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Math 5710, Test 3, Fall 2013 **Name**____________________

1. a) How do you compute the probability of an event that is described in terms of two discrete random variables X and Y?

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Let $D = \{(x,y) :$ event E occurs $\}$

$$P(E) = \sum_{(x,y) \text{ in } D} p(x,y) \leftarrow -2$$

b) How do you compute the probability of an event that is described in terms of two continuous random variables X and Y?

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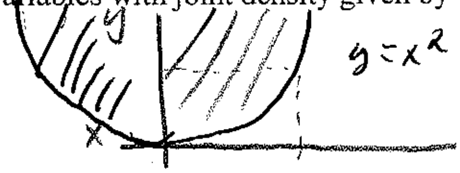
Let $D = \{(x,y) :$ event E occurs $\}$

$$P(E) = \iint_D f(x,y)\,dx\,dy \leftarrow -2$$

2. Suppose X, Y are continuous random variables with joint density given by

$$f(x,y) = \begin{cases} k\,e^{-13(xy)} & \text{for } 0 \le x^2 \le y \\ 0 & \text{otherwise} \end{cases}$$



$y = x^2$, $x = \sqrt{y}$

two intgs.

a) Find the density function for X and the density function for Y. Write each density in terms of an integral with appropriate limits of integration; do not evaluate.

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$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{x^2}^{\infty} k e^{-13xy}\,dy$$

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$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_{0}^{\sqrt{y}} k e^{-13xy}\,dx$$

or $-\sqrt{y}$

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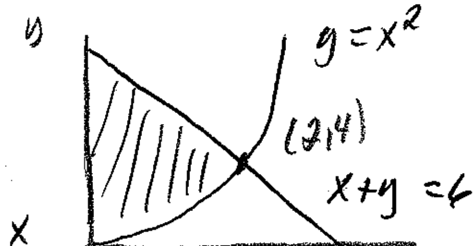
b) Find $P(X + Y \le 6)$. Write your answer as an integral; do not evaluate.



$$\int_0^2 \int_{x^2}^{6-x} k e^{-13xy}\,dy\,dx \quad \text{or}$$

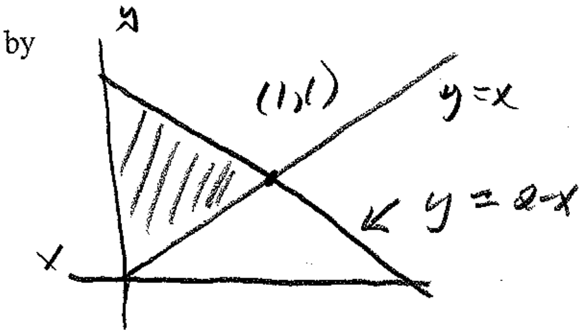
Type II

$x + x^2 = 6$, $x^2 + x - 6 = 0$, $(x+3)(x-2) = 0$

$x = 2$ or -3

3. The joint probability density function of X and Y is given by

$$f(x,y) = \begin{cases} 3xy & \text{if } 0 \le x \le y \le 2-x \\ 0 & \textit{otherwise} \end{cases}$$



a) Find the density function for X.

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$$f_X(x) = \int_x^{2-x} 3xy\,dy =$$

$$\left.\frac{3xy^2}{2}\right|_{y=x}^{y=2-x} = \frac{3}{2}x(2-x)^2 - \frac{3}{2}x\cdot x^2 = \frac{3}{2}(4x-4x^2)$$

$$= 6x - 6x^2$$

or $6x(1-x)$

b) Find the conditional density for Y given $X = x$.

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$$= \frac{f(x,y)}{f_X(x)} = \frac{3xy}{\frac{3}{2}(4x-4x^2)}, \quad x \le y \le 2-x$$

$$= \frac{2y}{4-4x}, \quad x \le y \le 2-x$$

$$= \frac{y}{2-2x}$$

for $0 \le x \le 1$.

c) Find $P(Y \le \sqrt{X}\,)$. Write your answer as an integral; do not evaluate.

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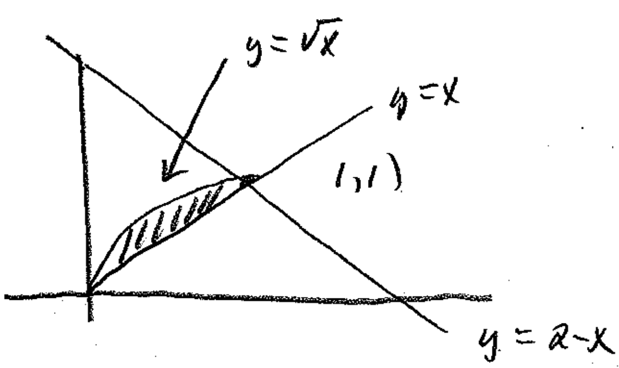

$$\int_0^1 \int_x^{\sqrt{x}} 3xy\,dy\,dx$$

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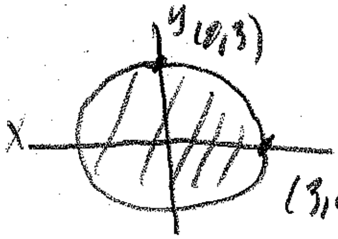
4. Suppose X *and* Y are independent standard normal random variables. Find $P(X^2+Y^2 \le 9)$. Write your answer as an integral; do not evaluate.

