$A: 87-100$

$B: 76-86$

$C: 60-75$

Mathematics 5710
Test 2, Fall 2013

1. Definitions:

a) What is a random variable? (2 pts) $X : \Omega \to R$

b) What is the cumulative distribution function, denoted by $F_X(x)$, for a random variable X? (2 pts)

$F_X(x) = P(X \le x)$

c) What is a discrete random variable? What is the probability frequency for a discrete random variable? (2 pts)

$P_X(x) = P(X = x)$

X is discrete if its values can be listed.

d) What is a continuous random variable? What is the probability density function (denoted by $f_X(x)$) of a continuous random variable? How are probabilities for a continuous random variable computed? How are $F_X(x)$ and $f_X(x)$ related? (4 pts)

X is continuous provided there exists $f : R \to [0, \infty)$ s.t. $P(a < X < b) = \int_a^b f(x)\,dx$. f is called the probability density function. $F'(x) = f(x)$.

2. Let box A = [1, 2, 3] and box B = [1, 2, 3, 4,]. Draw one number from each of the following boxes and let X be the minimum of the two numbers drawn.

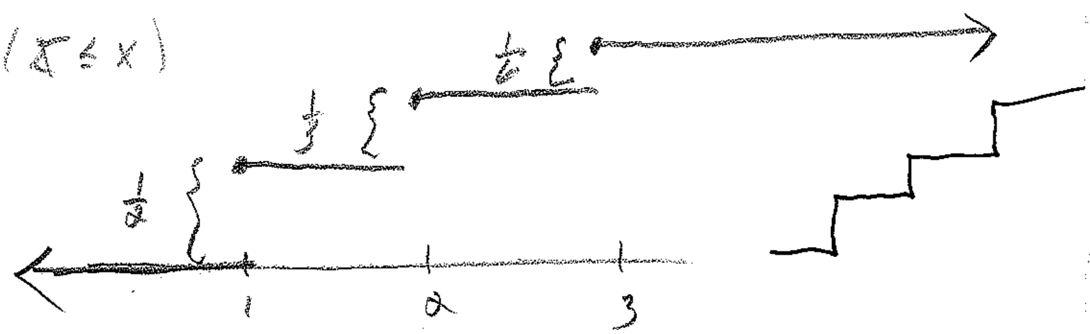
a) Find *p(x)*, the probability frequency function of *X*. (8 pts)

(1,1) (1,2) (1,3) (1,4)
(2,1) (2,2) (2,3) (2,4)
(3,1) (3,2) (3,3) (3,4)

X:	1	2	3
$p(x)$:	$\frac{6}{12}$	$\frac{4}{12}$	$\frac{2}{12}$
	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

b) Sketch *F(x)*, the cumulative distribution function of *X*. (8 pts)

$F(x) = P(X \le x)$



7. Suppose X is a normal random variable with $\mu = 5$ *and* $\sigma = 3$.

a) Express X in terms of Z (Z is standard normal, $\mu = 0$ *and* $\sigma = 1$) (4 pts)

$$\frac{X-5}{3} = Z \,, \quad X = 3Z+5$$

b) Find $P(\ -1 < 2X-3 < 1\ )$. Write your answer as an integral; do not evaluate. (6 pts)

$$= P(\ 2 < 2X < 4) = P(\ 1 < X < 2)$$

$$= \int_1^2 \frac{1}{3\sqrt{2\pi}} e^{-\frac{(x-5)}{18}} dx \quad \text{or}$$

$$P(-1 < 6Z+10-3 < 1) = P(-8 < 6Z < -6)$$

$$= P(-\tfrac{4}{3} < Z < -1) = \int_{-\frac{4}{3}}^{-1} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

8. Suppose the continuous random variable X has probability density function given by $f(x) = 4xe^{-2x^2}$, $x \geq 0$. Let $Y = \sqrt{X}$. Find the density function for Y. (12 pts)

1) $F_Y(t) = P(Y \leq t) = P(\sqrt{X} \leq t) = P(X \leq t^2)$

$= F_X(t^2)$,

2) $f_Y(t) = F_Y'(t) = F_X'(t^2) \cdot 2t =$

$f_X(t^2) \cdot 2t = 4t^2 e^{-2t^4} \cdot 2t$

$= 8t^3 e^{-2t^4}$.

