 4 - 3x - 2x^2$
2. $f(x) = x^2 + \sqrt{x^2} - 3$
3. $f(x) = x(x + 1)(x + 2)$
4. $f(x) = \sqrt{x^2 + 9}$

Exercises 5–10 ***Combining Functions*** *Use the polynomial functions f, g, and h, where*

$$f(x) = 3x + 2 \quad g(x) = 5 - x \quad h(x) = 2x^2 - x.$$

(a) *Determine an equation that describes the function obtained by combining f, g, and h.* ***(b)*** *If it is a polynomial function, give the degree, the leading coefficient, and the constant term.*

5. $f + g$
6. $f - h$
7. fg
8. $h \circ f$
9. $f \circ h$
10. $\frac{f}{g}$

Exercises 11–12 ***Which Window?*** *In order to determine the zeros of f, which window would you use?*

11. $f(x) = 0.3x^3 + 3x^2 - 7x - 6$; three zeros.
    (i) $[-10, 10] \times [-10, 10]$
    (ii) $[-8, 10] \times [-10, 50]$
    (iii) $[-15, 10] \times [-10, 80]$
12. $f(x) = x^4 - 11x^3 - 16x^2 + 44x + 400$; two zeros.
    (i) $[-10, 10] \times [-10, 10]$
    (ii) $[-10, 10] \times [-400, 400]$
    (iii) $[-5, 15] \times [-2500, 2000]$

Exercises 13–14 ***Which Window?*** *The graph of f contains a local maximum point and a local minimum point. Which window would you use to see this feature?*

13. $f(x) = x^3 - 16x^2 - 24x + 400$
    (i) $[-10, 10] \times [-10, 10]$
    (ii) $[-5, 10] \times [-200, 200]$
    (iii) $[-5, 15] \times [-480, 450]$
14. $f(x) = -x^3 - 20x^2 + 75x + 800$
    (i) $[-10, 10] \times [-10, 10]$
    (ii) $[-20, 10] \times [-2000, 1200]$
    (iii) $[-20, 5] \times [-1000, 1000]$

Exercises 15–18 ***Zero-product Principle*** *A formula for function p is given in factored form.* ***(a)*** *Express $p(x)$ in standard (expanded) form, give the leading coefficient, and constant term.* ***(b)*** *Use the zero-product principle to find the zeros of p.* ***(c)*** *Use cut points to find the solution set for $p(x) < 0$.*

15. $p(x) = x(x - 1)(x + 2)$
16. $p(x) = x^2(x - 2)(x - 1)$
17. $p(x) = (x - 1)(x + 1)(2x - 1)$
18. $p(x) = (2x^2 + x - 1)(x^2 + 2x - 3)$

Exercises 19–22 ***(a)*** *Factor and find all the zeros of f (including complex zeros).* ***(b)*** *Determine the end behavior.* ***(c)*** *Use a calculator graph to check your answers.*

19. $f(x) = x(x - 3) + 2x(x + 2)$
20. $f(x) = (x + 2)(x - 1) - (x + 2)(2x + 3)$
21. $f(x) = x^3 - 1$
22. $f(x) = x^3 - 2x^2 - 3x + 6$

Exercises 23–26 ***Zeros and Turning Points***
(a) *Read the discussion at the end of this section and tell how many real zeros f can possibly have. Do the same for turning points.* ***(b)*** *Draw a calculator graph and then tell how many zeros and how many turning points there actually are.* ***(c)*** *In what quadrants do the turning points lie?*

23. $f(x) = x^3 - 2x^2 - 3x - 1$
24. $f(x) = -x^4 + 5x^2 - x + 1$
25. $f(x) = -x^5 - 4x^4 + 6x^3 + 24x^2 - 5x - 20$
26. $f(x) = x^4 - 3x^3 + x^2 - 3x - 8$

Exercises 27–30 ***Approximating a Zero*** *Draw a graph and use it to find an approximation (1 decimal place) for the lar- gest zero of f.*

27. $f(x) = x^3 - 2x^2 - 5x + 3$

28. $f(x) = -x^3 - 2x^2 + 5x + 4$

29. $f(x) = x^4 - 7x^2 + x + 5$

30. $f(x) = x^4 - 7x^2 - x + 5$

Exercises 31–34 ***Local Maximum***
(a) *For the functions in Exercises 27–30, determine the coordinates of any local maximum points (1 decimal place).* ***(b)*** *Describe the end behavior for f.*