$$f_{X,Y}(x,y) = \frac{1}{2\pi} e^{-\frac{x^2}{2}} \cdot e^{-\frac{y^2}{2}} = \frac{1}{2\pi} e^{-\frac{(x^2+y^2)}{2}}$$



$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \frac{1}{2\pi} e^{-\frac{(x^2+y^2)}{2}}\, dy\, dx$$

or

$$\int_{0}^{2\pi}\int_{0}^{3} \frac{1}{2\pi} e^{-\frac{r^2}{2}}\, r\, dr\, d\theta$$

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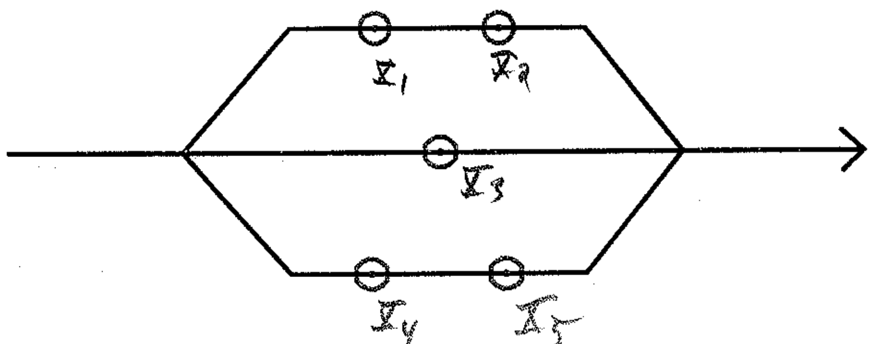
5. The time until failure of each component in the following system is an exponential random variable with parameter $\lambda = 2$. They operate independently of one another. Let W be the time until failure of the system. Find the density function for W.



$P(X_i \le t) = 1 - e^{-2t}$

$P(X_i > t) = e^{-2t}$

① $F_W(t) = P(W \le t) = (1-e^{-2t})(1-e^{-4t})^2$

② $f_W(t) = (1-e^{-2t})\, 2(1-e^{-4t}) \cdot 4e^{-4t} +$

$(1-e^{-4t})^2\, 2e^{-2t}$

+ simplify

(1,2) (2,2) (3,2) (4,2) (5,2)
(1,3) (2,3) (3,3) (4,3) (5,3)
(1,4) (2,4) (3,4) (4,4) (5,4)

6. Let box A = [1, 2, 3, 4, 5] and box B = [2, 3, 4]. Draw one number from each of the following boxes and let X be the minimum of the two numbers drawn.

a) Find the probability frequency function of X .

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X :	1	2	3	4
p_X :	$\frac{3}{15}$	$\frac{6}{15}$	$\frac{4}{15}$	$\frac{2}{15}$

$\frac{1}{5}$, $\frac{2}{5}$ $\frac{4}{15}$ $\frac{2}{15}$

b) Find the moment generating function for X and use it to find the expected value of X.

$Note: M_X(t) = E\left(e^{tX}\right) = \sum_x e^{tx} p(x)$

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$$= \frac{1}{5}e^{t} + \frac{2}{5}e^{2t} + \frac{4}{15}e^{3t} + \frac{2}{15}e^{4t}$$

$$E(X) = M_X'(0), \quad M_X'(t) = \frac{e^{t}}{5} + \frac{4e^{2t}}{5} + \frac{12e^{3t}}{15} + \frac{8e^{4t}}{15}$$

$$E(X) = \frac{1}{5} + \frac{4}{5} + \frac{12}{15} + \frac{8}{15} = \frac{35}{15} = \frac{7}{3}$$

7. Suppose X is a continuous random variable with density function given by $f(x) = 7e^{-7x}$, $x \geq 0$. Derive the moment generating function for X and use it to find the expected value and variance of X. $Note: M_X(t) = E\left(e^{tX}\right)$

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$$M_X(t) = E\left[e^{tX}\right] = \int_0^\infty e^{tx}\cdot 7e^{-7x}\,dx = \int_0^\infty 7e^{(t-7)x}\,dx$$

$$= \frac{7e^{(t-7)x}}{t-7}\Bigg|_{x=0}^{\infty} = \frac{-7}{t-7} = \frac{7}{7-t}$$

$$M_X'(t) = \frac{+7}{(7-t)^2}, \quad M_X''(t) = \frac{7\cdot 2(7-t)}{(7-t)^4} = \frac{14}{(7-t)^3}$$

$$M_X'(0) = \frac{1}{7}, \quad M_X''(0) = \frac{2}{49}$$

$$E(X) = \frac{1}{7}, \quad \text{var}\, X = \frac{2}{49} - \frac{1}{49} = \frac{1}{49}$$