5. Let $f(x) = cx^3$ for $0 < x < 4$.

a) Find the value of c that makes f a density function for a random variable X. (4 pts)

$$\int_0^4 cx^3\,dx = 1, \quad c\frac{x^4}{4}\Big|_0^4 = 1, \quad 64c = 1, \quad c = \frac{1}{64}$$

b) Find the cumulative distribution function for X. (4 pts)

$$F(x) = \frac{x^4}{256}, \quad 0 < x < 4$$

c) Find $P(X^2 < 2)$ (4 pts)

$$= P(0 < X < \sqrt{2}) = \int_0^{\sqrt{2}} \frac{x^3}{64}\,dx = \frac{x^4}{256}\Big|_0^{\sqrt{2}} = \frac{1}{64}$$

6. When a certain component of a manufacturing process breaks down, the time that it takes to fix it (in hours) is a random variable with the density function

$$f(x) = \begin{cases} ce^{-5x} & \text{if } 0 \le x < \infty \\ 0 & \text{otherwise.} \end{cases}$$

$$f(x) = 5e^{-5x}, \quad 0 \le x < \infty$$

a) Calculate the value of c. (4 pts)

$$\int_0^\infty ce^{-5x}\,dx = -\frac{ce^{-5x}}{5}\Big|_0^\infty = \frac{c}{5} = 1, \quad c = 5$$

b) Find $F_X(x)$. (6 pts)

$$F(x) = 1 - e^{-5x}$$

$$F_X(x) = P(X \le x) = \int_0^x 5e^{-5t}\,dt = -e^{-5t}\Big|_{t=0}^{t=x}$$

c) Find the probability that, when this component breaks down, it takes at least 2 hours to fix it. (6 pts))

$$P(X \ge 2) = 1 - P(X \le 2) = 1 - F(2) = e^{-10}$$

$$\int_2^\infty 5e^{-5x}\,dx = -e^{-5x}\Big|_2^\infty = e^{-10}.$$

3. Suppose X is a random variable giving the number of tosses necessary for a biased coin to turn up heads where, on any single toss, the coin has probability 2/3 for "heads" and 1/3 fór "tails". Find the probability that X is an odd positive integer. (12 pts)

H , TTH , $TTTTH$, ...

$\frac{2}{3} + \left(\frac{1}{3}\right)^2 \cdot \frac{2}{3} + \left(\frac{1}{3}\right)^4 \cdot \frac{2}{3} + \ldots$ geometric series

$a = \frac{2}{3}$, $r = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$

$$\text{sum} = \frac{a}{1-r} = \frac{\frac{2}{3}}{1-\frac{1}{9}} = \frac{\frac{2}{3}}{\frac{8}{9}} = \frac{2}{3}\cdot\frac{9}{8} = \boxed{\frac{3}{4}}$$

4. The number of times that a person contracts a cold in a given year is approximately a Poisson random variable. On average, a person should expect 2 colds per year. Find the probability of getting more than three colds during the semester (a four-month period). (12 pts)

$X \sim \text{Poisson}$, $\lambda = \frac{2}{3}$

$$P(X=k) = \frac{\lambda^k e^{-\lambda}}{k!} = \frac{\left(\frac{2}{3}\right)^k e^{-\frac{2}{3}}}{k!} \quad , \quad k = 0, 1, 2, \ldots$$

$$P(X>3) = 1 - \left[e^{-\frac{2}{3}} + \frac{2e^{-\frac{2}{3}}}{3} + \frac{2}{9}e^{-\frac{2}{3}} + \frac{4}{81}e^{-\frac{2}{3}}\right]$$

$\approx .0048$