Exercises 35–38 ***Turning Points*** *Determine the coordinates of the turning point in the given quadrant (1 decimal place).*

35. $f(x) = x^3 - 2x^2 - 5x + 3$; *QII*

36. $f(x) = x^3 - 2x^2 - 5x + 3$; *QIV*

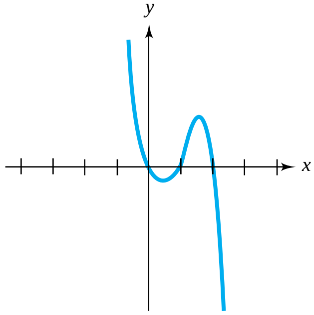
37. $f(x) = 3 + 5x - 2x^2 - x^3$; *QIII*

38. $f(x) = 3 + 5x - 2x^2 - x^3$; *QI*

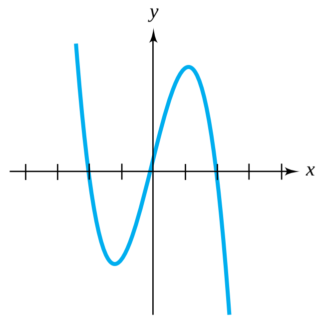
Exercises 39–43 ***Graph to Formula*** *A graph of a polynomial function is given, where the vertical scale is not necessarily the same as the horizontal scale. From the following list of polynomials, select the one that most nearly corresponds to the given graph. As a check draw a calculator graph of your selection and see if it agrees with the given graph.*

(a) $f(x) = x^2(x - 1)(x - 3)$
(b) $f(x) = x^2 + 3x$
(c) $f(x) = x^2(x - 2)^2$
(d) $f(x) = x(x - 1)^2(x + 2)^2$
(e) $f(x) = 4x - x^3$
(f) $f(x) = x(x - 1)(3 - x)$
(g) $f(x) = x(1 - x)(3 - x)$
(h) $f(x) = x^4 - 5x^2 + 4$

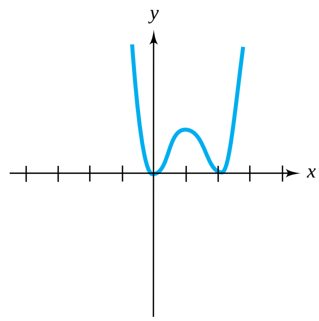
39.



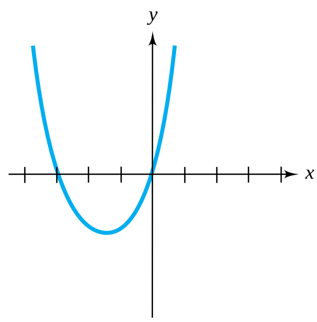
40.



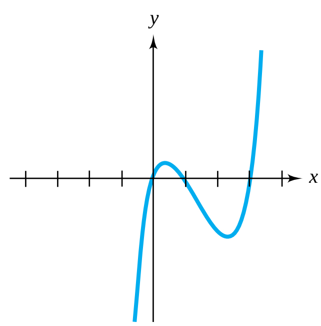
41.



42.



43.



Exercises 44–47 ***Related Graphs*** *Draw graphs of f and g. From the graphs, make a guess about how the graphs are related. Prove algebraically.*

44. $f(x) = x^3 + x^2 - 6x$, $g(x) = x^3 + 7x^2 + 10x$

45. $f(x) = x^3 + 3x^2 - x - 3$,
$g(x) = x^3 + 6x^2 + 8x$

46. $f(x) = x^3 + 3x^2 - x - 3$,
$g(x) = x^3 - 3x^2 - x + 3$

47. $f(x) = x^3 - x^2 - 6x$, $g(x) = x^3 - 7x^2 + 10x$

48. Determine all integer values of k for which $f(x) = x^3 - x^2 - 5x + k$ will have three real zeros. (*Hint:* Locate the local maximum and local minimum points for the graph of $g(x) = x^3 - x^2 - 5x$. Then consider vertical translations.)

49. Solve Exercise 48 for $f(x) = x^3 - 2x^2 - 5x + k$.

50. Solve Exercise 48 for $f(x) = -x^3 - x^2 + x + k$.

51. For what integer value(s) of k will f have one negative and two positive zeros where

$$f(x) = (x - k)^3 + 5(x - k)^2 + 3(x - k) - 1?$$

(*Hint:* Draw a graph of $y = x^3 + 5x^2 + 3x - 1$ and then consider horizontal translations.)

52. Solve Exercise 51 for

$$f(x) = -(x - k)^3 - 5(x - k)^2 + 5.$$

Exercises 53–54 ***Your Choice*** *Draw a rough sketch of a graph of a polynomial function satisfying the specified conditions. The answer is not unique.*

53. Function f has exactly 3 distinct zeros and $f(x) \to \infty$ as $x \to -\infty$.

54. The degree of f is 3, f has one real zero, and $f(x) \to -\infty$ as $x \to \infty$.

55. The base of a rectangle is on the x-axis and its upper two vertices are on the parabola $y = 16 - x^2$. Of all such rectangles, what are the dimensions, (1 decimal place) of the one with greatest area?

56. Solve the problem in Exercise 55 where $y = 16 - x^4$.

57. A rectangular box without a top is to be made from a rectangular piece of cardboard 12×15 inches by cutting a square from each corner and bending up the sides of the remaining piece. Of all such boxes, find the dimensions (1 decimal place) of the one having the largest volume. See illustration on p. 179.

58. ***Maximum Strength*** At a lumber mill a beam with rectangular cross section is cut from a log having cylindrical shape of diameter 12 inches. Assuming that the strength S of the beam is the product of its width w and the square of the depth d, what are the dimensions (1 decimal place) of the cross section that will give a beam of greatest strength.

59. If a polynomial function of degree 3 has no local extrema, explain why it must be one-one and therefore have an inverse.

60. **(a)** Draw a graph of $f(x) = x^3 - 3x^2 + 9x + 2$.
(b) Is it reasonable to conclude that f is one–one, and so it has an inverse given by the equation $x = y^3 - 3y^2 + 9y + 2$ giving $y = f^{-1}(x)$? Give reason.
(c) Find a decimal approximation (1 decimal place) of $f^{-1}(3)$. That is, solve the equation $y^3 - 3y^2 + 9y + 2 = 3$

61. Solve Exercise 60 for the function
$f(x) = x^3 - 3x^2 + 9x - 2$.

62. Explain why a fourth degree polynomial function cannot be one-one. Consider end behavior.

Exercises 63–64 ***Point of Intersection*** *On the same screen, draw graphs of f and g. The two graphs intersect at a single point. Find the coordinates of that point (1 decimal place).*

63. $f(x) = x^3 - 2x^2 + 5x + 4$, $g(x) = x\sqrt{x + 4}$.

64. $f(x) = x^3 + 2x^2 + 3x - 5$,
$g(x) = x^2 - 8x + 15$.

Exercises 65–66 ***Intersecting Graphs*** *The graph of f and the half circle intersect at a single point. Use calculator graphs to help you find the coordinates of the point of intersection (1 decimal place).*

65. $f(x) = x^3 - 3x^2 + 5x - 8$;
upper half of circle $(x - 2)^2 + y^2 = 25$.

66. $f(x) = x^3 - 3x^2 + 5x + 8$;
lower half of circle $(x + 1)^2 + y^2 = 9$.

Exercises 67–72 Determine the end behavior of the graph of the function when $x \to \infty$ and when $x \to -\infty$.

67. $f(x) = 2x - 3x^2$

68. $g(x) = x^4 - 3x^2 + 4$

69. $h(x) = 1 + 3.4x - 5.2x^3 - 2x^5$

70. $f(x) = -\dfrac{2}{5}(2x - x^3)$

71. $g(x) = (3 - 2x)(4 - x^3)$

72. $h(x) = (1 - 2x)(3 - 4x^2)$

Exercises 73–76 ***Looking Ahead to Calculus*** *In calculus you will learn that for the given function f there is an associated function g such that the real zeros of g are the x-coordinates of the local extrema points of f. If g has no real zeros then f has no local extrema points.*
(a) *Find the zeros of g.*
(b) *Find the coordinates of the local extrema points of f, if there are any.*
(c) *Use a graph as a check.*

73. $f(x) = 2x^3 + 3x^2 - 12x + 3$,
$g(x) = x^2 + x - 2$

74. $f(x) = x^3 - x^2 - 8x + 1$,
$g(x) = 3x^2 - 2x - 8$

75. $f(x) = x^3 - 3x^2 + 12x + 1$,
$g(x) = x^2 - 2x + 4$

76. $f(x) = x^3 - 6x^2 + 9x + 4$,
$g(x) = x^2 - 4x + 3$